GLIMPSES OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN THE TWENTIETH CENTURY. A PRIORI ESTIMATES AND THE BERNSTEIN PROBLEM

NEIL S. TRUDINGER

School of Mathematical Sciences, The Australian National University, CANBERRA, ACT 0200, AUSTRALIA

E-mail: Neil.Trudinger@maths.anu.edu.au

1 Introduction

The basic models of linear partial differential equations were formulated in the eighteenth and early nineteenth centuries. In order of appearance these were: the **wave equation**, (D'Alembert 1752),

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,; \tag{1.1}$$

the **Laplace equation**, (Laplace 1780),

$$\Delta u \;:= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,; \tag{1.2}$$

and the **heat equation**, (Fourier 1810),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 y}{\partial x^2}\,. \tag{1.3}$$

In these equations, u denotes the solution and x, y and t the independent variables. In applications, x and y indicate position and t time. These basic models in two variables were followed quickly by their higher dimensional analogues. Approaches to finding solutions and some basic theory were developed in the nineteenth century, with more rigorous treatments appearing in the late nineteenth and early twentieth century. The study of these equations is the basic fodder of introductory undergraduate courses in partial differential equations and well documented in many text books such as Courant and Hilbert [16], Petrowski [42], Garabedian [23], Weinberger [54], John [28], and Strauss [52].

The most famous **nonlinear** partial differential equations also arose around the same time. These were: the **Euler equations**, (Euler 1755),

$$\frac{\partial \widetilde{u}}{\partial t} + \widetilde{u} \bullet \nabla \widetilde{u} = -\nabla p\,, \quad \operatorname{div} \widetilde{u} = 0\,, \tag{1.4}$$

the **Navier Stokes equations**, (Navier 1822, Stokes 1845),

$$\frac{\partial \widetilde{u}}{\partial t} + \widetilde{u} \bullet \nabla \widetilde{u} - \Delta \widetilde{u} = -\nabla p\,, \operatorname{div} \widehat{u} = 0\,, \tag{1.5}$$

of fluid mechanics; the **minimal surface equation**, (Lagrange 1760),

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{1.6}$$

satisfied by functions whose graphs minimize surface area, and the **Monge-Ampère equations**, (Monge 1775),

$$\det D^2 u = f, \tag{1.7}$$

which arises in various geometric problems.

We have written these equations in their n dimensional forms. The independent variables are vectors $x = (x_1, \ldots, x_n)$ in Euclidean n-space. $\mathbb{R}^n$ and time t, in the case of (1.4) and (1.5). In the fluid mechanics equations (1.4) and (1.5), the solution $\widetilde{u} = (u^1, \ldots, u^n)$ is a vector function of x and t, (corresponding to velocity at point x and time t, while the function p, (corresponding to pressure at point x and time t) is also to be determined from the equations. What we really have here is thus a system of $n+1$ equations in $n+1$ unknown functions $u^1, \ldots, u^n, p$. For scalar functions u of n variables, we have employed the usual notation for the **gradient**,

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$$

Laplacian

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

and **Hessian**

$$D^2 u = \left[\frac{\partial^2 u}{\partial x_i \, \partial x_j}\right]_{i,j=1,\ldots,n},$$

while for vector functions $\widetilde{u} = (u^1 \ldots u^n)$, the **divergence**,

$$\operatorname{div} \widetilde{u} = \sum_{i=1}^{n} \frac{\partial u^i}{\partial x_i}.$$

The function f in the Monge-Ampère equation (1.7) is prescribed and we have abbreviated determinant to det.

Stemming from the three basic linear models (1.1), (1.2) and (1.3), partial differential equations (at least of second order) are classified respectively as hyperbolic, elliptic or parabolic. The main objects of our presentation are the **elliptic** equations, and our particular interest concerns their invariance properties. Accordingly, we rewrite the relevant examples from the above list.

1. **Laplace equation**

$$\Delta u \ :=\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \qquad (1.8)$$

This equation is **orthogonally invariant** in $\mathbf{R}^n$, in that its form is preserved by an orthogonal change of independent variables $x \in \mathbf{R}^n$

2. **Minimal surface equation**
Carrying out the differentiation in (1.6), we obtain

$$(1+|\nabla u|^2)\Delta u - \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \qquad (1.9)$$

This is a **quasilinear** equation, because it is linear in its highest order derivatives, and it is **elliptic** because its principal coefficient matrix

$$A = (1+|\nabla u|^2)I - \nabla u \otimes \nabla u$$

is positive. It possesses a significantly stronger invariance property than the Laplace equation, namely it is orthogonally invariant in $\mathbf{R}^{n+1}$, that is under an orthogonal transformation of $\mathbf{R}^{n+1}$, the graph $M_u = \{x, u(x)\}$ of a solution is transformed into a surface, which can be locally represented as the graph of a solution of the same equation. This invariance is immediately evident from the fact that the minimal surface equation is equivalent to the vanishing of the **mean curvature** of M_u.

