Maths 260 Lecture 28

- **Topics for today:**
  - More on using nullclines to sketch phase portraits for nonlinear systems
  - Modelling using systems
- **Reading for this lecture:** BDH Section 2.1
- **Suggested exercises:** BDH Section 2.1, #1-4,9,10
- **Reading for next lecture:** None

Result from the last lecture

- Linearisation can tell us about the behaviour of solutions near equilibria but is unhelpful for solutions far away from equilibria.
- Nullclines can help us to sketch the complete phase portrait for a nonlinear system (both near equilibria and far from equilibria).

Remember:

- The x-nullcline is the set of points (x, y) where $\frac{dx}{dt} = 0$ and tells us where the solution curves are vertical.
- The y-nullcline is the set of points (x, y) where $\frac{dy}{dt} = 0$ and tells us where the solution curves are horizontal.

Sketching a phase portrait

To sketch a phase portrait for a nonlinear system:

1. Find all equilibria. Where possible, use linearisation to determine their types (e.g., saddle, spiral source).
2. Draw the nullclines. Determine the direction of solutions in the regions between nullclines. Determine the direction of solutions on the nullclines.
3. Sketch some representative solution curves. Make sure the solution curves you sketch go in the directions determined by the nullclines and behave like the appropriate linearised system near any equilibrium.

Example 1

Use nullclines to sketch the phase portrait for the system

$$\frac{dx}{dt} = x - y^2 + 2$$
$$\frac{dy}{dt} = y - x.$$

Example 1

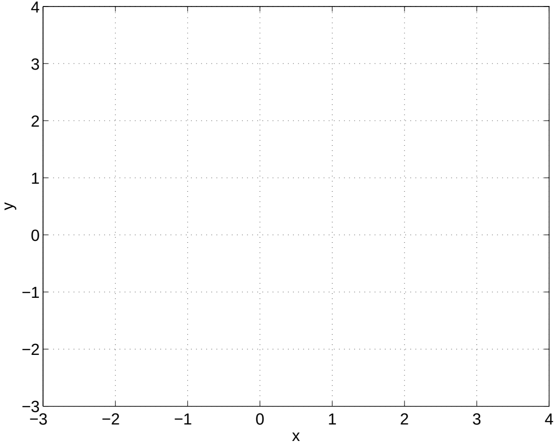
$$J = \begin{pmatrix} 1 & -2y \\ -1 & 1 \end{pmatrix}$$

$$J(-1,-1) =$$

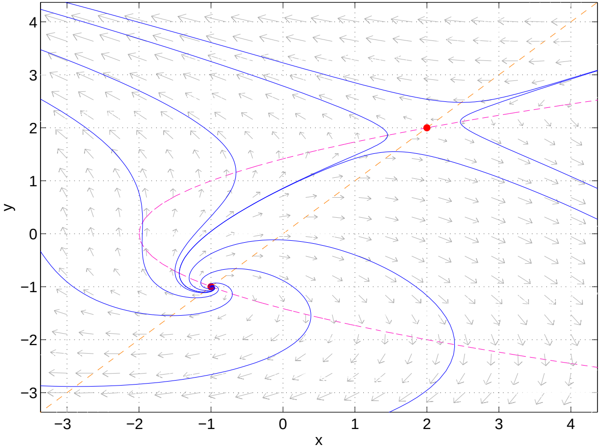
$$J(2,2) =$$

Example 1

Sketch the phase portrait using the nullclines:



The approximate phase portrait obtained using nullclines looks very like the phase portrait obtained with pplane:



Modelling - Predator/prey system example

The following equations give a typical simple model of two populations where animals of one type (known as the **predators**) eat animals of the other type (known as the **prey**).

Let $R(t) =$ number of prey (e.g., rabbits), in 1000's

Let $F(t) =$ number of predators (e.g., foxes) in 1000's.

