Maths 260 2010S Assignment 2 Answers

September 3, 2010

1. (a) Write $f(y) = y^2 + 3y + 1$. Then equilibrium solutions are at y values satisfying $f(y) = 0$, i.e.,

$$y = \frac{-3 \pm \sqrt{5}}{2}.$$

Also,

$$\frac{\partial f}{\partial y} = 2y + 3,$$

so

$$\left.\frac{\partial f}{\partial y}\right|_{y=(-3+\sqrt{5})/2} = \sqrt{5} > 0, \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{y=(-3-\sqrt{5})/2} = -\sqrt{5} < 0.$$

This means that $y = (-3+\sqrt{5})/2$ is a source and $y = (-3-\sqrt{5})/2$ is a sink. The phase line is shown on the left in the figure on the next page.

(b) Write $f(y) = -y^2$. There is just one equilibrium, at $y = 0$. Then,

$$\frac{\partial f}{\partial y} = -2y,$$

so

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = 0$$

and the linearisation theorem tells us nothing about the type of equilibrium at $y = 0$. However, $f(y) \leq 0$ and so all solutions except the equilibrium decrease as time increases. This means that $y = 0$ is a node. The phase line is shown in the middle in the figure on the next page.

(c) Write $f(y) = -\cos(y)$. Then equilibrium solutions are at y values satisfying $f(y) = 0$, i.e.,

$$y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Also,

$$\frac{\partial f}{\partial y} = \sin(y).$$

Now,

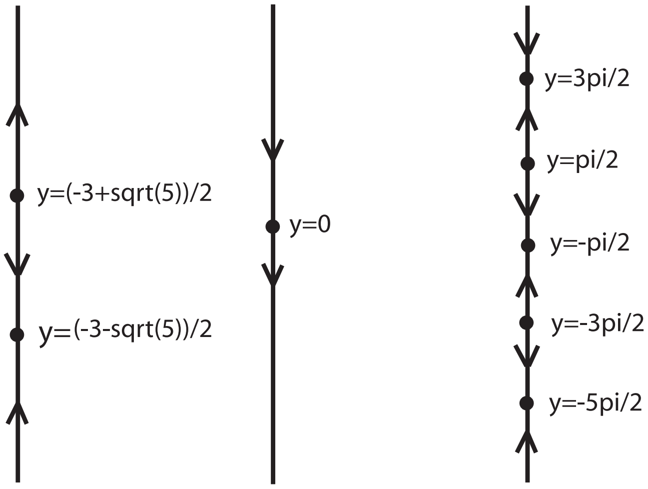
$$\sin(y) = 1 \text{ if } y = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots \text{ or } y = -\frac{3\pi}{2}, -\frac{7\pi}{2}, -\frac{11\pi}{2}, \ldots$$

so at these values, the equilibrium is a source. Similarly,

$$-\sin(y) = -1 \text{ if } y = -\frac{\pi}{2}, -\frac{5\pi}{2}, -\frac{9\pi}{2}, \ldots \text{ or } y = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$$

so at these values, the equilibrium is a sink.

Part of the phase line is shown on the right in the figure on the next page. Note that there are infinitely many equilibria in the full phase line, and they alternate between sinks and sources.



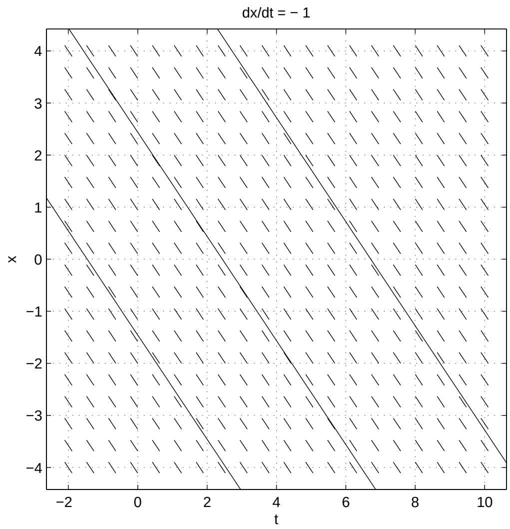
2. There are many possible answers to parts (a) and (c) of this question. I give only one.