3. **Monge-Ampère equation**

$$\det D^2 u = f. \qquad (1.7)$$

This is a **fully nonlinear** equation because the highest order derivatives occur nonlinearly. It will be elliptic with respect to solutions whose Hessian is positive, that is with respect to solutions which are locally uniformly convex. More generally we note that any second order partial differential equation of form

$$F(D^2u, \nabla u, u, x) = 0 \qquad (1.10)$$

is **elliptic**, with respect to a function u if the linearized principal coefficient matrix,

$$A = \frac{\partial F}{\partial(D^2 u)}$$

is positive. For the Monge-Ampère equation (1.7), A is the cofactor matrix $[U^{ij}]$ of the Hessian matrix D^2u, whose positivity is equivalent to that of D^2u itself. The Monge-Ampère equation is invariant with respect to **unimodular transformations** in $\mathbf{R}^n$, that is transformations of the independent variable x of the form

$$x \mapsto Tx, \quad \det T = 1,$$

so again it has a significantly stronger invariance property than Laplace's equation (where T is orthogonal).

Now what about equations which enjoy (or perhaps are cursed by) both the invariances of (1.10) and (1.7). It turns out that we have to go to fourth order to find these and such an equation figures in our last glimpse of the twentieth century.

2 Bernstein's Theorem

Our first glimpse in the twentieth century is a surprising and truly nonlinear result by Sergei Bernstein 1915 [7], one of the great pioneers of the modern theory of nonlinear elliptic equations. To motivate his theorem, let us recall that there are an abundance of solutions of Laplace's equation, defined on all of space, in more than one dimension. Typical examples, say in two dimensions, are the trivial examples of linear functions, quadratic functions such as xy, $x^2 - y^2$ etc. In fact, it is well known that the real part of any analytic function of a complex variable is harmonic, (that is a solution of Laplace's equation). But this is not the case for the minimal surface equation (1.9) in two dimensions, which we write in the form,

$$\left(1+\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\,\partial y} + \left(1+\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2} = 0 \qquad (2.1)$$

Bernstein's Theorem. *Let u be a solution of the minimal surface equation (2.1) in the plane. Then u is an affine function, that is its graph is a plane. Consequently a complete minimal surface in three dimensional space, which is a graph in some direction, must be a plane!*

A minimal surface is a surface which locally minimizes area. If imbedded, in three dimensional space, then locally at least it can be represented as a graph of a solution of the minimal surface equation. Bernstein's theorem shows that a global representation as a graph is only possible for the trivial examples. The simplest example of a complete minimal surface that is not a plane is the **catenoid**

$$x^2 + y^2 = (\cosh z)^2 \tag{2.2}$$

which is the surface of revolution of the catenary, $x = \cosh z$ about the z axis. Despite its restrictive aspect, Bernstein's Theorem turned out to be only a minor hiccup in the beautiful and rich **geometric** theory of two dimensional minimal surfaces that was subsequently developed in the twentieth century and which is presented for example, in the monographs by Osserman [40], Nitsche [30], Dierkes, Hildebrandt, Küster and Wohlrab [19] and Fang [20]. Fascinating examples of complete minimal surfaces include the helicoid (given parametrically by $x = t\sin s$, $y = -t\cos s$, $z = s$, $-\infty < s, t < \infty$ and the Costa-Hoffman-Meeks surface, which was the first example, (discovered only in 1985!), of an imbedded complete minimal surface of finite topological type other than the plane, catenoid or helicoid. The interested reader is referred to the book of Fang [20], supplemented by the [3] Filmstrip of Palais, available on the internet [41].

Bernstein's proof.