A possible model of change in the two populations is given by

$$\dot{R} = 0.4R - 0.1RF,$$

$$\dot{F} = -0.5F + 0.1RF, \qquad R \geq 0, \; F \geq 0.$$

Physical significance of terms in the DEs

- The term $0.4R$ in the $\dot{R}$ equation gives unlimited growth of prey population if there are no predators.
- The term $-0.5F$ in the $\dot{F}$ equation gives exponential decay in the predator population if there are no prey.
- The term $-0.1RF$ in the $\dot{R}$ equation models the negative effect on prey population of 'interactions' between prey and predators, (i.e., predators eat prey and prey population decreases).
- The term $0.1RF$ in the $\dot{F}$ equation models the positive effect on predator population of interactions between prey and predators, (i.e., predators eat prey and predator population increases).

Equilibrium solutions to the predator/prey system

Rewrite the system as:

$$\dot{R} = R(0.4 - 0.1F),$$
$$\dot{F} = F(0.1R - 0.5),$$

It is easy to see that $(R, F) = (0, 0)$, is an equilibrium solution.

- What does this mean physically?

We also see that $(R, F) = (5, 4)$ is an equilibrium solution.

- Physically, this tells us that a prey population of 5000 and a predator population of 4000 is perfectly balanced; neither population increases or decreases over time.

Types of equilibria

The Jacobian is $J =$

$J(0,0) =$

$J(5,4) =$

Some other special cases

If $F(t_0) = 0$, then $dF/dt = 0$, and so $F(t) = 0$ for all time, regardless of the behaviour of R.

However, if $F(t) = 0$, then $dR/dt = 0.4R$, which implies

$$R(t) = R(0)e^{0.4t},$$

i.e., if there are no predators, the prey population grows exponentially.

Some other special cases

Similarly, if $R(t_0) = 0$, then $dR/dt = 0$, and so $R(t) = 0$ for all time, regardless of the behaviour of F.

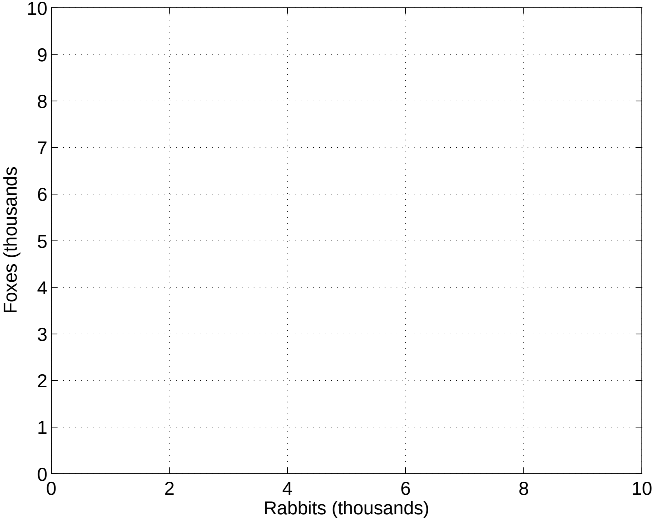
However, if $R(t) = 0$, then $dF/dt = -0.5F$, which implies

$$F(t) = F(0)e^{-0.5t},$$

i.e., if there are no prey, the predator population decreases exponentially.

Nullclines

Find and sketch the nullclines

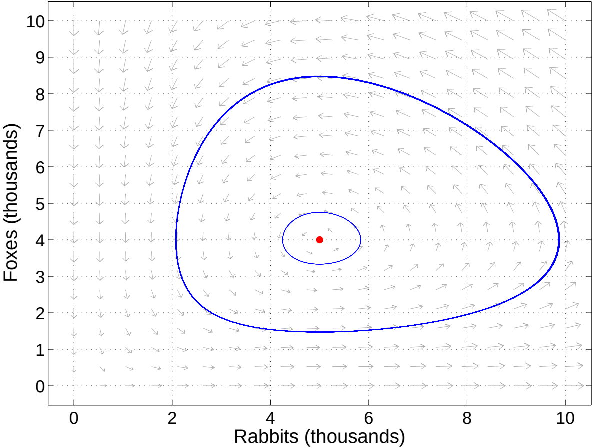


Experiments with *pplane* confirm that:

- There are equilibrium solutions at $(R, F) = (0, 0)$ and at $(R, F) = (5, 4)$;
- If $F(0) = 0$ and $R(0) > 0$, then R increases exponentially;
- If $R(0) = 0$ and $F(0) > 0$, then F decreases exponentially;
- All other solutions with $R(0) > 0$ and $F(0) > 0$ are periodic with R and F having the same period as each other.

Phase portrait

The phase portrait for the predator/prey system is:



This simple predator-prey model is known as the Lotka-Volterra model (1925).

What else can we model?

- Infectious diseases
  1. Think about the different populations involved (infected, immune, susceptible, dead, ...)
  2. How do they affect each other?

- Two species in competition for the same resources
  1. Can both species survive?
  2. Can one species become extinct and the other species survive?
  3. Can both species become extinct?

- What about species that are mutually beneficial?
  1. Here, each species helps the other one survive
  2. Populations should not be able to grow indefinitely as there are limits on natural resources

- And lots and lots more ....

Modelling mutually beneficial species - a quick example

Consider the nonlinear system

$$\frac{dx}{dt} = x(1 - 0.5x + 0.1y)$$
$$\frac{dy}{dt} = y(1 - 0.8y + 0.5x)$$

where $x(t), y(t) \geq 0$.

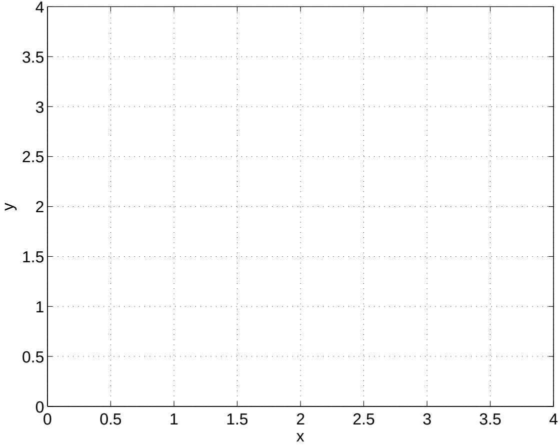
Think of $x(t)$ and $y(t)$ as two different populations that help each other. How can we tell (from the equations) that they help each other?

Getting ready to sketch the phase portrait

Let's see how far we can get with just nullclines:

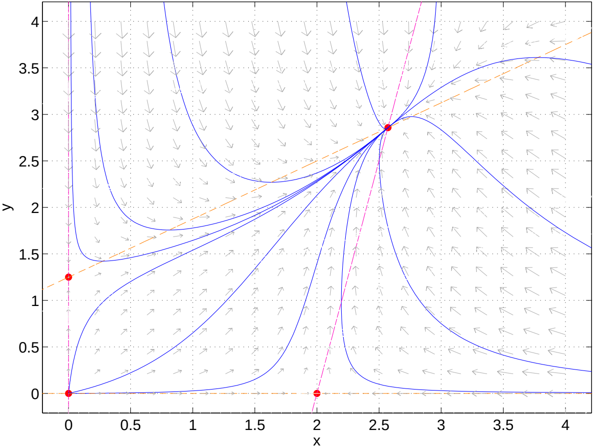
Phase portrait

Sketch the phase portrait:



Phase portrait from Matlab

Our sketch of the phase portrait agrees with *pplane:*



Important ideas from today's lecture:

- Simple population interactions can be modelled using a system of nonlinear differential equations.
- Some examples are models of predator-prey systems, competitive species and infectious diseases (epidemics).
- Simple models might not be totally accurate but they can be very useful - and not just for modelling population interactions.