Phase line 1:

(a) $dy/dy = -1$

(b) All solutions tend to $-\infty$ as t increases. (Not necessarily as $t \to \infty$ though.)

(c)

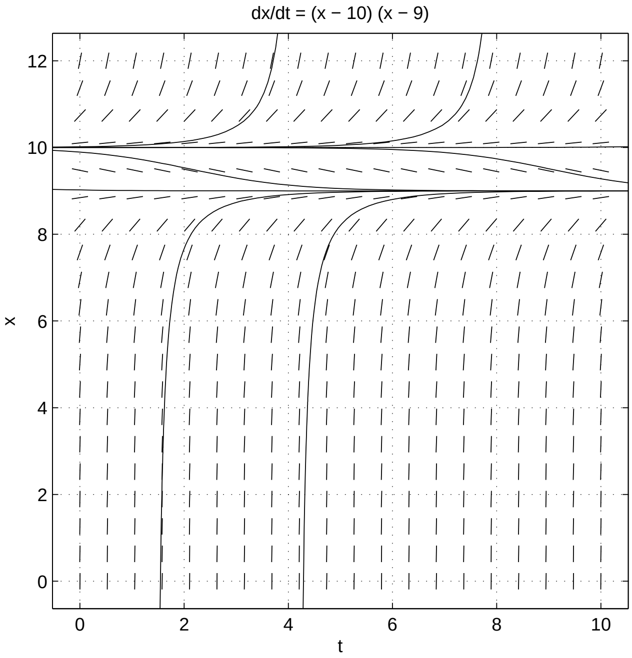


Phase line 2:

(a) $dy/dy = (y-10)(y-9)$

(b) If $y(0) < 9$ then $y(t) \to 9$ as $t \to \infty$. If $9 < y(0) < 10$ then $y(t) \to 9$ as $t \to \infty$. If $y(0) > 10$, then $y(t) \to \infty$ as t increases. Also, $y(t) = 9$ and $y(t) = 10$ are equilibrium solutions.

(c) See next page.

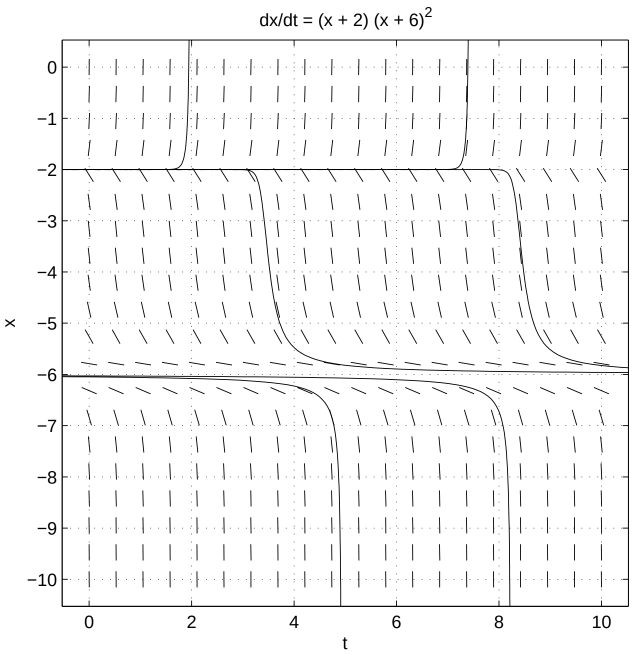


Phase line 3:

(a) $dy/dy = (y + 2))(y + 6)^2$

(b) If $y(0) < -6$ then $y(t) \to -\infty$ as t increases. If $-6 < y(0) < -2$ then $y(t) \to -6$ as $t \to \infty$. If $y(0) > -2$, then $y(t) \to \infty$ as t increases. Also, $y(t) = -2$ and $y(t) = -6$ are equilibrium solutions.

(c)



3. (a) The DE is

$$\frac{dy}{dt} = y(-8 + y^2),$$

and this has three equilibrium solutions, i.e., $y(t) = 0$, $y(t) = 2\sqrt{2}$ and $y(t) = -2\sqrt{2}$. Also, writing $f(y) = y(-8 + y^2)$, we have

$$\frac{\partial f}{\partial y} = -8 + 3y^2$$

so

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = -8, \qquad \left.\frac{\partial f}{\partial y}\right|_{y=2\sqrt{2}} = -8 + 3(2\sqrt{2})^2 = 16, \qquad \left.\frac{\partial f}{\partial y}\right|_{y=-2\sqrt{2}} = -8 + 3(-2\sqrt{2})^2 = 16.$$

Hence, $y = 0$ is a sink and $y = \pm 2\sqrt{2}$ are sources. The phase line is shown in the figure below, on the left.