By calculation, the functions

$$\varphi = \arctan\frac{\partial u}{\partial x}, \qquad \psi = \arctan\frac{\partial u}{\partial y} \tag{2.3}$$

each satisfy the **linear elliptic** equation,

$$\left(1+\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2\varphi}{\partial x^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2\varphi}{\partial x\,\partial y} + \left(1+\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2\varphi}{\partial y^2} = 0. \tag{2.4}$$

with $|\varphi|, |\psi| < \pi/2$. In modern terminology, equation (2.4) means that the functions φ and ψ are harmonic on the graph of u. If (2.4) were Laplace's equation, we would conclude our result immediately from the classical Liouville theorem, which asserts that **bounded** harmonic vector functions in $\mathbf{R}^n$ are necessarily constant. Bernstein amazingly, and even with unbounded coefficients, proved a Liouville theorem for solutions of elliptic equations of the form

$$L\varphi = A \bullet D^2\varphi \tag{2.5}$$

in the plane, where the only conditions are that L is elliptic (that is the coefficient matrix A is positive).and the solution $\varphi = o(\sqrt{x^2+y^2})$ as $x, y \to \infty$. This would give us $\nabla\varphi = \nabla\psi = 0$, and hence $\nabla u =$ constant as required. More generally, Bernstein considered functions φ satisfying

$$\det D^2\varphi = \frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2} - \left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2 \le 0, \tag{2.6}$$

since any solution of (2.5) satisfies (2.6) by virtue of the positivity of A. Unfortunately there was a gap of a topological nature in Bernstein's argument, which was eventually fixed by E. Hopf in 1950 [27]. Many other proofs of Bernstein's Theorem were subsequently found by Bers [8], Finn [21], Nitsche [38], Fleming [22], Giusti [25], Simon [50] and we refer the reader to the survey [50] for further information. So far no alternative proofs have been found of the Liouville theorem employed by Bernstein and Hopf.

3 Jorgens' Theorem

Bernstein's Theorem has an analogue for the Monge-Ampère equation (1.7), discovered by Jorgens, 1954 [29].

Jorgens' Theorem *Let u be a solution of the Monge-Ampère equation,*

$$\det D^2\varphi = \frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2} - \left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2 = 1, \tag{3.1}$$

in $\mathbf{R}^2$. Then u is a second degree polynomial, that is the graph of u is a paraboloid.

We observe that equation (3.1) tells us that the eigenvalues of the Hessian D^2u are either both positive or both negative and hence a solution u is either convex or concave, whence the resultant paraboloid must be elliptic.

Bernstein's and Jorgen's Theorems are connected in that the former follows readily from the latter. We show this now, together with a simple complex variables proof of Jorgen's Theorem, following Nitsche [38].

Proof of Jorgen's Theorem

From our remark above, we can assume that a solution u of (3.1) is convex so that the mapping given by

$$\begin{cases} \xi = x + \dfrac{\partial u}{\partial x} \\ \eta = y + \dfrac{\partial u}{\partial y} \end{cases} \tag{3.2}$$

is a diffeormorphism of $\mathbf{R}^2$ onto itself. Now consider a complex variable

$$\zeta = \xi + i\eta$$

and define a complex function w by

$$w(\zeta) = x - \frac{\partial u}{\partial x} - i\left(y - \frac{\partial u}{\partial y}\right) \tag{3.3}$$

where x and y are related to ξ and η by inverting the transformation (3.2). By calculation we can show that the real and imaginary parts of w satisfy the Cauchy-Riemann equation,

$$\frac{\partial\, Rew}{\partial \xi} = \frac{\partial\, Imw}{\partial \eta}, \quad \frac{\partial\, Rew}{\partial \eta} = -\frac{\partial\, Imw}{\partial \xi}, \tag{3.4}$$

which means that w is an analytic function of ζ, along with its complex derivatives. Furthermore

$$|w'(\varphi)|^2 = \frac{\Delta u - 2}{\Delta u + 2} < 1 \tag{3.5}$$

and hence by the classical Liouville Theorem for analytic functions, (which is equivalent to that for harmonic functions in the plane), we have that w' is constant which implies that D^2u is constant and hence u is a second degree polynomial.

Jorgens' implies Bernstein

We define three functions f, g and h on $\mathbf{R}^2$ by

$$\begin{aligned} f &= \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)/v \\ g &= \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right)/v \\ h &= \left(1 + \left(\frac{\partial u}{\partial y}\right)^2\right)/v, \end{aligned} \tag{3.6}$$

where $v = \sqrt{1 + |\nabla u|^2}$. The minimal surface equation (2.1) implies that

$$\begin{aligned} \operatorname{curl}(f,g) &= \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = 0, \\ \operatorname{curl}(g,h) &= \frac{\partial g}{\partial y} - \frac{\partial h}{\partial x} = 0, \end{aligned} \tag{3.7}$$

as that by basic vector calculus, there exists a function φ with second derivatives

$$\frac{\partial^2 \varphi}{\partial x^2} = f, \quad \frac{\partial^2 \varphi}{\partial x \partial y} = g, \quad \frac{\partial^2 \varphi}{\partial y^2} = h. \tag{3.8}$$

But from (3.6) we have

$$\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} - \left(\frac{\partial^2 \varphi}{\partial x \partial y}\right)^2 = 1,$$

which implies, by Jorgens' Theorem, $\nabla u = \text{constant}$.