(b) The DE is

$$\frac{dy}{dt} = y(8 + y^2),$$

and this has just one equilibrium solution, i.e., $y(t) = 0$. Writing $f(y) = y(8 + y^2)$, we have

$$\frac{\partial f}{\partial y} = 8 + 3y^2$$

so

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = 8$$

so $y = 0$ is a source. The phase line is shown in the figure below, on the right.

(c) The DE is

$$\frac{dy}{dt} = y^3,$$

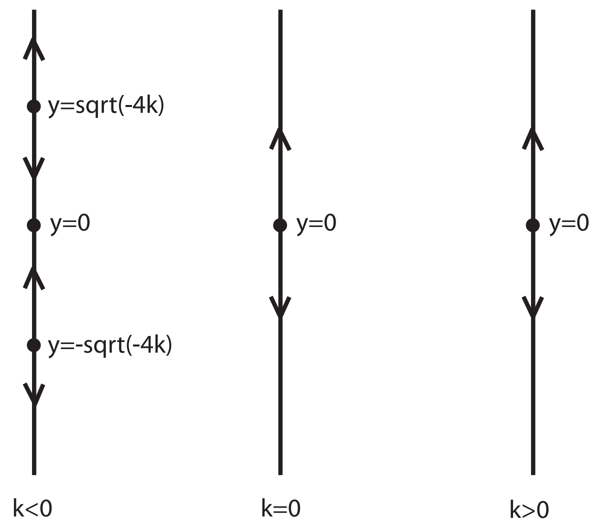
and this has just one equilibrium solution, at $y(t) = 0$. Writing $f(y) = y^3$, we have

$$\frac{\partial f}{\partial y} = 3y^2$$

so

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = 0.$$

Hence the type of equilibrium cannot be determined from linearisation. However, $y^3 > 0$ if $y > 0$ and $y^3 < 0$ if $y < 0$ so the phase line is as shown in the middle case of the figure below and $y = 0$ is a source.



(d) i. The DE

$$\frac{dy}{dt} = y(4k + y^2),$$

has equilibrium solutions at $y = 0$ (for all k) and $y = \pm\sqrt{-4k}$ (for $k \leq 0$). Thus there are three equilibrium solutions for $k < 0$ (i.e., $y = 0$, $y = \sqrt{-4k}$ and $y = -\sqrt{-4k}$) and one equilibrium for $k \geq 0$ (i.e., $y = 0$).

Writing $f(y) = y(4k + y^2)$, we have

$$\frac{\partial f}{\partial y} = 4k + 3y^2$$

so

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = 4k.$$

Thus $y = 0$ is a sink if $k < 0$ and a source if $k > 0$. The case for $k = 0$ must be considered separately (see below) because linearisation fails here.

Also,

$$\left.\frac{\partial f}{\partial y}\right|_{y=\sqrt{-4k}} = 4k + 3(\sqrt{-4k})^2 = -8k$$