4 The Bernstein problem

The Bernstein problem in higher dimensions, whether entire minimal graphs in Euclidean space are hyperplanes, became one of the most celebrated problems in partial differential equations in the twentieth century. Its study was pivotal in the development of higher dimensional minimal surface theory and geometric measure theory. It was solved in the affirmative by De Giorgi [18] in 1965 for three dimensional graphs, and subsequently by Almgren [2] in 1966 for four dimensions and Simons [51] in 1968 for five to seven dimensions. Fleming's proof [22] of the two dimensional case turned out to be a trail blazer for the higher dimensional theory, through its connection with minimal cones. The Bernstein problem was finally settled in all dimensions through an amazing piece of work by Bombieri, De Giorgi and Giusti in 1969, who showed that in dimensions larger than seven, there do exist entire solutions of the minimal surface equation (1.9), whose graphs are not hyperplanes. Further important contributions by Schoen, Simon and Yau [46] in 1975 provided curvature estimates for minimal graphs in dimensions up to seven, which implied the Bernstein property and later Simon [49] in 1989 found and classified many examples of non-affine minimal graphs in dimensions higher than seven. For further information, the reader is referred to the excellent survey of Simon [50].

The higher dimensional Jorgens' theorem turned out to be more straightforward. Indeed, for the higher dimensional Monge-Ampère equations,

$$\det D^2 u = 1 \tag{4.1}$$

a convex entire solution in $\mathbf{R}^n$ is a second degree polynomial, that is its graph is a paraboloid, for all dimensions n. This was shown by Calabi [13] for $n \leq 5$ in 1958 and then for all n by Pogorelov [43], 1972. Their proofs used a priori interior estimates for third derivatives of solutions of Monge-Ampère equations, which were important in the early development of the theory of Monge-Ampère equations. In affine geometry, Jorgens' theorem implies that complete parabolic affine hyperspheres are elliptic paraboloids.

5 A priori estimates

A priori estimates in partial differential equations are estimates for prospective solutions, which are independent of any knowledge about their existence. Their fundamental importance lies in the crucial role they often play in establishing existence, uniqueness, regularity and other qualitative properties of solutions. Bernstein, in papers starting from 1906, [4], [5], [6], was the great pioneer of a priori estimates for solutions of nonlinear elliptic equations in the twentieth century. The scope of his work was limited as it preceded the discovery of much of the linear theory but nevertheless his ideas for gradient estimates in particular were used extensively throughout the century. These ideas included transformations of the dependent variable $u \mapsto \varphi(u)$ and application of the operator $\nabla u \bullet \nabla$ to get a differential inequality for $|\nabla u|^2$. Their execution in higher dimensions by Ladyzhenskaya and Uraltśeva [33], Serrin [48] and others led to a definitive quasilinear theory, which is also described in the book [24]. The basic linear theory to underpin the quasilinear theory was provided by the Hopf maximum principle [26] and the Schauder theory of Hölder estimates of second derivatives for linear elliptic equations with Hölder continuous coefficients [45]. The reader is referred also to [24] for a presentation of this theory.

The critical results needed to link nonlinear equations to the linear Schauder theory were Hölder estimates for linear equations with possibly bad coefficients as such equations arose through differentiation. The first big breakthrough in this direction was made by Charles Morrey Jnr. [34] in 1938 for two dimensions. Morrey considered linear elliptic equations of the form

$$Lu := \sum_{i,j=1}^{2} a^{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} = 0 \tag{5.1}$$

in domains $\Omega \subset \mathbf{R}^2$, with coefficient matrix $A = [a^{ij}]$ satisfying

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{2} a^{ij} \xi_i \xi_j \leq \Lambda|\xi|^2 \tag{5.2}$$

for all vectors $\xi \in \mathbf{R}^2$ and some positive constants λ, Λ. He proved a Hölder estimate for the gradient of solutions u, namely for any strictly contained subdomain $\Omega' \subset\subset \Omega$, there exists a constant $\alpha > 0$ depending only on Λ/λ and a constant $C > 0$ depending additionally on $\mathrm{dist}(\Omega', \partial\Omega)$ such that

$$|\nabla u(x) - \nabla u(y)| \leq C|x-y|^\alpha \sup_\Omega |u| \tag{5.3}$$

for all $x, y \in \Omega'$. Morrey's estimate led to a fairly definitive theory of *two* dimensional quasilinear and fully nonlinear elliptic equations, [37] , [24].

In higher dimensions the big breakthrough came through the independent discoveries by De Giorgi [17] in 1957 and Nash [36] in 1958 of Hölder estimates for solutions of linear elliptic equations in **divergence** form,

$$Lu := \mathrm{div}(A\,\nabla u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial u}{\partial x_j}\right) = 0, \tag{5.4}$$

with coefficient matrix $A = [a^{ij}]$ satisfying, corresponding to (5.2),

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \leq \Lambda|\xi|^2 \tag{5.5}$$

for all $\xi \in \mathbf{R}^n$ and some positive constants λ, Λ. For solutions u of equation (5.4) in domains $\Omega \subset \mathbf{R}^n$, the De Giorgi-Nash result provides a Hölder estimate

$$|u(x) - u(y)| \leq C|x-y|^\alpha \sup_\Omega |u| \tag{5.6}$$

for $x, y \in \Omega'$ for any subdomain $\Omega' \subset\subset \Omega$, where α is a positive constant depending only on Λ/λ and n and C is a constant depending additionally on $\mathrm{dist}(\Omega', \partial\Omega)$. This result was applied immediately to derivatives of solutions of quasilinear equations such as the minimal surface equation, and in tandem with the gradient estimates arising from the above mentioned ideas of Bernstein led to a fairly definitive theory of quasilinear equations and scalar variational problems, [24], [33], [48].

For linear elliptic equations in the general form, (5.1),

$$Lu = \sum_{i,j=1}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} = 0 \tag{5.7}$$

the major breakthrough was made by Krylov and Safonov [31] in 1979. For solutions u in domains $\Omega \subset \mathbf{R}^n$, and coefficient matrix A satisfying (5.5), they derived a Hölder estimate of the form (5.5). Their approach was fundamental for the development of the theory of fully nonlinear elliptic equations in higher dimensions, including the Monge-Ampère equation [24], [30], [11]. Interestingly, it rested upon a result coming from earlier considerations of the Monge-Ampère equation, namely the Aleksandrov-Bakel'man maximum principle which had been discovered about twenty years earlier, [1], [3], [24] and provides an estimate

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + C \operatorname{diam}(\Omega) \left\{ \int_{\Omega} (Lu)^{-} / \det A \right\}^{1/n} \tag{5.8}$$

where C is a constant depending on n.

The Hölder estimates for solutions of the equations (5.1), (5.4) and (5.7) were accompanied by Harnack inequalities

$$\sup_{\Omega'} u \le C \inf_{\Omega'} u \tag{5.9}$$

for non-negative solutions u and domains $\Omega' \subset\subset \Omega$, where C is a constant depending on n, Λ/λ, and $\operatorname{dist}(\Omega', \partial\Omega)$. These were discovered by Serrin [47] for (5.1), Moser [35] for (5.2) and Krylov and Safonov [31] for (5.7). From the Harnack inequalities followed Liouville theorems asserting that entire solutions, bounded at least from above or below, are constant. Discrete versions of these results on general meshes were found by Kuo and Trudinger [32].

6 The affine Bernstein problem

The affine maximal surface equation was introduced by Chern [15] in 1977. To formulate it, we first recall that a smooth convex function u in $\mathbf{R}^n$ has non-negative Hessian matrix D^2u. Let us call u **strongly convex** if also $\det D^2u$ is positive. We then set

$$w = \left(\det D^2 u\right)^{-\frac{n+1}{n+2}} \tag{6.1}$$

and let $U = [U^{ij}]$ be the cofactor matrix of D^2u. The **affine maximal surface equation** can be written as

$$U \bullet D^2 w = 0 \tag{6.2}$$

It is a nonlinear, fourth order partial differential equation, elliptic with respect to strongly convex solutions u. Furthermore, it is invariant under uni-modular affine transformations of $\mathbf{R}^{n+1}$. The expression on the left of (6.2) is the mean curvature of the graph of u with respect to its affine metric [15]. Simple examples of solutions are second degree polynomials, for which the function w in (6.1) is constant. We also know from Jorgens' theorem that in fact **all** the trivial entire solutions for which w is constant are second degree polynomials. Chern [15] conjectured in 1977 that all entire solutions of the affine maximal surface equation (6.2), in two dimensions, are second degree polynomials.