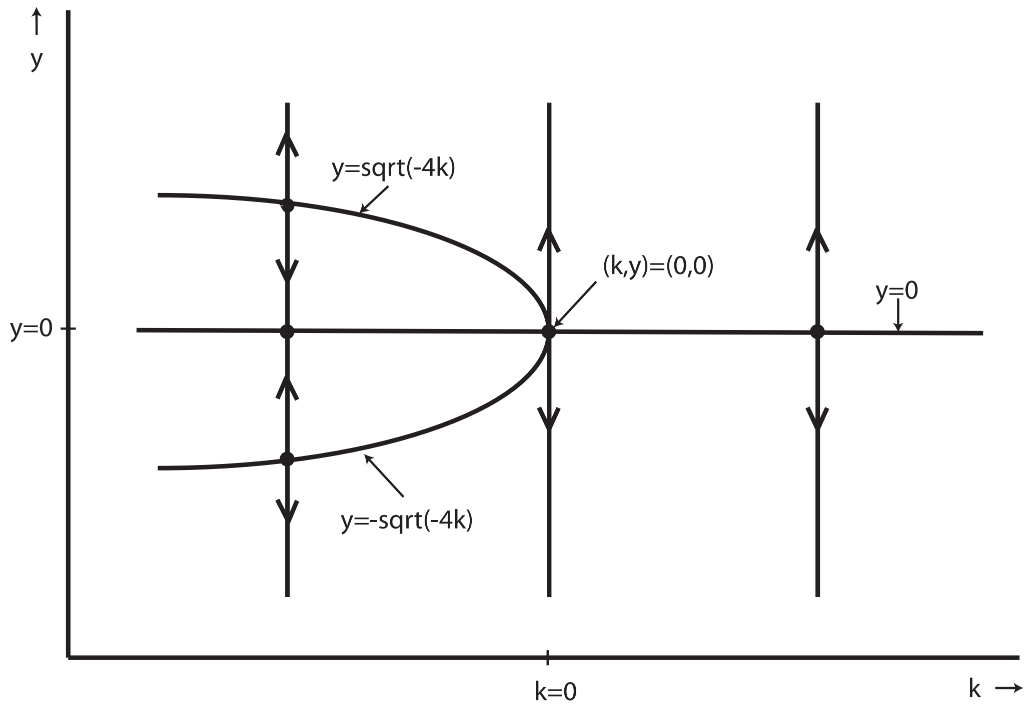
and $-8k > 0$ for $k < 0$, i.e., for the non-zero k values for which this equilibrium solution exists. Hence $y = \sqrt{-4k}$ is a source for $k < 0$. Similarly,

$$\left.\frac{\partial f}{\partial y}\right|_{y=-\sqrt{-4k}} = 4k + 3(-\sqrt{-4k})^2 = -8k$$

and so $y = -\sqrt{-4k}$ is also a source for $k < 0$.

The case $k = 0$ was considered in part (c) above. At this k value there is just one equilibrium, and it is a source as shown above.

ii.



4. (a) Rewrite the equation in standard form as:

$$\frac{dy}{dt} - \frac{y}{t} = t^2 \exp(t)$$

Then find an integrating factor:

$$\text{IF} = \exp\left(\int \frac{-1}{t}\, dt\right) = \exp(-\ln t) = \frac{1}{t}.$$

Here the constant of integration has been chosen to be zero. Multiplying both sides of the DE by the IF gives

$$\frac{1}{t}\frac{dy}{dt} - \frac{y}{t^2} = t\exp(t)$$

which we write as

$$\frac{d}{dt}\left(\frac{y}{t}\right) = t\exp(t). \quad (*)$$

Integrating,

$$\frac{y}{t} + c_1 = \int t\exp(t)\; dt$$

We integrate the RHS by parts:

$$\int t\exp(t)dt = t\exp(t) - \int \exp(t)\; dt = t\exp(t) - \exp(t) + c_2$$

Substituting this into the RHS of $(*)$ gives

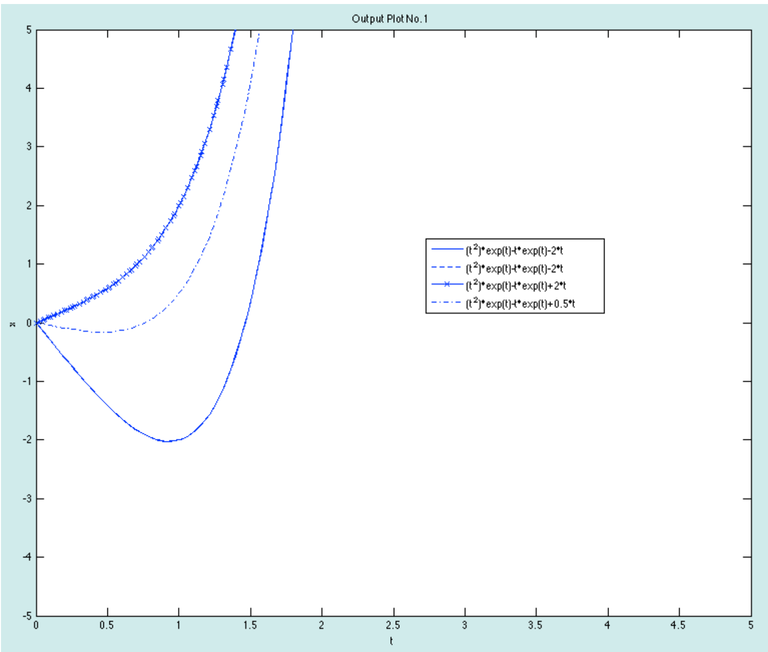
$$\frac{y}{t} = t\exp(t) - \exp(t) + C$$

where $C = c_2 - c_1$. Finally re-arranging gives the one-parameter family of solutions

$$y = t^2\exp(t) - t\exp(t) + Ct$$

where C is an arbitrary constant.

(b) Example of three solutions are shown below. Note that solutions are valid for $t > 0$ only (since the DE is defined for $t > 0$ only) and so nothing should be plotted for $t < 0$.



(c) All solutions tend to positive infinity as $t \to \infty$. This can be seen by taking the limit as $t \to \infty$ of $y = t^2 \exp(t) - t\exp(t) + Ct$. In this expression, the first term grows the fastest and so $y \to t^2 \exp(t) \to \infty$ as $t \to \infty$ regardless of the value of C.

5. (a) Sample solution curves are shown below.

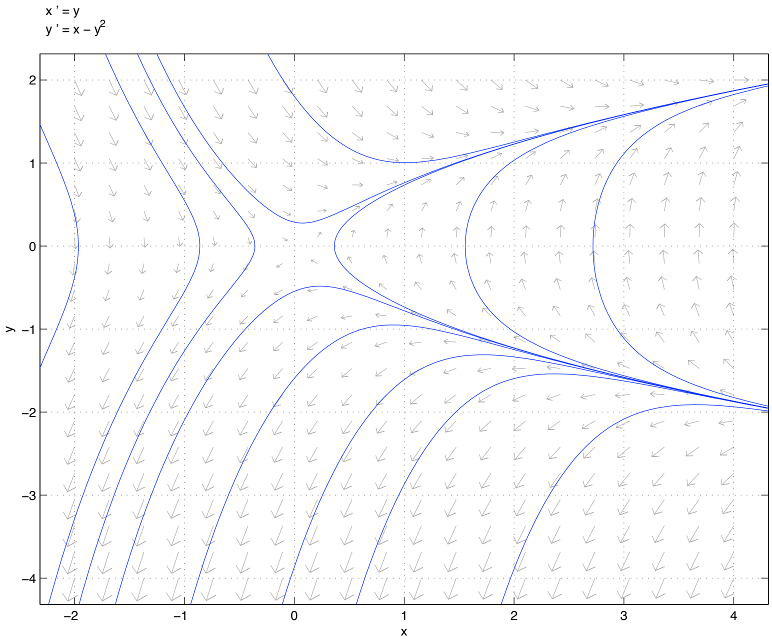


Figure 1: $\dot{x} = y$, $\dot{y} = x - y^2$

(b) Sample solution curves are shown below, with arrows indicating the direction moved as t increases. The direction of the arrows is computed by looking at the sign of $\dot{x}$. We are told that $\dot{x} = y - x$. Thus, $\dot{x} > 0$ if $y > x$, i.e., solutions move to the right above the diagonal line $y = x$ and so the arrows are as shown.