The problem as to whether this is true, which can be posed in any dimension, became known as the **affine Bernstein problem.**

In a later paper in 1982, Calabi [14] established the connection between the affine maximal surface equation and affine area. If M is a strongly convex, smooth hypersurface in $\mathbf{R}^{n+1}$ and K its Gauss curvature, (normalized to be positive), then the affine area of M is given by

$$A(M) = \int_M K^{\frac{1}{n+2}} \tag{6.3}$$

When M is the graph of a strongly convex function u on a domain $\Omega \subset \mathbf{R}^n$, we call

$$A(u) := A(M) = \int_{\Omega} (\det D^2 u)^{\frac{1}{n+2}} \tag{6.4}$$

the **affine area functional** of u. The functional A is readily shown to be invariant under uni-modular affine transformations in $\mathbf{R}^{n+1}$. The affine maximal surface equation can now be characterized as the Euler-Lagrange equation of the affine area functional. Calabi showed moreover that (6.2) is both necessary and sufficient for the graph of u to locally maximize affine area. Calabi also reformulated the Chern conjecture as whether Euclidean complete, affine maximal hypersurfaces are paraboloids and showed by geometric arguments that this is true for two dimensional hypersurfaces which are also affine complete.

The Chern conjecture was recently resolved by Trudinger and Wang [53], who proved:

Theorem. *A complete, strongly convex affine maximal surface in* $\mathbf{R}^3$ *must be an elliptic paraboloid.*

For higher dimensions, the Bernstein problem was reduced to estimation of the strict convexity of solutions, [53].

One interesting feature of the affine maximal surface quation (6.2) is that it also has a divergence form (5.4)

$$\operatorname{div}(UDw) = 0 \tag{6.5}$$

so that if the Hessian matrix D^2u satisfies (5.5) we deduce from either the De Giorgi–Nash estimates or the Krylov–Safonov estimates, Hölder estimates for the function w. Recently, Caffarelli and Gutierrez [12], exploiting the affine invariance of the second order operator, $L := U \bullet D^2$, in $\mathbf{R}^n$ obtained much stronger results. To formulate their estimates, we define for $x \in \mathbf{R}^n$ and $h > 0$, the **section** $S(x,h)$ of the convex function u by

$$S(x,h) = \{y \in \mathbf{R}^n \,|\, u(y) \le u(x) + \nabla u(x) \bullet (y - x) + h\} \tag{6.6}$$

Then, replacing (5.5) by,

$$0 < \lambda \le \det D^2 u \le \Lambda \tag{6.7}$$

for constants λ, Λ, we obtain for solutions w of (6.2) in sections $S(x, h_0) \subset \mathbb{R}^n$, "Hölder" estimates,

$$\operatorname*{osc}_{S(x,h)} w \le C \left(\frac{h}{h_0}\right)^{\alpha} \operatorname*{osc}_{S(x,h_0)} w \tag{6.8}$$

where C and α are positive constants depending on n, Λ/λ, and for non-negative solutions w, Harnack inequalities,

$$\sup_{S(x,h)} w \le C \inf_{S(x,h)} w, \tag{6.9}$$

with constant C depending on n, Λ/λ and h/h_0. Consequently, if we can establish (6.7) for solutions u of the affine maximal surface equation, using the definition (6.1), we may conclude from either (6.8) or (6.9) or Bernstein's Liouville theorem [7], [27] in the case $n = 2$, that w is constant and hence u is a second degree polynomial by Jorgens' theorem. The approach in [53] differs slightly from this, but its essential ingredients still reduce to a priori estimation of the form (6.7).

7 Conclusion

The theory of nonlinear partial differential equations has been a massive development of twentieth century mathematics, impacting upon other areas of mathematics and a diverse range of applications; (see [10]). In a small article, we can only sample fragments of this amazing development. Our first glimpse was the two dimensional Bernstein problem for minimal graphs, initiated by Bernstein in 1915, whose higher dimensional version provided one of the major challenges for nonlinear partial differential equations in the twentieth century. The corresponding affine problem in two dimensions, solved at the end of the twentieth century, also provided a glimpse into the fascinating world of nonlinear, higher order, geometric partial differential equations. The higher dimensional affine Bernstein problem clearly becomes a challenge for the twenty first century. One aspect of this challenge lies in interpreting the example in dimension ten, namely

$$u(x) = \sqrt{|x'|^9 + x_{10}^2}, \quad x' = (x_1, \dots, x_9), \tag{7.1}$$

of a non-smooth entire affine maximal graph, (see [53]). Indeed could it happen that the affine Bernstein property is true to dimension nine and then fails for higher dimensions?