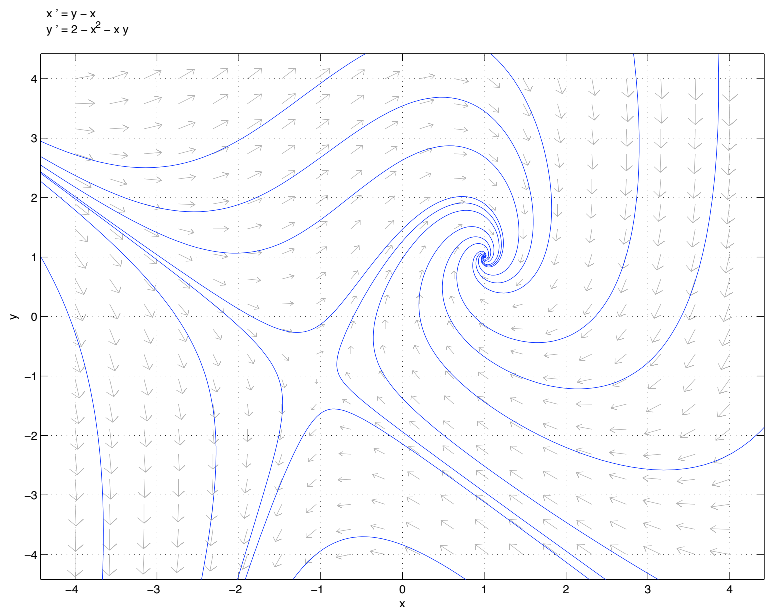
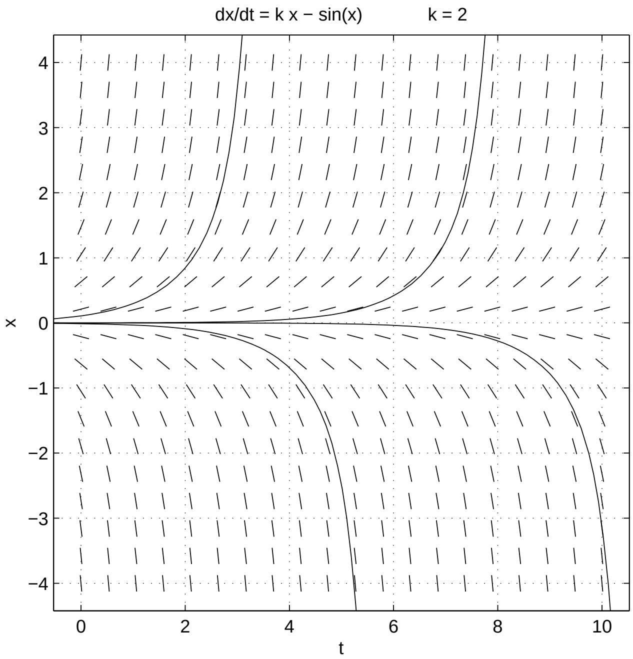


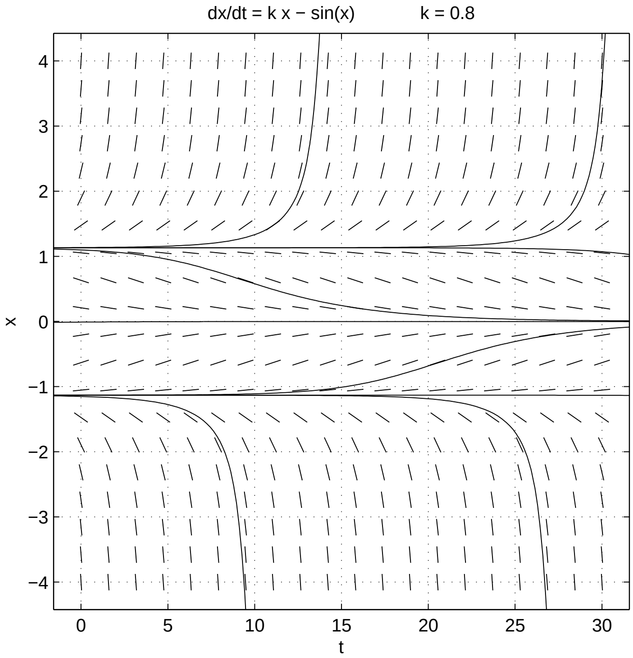
Figure 2: $\dot{x} = y - x$, $\dot{y} = 2 - x^2 - xy$