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References

1. A.D. Aleksandrov, Certain estimates for the Dirichlet problem. *Dokl. Akad. Nauk*, SSSR 134, 1001–1004 (1960). [Russian]. Eng. trans. in *Soviet Math. Dokl. 1*, 1151–1154 (1960).
2. F.J. Almgren, Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math. (2)*, 84, 277–292 (1966).
3. I.Ya. Bakel'man, Theory of quasilinear elliptic equations. *Sibirsk. Mat. Ž.*, 2, 179–186 (1961) [Russian].
4. S. Bernstein, Sur la généralisation du problème de Dirichlet. *I. Math. Ann.*, 62, 253–271 (1906).
5. S. Bernstein, Méthode générale pour la resolution du problème de Dirichlet. *C.R. Acad. Sci. Paris*, 144, 1025–1027 (1907).
6. S. Bernstein, Sur la généralisation du problème de Dirichlet II. *Math. Ann.*, 69, 82–136 (1910).
7. S. Bernstein, Sur un théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique. *Comm. Soc. Math. de Kharkov 2*, 15, 38–45 (1915–1917). German trans. *Math. Zeit.*, 26, 551–558 (1927).
8. L. Bers, Non–linear elliptic equations without non–linear entire solutions. *J. Rational Mech. Anal.*, 3, 767–787 (1954).
9. E. Bombieri, E. De Giorgi and E. Giusti, Minimal cones and the Bernstein problem. *Invent. Math.*, 7, 243–268 (1969).
10. H. Brezis and F. Browder, Partial Differential Equations in the 20th Century. *Adv. Math.*, 135, 76–144 (1998).
11. L. Caffarelli and X. Cabré, Fully nonlinear elliptic equations. *Amer. Math. Soc. Colloquium Publications*, 43 (1995).
12. L. Caffarelli and C. Gutiérrez, Properties of the solutions of the linearized Monge-Ampère equations. *Amer. J. Math.*, 119, 423–465 (1997).
13. E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Mich. Math. J.*, 5, 1005–126 (1958).
14. E. Calabi, Hypersurfaces with maximal affinely invariant area. *Amer. J.*

Math., 104, 91–126 (1982).

15. S.S. Chern, Affine minimal hypersurfaces. In Minimal submanifolds and geodesics, *Proc. Japan–United States Sem., Tokyo*, 17–30 (1977); see also Selected papers of S.S. Chern, Volume III, Springer, 425–438 (1989).
16. R. Courant and D. Hilbert, Methods of Mathematical Physics. Volumes I, II. New York: *Interscience*, (1953), (1962).
17. E. De Giorgi, Sulla differenziabilità e l'annaliticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 3*, 3, 25–43 (1957).
18. E. De Giorgi, Una estensione del teorema di Bernstein. *Ann. Scuola Norm. Sup. Pisa 3*, 19, 79–85 (1965).
19. U. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab, Minimal surfaces, Vols. I, II. Springer–Verlag, Berlin–Heidelberg–New York (1992).
20. Y. Fang, Lectures on minimal surfaces in $\mathbf{R}^3$. *Proc. Centre for Mathematics and its Applications, Aust. Nat. Univ.*, 35, (1996).
21. R. Finn, On equations of minimal surface type. *Ann. of Math.*, 60, 397–416 (1954)
22. W. Fleming, On the oriented Plateau problem. *Rend. Circ. Mat. Palermo*, 11, 69–90 (1962).
23. P. Garabedian, Partial differential equations. *J. Wiley*, New York, (1964).
24. D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order (2nd Edition). Springer–Verlag, Berlin–Heidelberg–New York (1983).
25. E. Giusti, Minimal surfaces and functions of bounded variation. Birkhäuser, Boston–Basel–Stuttgart (1984).
26. E. Hopf, Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. *Sitz. Ber. Preuss. Akad. Wissensch.*, Berlin, *Math.–Phys. K1.*, 19, 147–152 (1927).
27. E. Hopf, On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature. *Proc. Amer. Math. Soc.*, 1, 80–85 (1950).
28. F. John, Partial differential equations. Springer–Verlag (1971).
29. K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$. *Math. Ann.*, 127, 130–134 (1954).
30. N. Krylov, Nonlinear elliptic and parabolic equations of the second order. *Mathematics and its Applications*, Reidel (1987).
31. N.V. Krylov and M.V. Safonov, Certain properties of solutions of parabolic equations with measurable coefficients. *Izvestia Akad. Nauk.*, SSSR 40, 161–175 (1980) [Russian].
32. H.J. Kuo and N.S. Trudinger, Positive difference operators on general meshes. *Duke Math. J.*, 83, 415–433 (1996).
33. O.A. Ladyzkenskaya and N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations. Moscow: *Izdat. "Nauka"*, (1964) [Russian]. English Translation: New York: *Academic Press* (1968), 2nd Russian ed. (1973).
34. C.B. Morrey Jr., On the solutions of quasi–linear elliptic partial differential equations. *Trans. Amer. Math. Soc.*, 43, 126–166 (1938).
35. J.K. Moser, On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14, 577–591 (1961).
36. J. Nash, Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.*, 80, 931–954 (1958).
37. L. Nirenberg, On nonlinear elliptic partial differential equations and Hölder continuity. *Comm. Pure Appl. Math.*, 6, 103–156 (1953).
38. J.C.C. Nitsche, Elementary proof of Bernstein's theorem on minimal surfaces. *Ann. of Math.*, 66, 543–544 (1957).
39. J.C.C. Nitsche, Vorlesungen über Minimalflächen. Springer–Verlag, Berlin–Heidelberg–New York (1975).
40. R. Osserman, A survey of minimal surfaces. *Dover*, New York (1986).
41. R. Palais, 3-D Filmstrip, (see web address: http://rsp.math.brandeis.edu/3D-Filmstrip.html/3D-FilmstripHomePage.html).
42. I.G. Petrovski, Partial differential equations. *Saunders* (1967).
43. A.V. Pogorelov, On the improper affine hyperspheres. *Geom. Dedicata*, 1, 33–46 (1972).
44. A.V. Pogorelov, The Minkowski multidimensional problem. *J. Wiley*, New York (1978).
45. J. Schauder, Über lineare elliptische Differentialgleichungen zweiter Ordnung. *Math. Z.*, 38, 257–282 (1934).
46. R. Schoen, L. Simon and S.-T. Yau, Curvature estimates for minimal hyper–surfaces. *Acta Math.*, 134, 276–288 (1975).
47. J. Serrin, On the Harnack inequality for linear elliptic equations. *J. Analyse Math.*, 4, 292–308 (1955/56).
48. J. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A*, 264, 413–496 (1969).
49. L.M. Simon, Entire solutions of the minimal surface equation. *J. Differential Geom.*, 30, 643–688 (1989).
50. L.M. Simon, The minimal surface equation. *Geometry V, Encyclopaedia Math. 90*, Springer–Verlag (1997).
51. J. Simons, Minimal varieties in riemannian manifolds. *Ann. of Math. 2*, 88, 62–105 (1968).
52. W.A. Strauss, Partial differential equations. An introduction. *J. Wiley*,