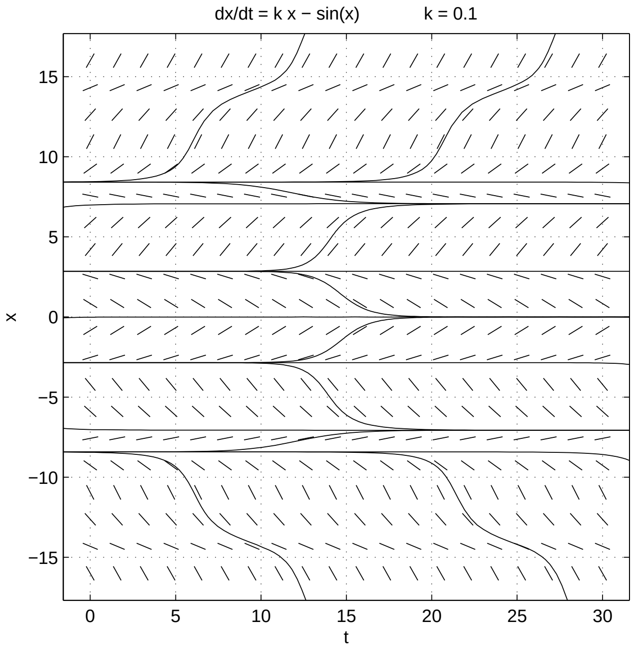
6. (a) Equilibrium solutions to this DE occur when $ky = \sin(y)$. Clearly $y = 0$ is an equilibrium for all k. Writing $f(y) = ky - \sin(y)$ I find that $\partial f/\partial y = k - \cos(y) = k - 1$ at $y = 0$. Thus $y = 0$ is a sink if $k < 1$ and a source if $k > 1$. This means there is a bifurcation at $k = 1.0$. It is also possible to find this bifurcation using *dfield*, although it is a bit hard to figure out the bifurcation value to two significant figures with *dfield*.

To find a second bifurcation, I use *dfield*. I find, by trial and error, that for $k = 0.128$ there are seven equilibrium solutions while for $k = 0.129$ there are three equilibrium solutions. Therefore there is a bifurcation at $k \approx 0.1285$ or, to two significant figures, at $k = 0.13$.

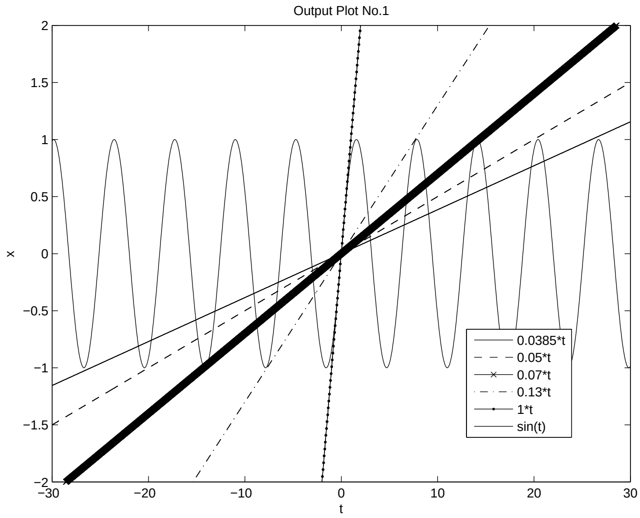
Thus I have found three qualitatively different types of behaviour for $k \in [0.1, 2]$. For $1 < k < 2$ there is one equilibrium of source type. For $0.13 < k < 1$ there are three equilibria (middle one, at $y = 0$, is a sink, the other two are sources). For $0.1 < k < 0.13$ there are seven equilibria (middle one, at $y = 0$ is a sink, others alternate between source and sink as in the figure below). The following figures show solutions for $k = 2.0$, $k = 0.8$ and $k = 0.1$; these figures are typical of the three qualitatively different types of behaviour described above.







(b) I used *analyzer* to plot the functions $h(y) = \sin(y)$ and $g(y) = ky$ on the same axes for several choices of k. I found that as I reduced k there could be different numbers of intersections of the two graphs. In particular I found that at $k \approx 0.07$ the number of intersections (i.e., equilibrium solutions to the DE) changed from 7 to 11, at $k \approx 0.05$ the number of intersections changed from 11 to 15, and at $k =\approx 0.038$ the number of intersections changed from 15 to 19. Graphs at the transition (bifurcation) values of k are shown on the next page. It can be seen that a bifurcation occurs at each value of k for which there is a tangency between the graphs of h and g. In fact there will be an infinite number of bifurcations like this as $k \to 0$, and so I cannot find them all. At $k = 0$ there will be infinitely many equilibrium solutions. The types of the equilibria can be determined using *dfield*. If k is not a bifurcation value, the equilibria will alternate between sinks and sources, starting with a sink at $y = 0$ (at least when $k < 1$).



(c) The bifurcation diagram will look qualitatively like the following picture.