New York, 1992.
53. N.S. Trudinger and X.-J. Wang, The Bernstein problem for affine maximal hypersurfaces. *Invent. Math.*, 140, 399–422 (2000).
54. H. Weinberger, A first course in partial differential equations, *Blaisdell*, Waltham, Mass., 1965.

THREE-DIMENSIONAL SUBGROUPS AND UNITARY REPRESENTATIONS

DAVID A. VOGAN, JR.
Department of Mathematics, MIT, Cambridge, MA 02139, USA
E-mail: dav@math.mit.edu

The simplest noncommutative compact Lie group is the group $SU(2)$ of unit quaternions. If K is a compact Lie group, write $D(K)$ for the set of conjugacy classes of homomorphisms of $SU(2)$ into K. Dynkin showed in the 1950s that $D(K)$ is a finite set, and calculated it in all cases.

A fundamental unsolved problem is to parametrize the "purely real" unramified unitary representations of a split reductive group G over a local field. Such representations are parametrized by a compact polytope $P(G)$. When G and K are "Langlands dual" to each other, a conjecture of Arthur realizes $D(K)$ as a subset of $P(G)$. We discuss the status of this conjecture, and how Dynkin's problem illuminates the representation-theoretic one.

1 Introduction

One of the purposes of representation theory is to provide tools for harmonic analysis problems. The idea is to understand actions of groups on geometric objects by understanding first the possible representations of the group (by linear operators). Formally the simplest examples are finite groups: no sophisticated analytical tools are needed to study them. Nevertheless the (finite set) of irreducible representations of a finite group can be extraordinarily complicated from a combinatorial point of view. In some respects the representation theory of (connected) Lie groups is actually simpler than that of finite groups, because the geometric structure of a Lie group constrains the multiplication law to be nearly commutative.

The purpose of this paper is to examine a classical problem in the representation theory of Lie groups (formulated as (23) below). The problem is still unsolved. I'll explain a conjecture due to James Arthur that relates this representation theory problem to a structural problem for compact groups. The structural problem was solved by Eugene Dynkin in the 1950s. Connecting the two problems requires the classification of compact Lie groups in the beautiful form given to it by Michel Demazure and Alexandre Grothendieck (elaborating on previous constructions). I will recall that classification in Sec. 2. The solution to Dynkin's problem appears in Sec. 3. In Sec. 4 I will formulate the representation-theoretic problem, and state Arthur's conjecture