The Q-Curvature Equation in Conformal Geometry

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Table of Contents

Conformally covariant operators

On (M^n, g), compact Riemannian manifold, A metric $\bar{g}$ is conformal to g, if $\bar{g} = e^{2w} g$.

An operator A is a conformally covariant operator of bidegree (a, b), if under $g_w = e^{2w} g$,

$$A_{g_w}(\phi) = e^{-bw} A(e^{aw} \phi) \quad \forall \phi \in \mathcal{C}^\infty(M^n) .$$

Examples:

1. when $n = 2$, Δ of bidegree $(0, 2)$.

Properties of Δ_g on (M^2, g)

a. $\Delta_{g_w} = e^{-2w} \Delta_g$.

b. $-\Delta_g w + K_g = K_{g_w} e^w$.

Gauss-Bonnet formula:

$$2\pi \chi(M) = \int_M K_g dv_g$$

Uniformization of Surfaces

The Gauss curvature equation and the Gauss-Bonnet formula provide a PDE approach to the uniformization problem:

Given a conformal structure on a surface (M, g), solve for a conformal metric $g_w = e^{2w}g$ so that the Gauss curvature of $g_w = 1$, 0, or -1 according to the sign of the euler number.

The conformal Laplacian

$$L = -\frac{4(n-1)}{n-2}\Delta + R$$

is conformally covariant of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$.

The Yamabe equation:

$$(*) \quad Lu = \bar{R}u^{(n+2)/(n-2)}$$

Yamabe Problem Yamabe-Trudinger-Aubin-Schoen

The equation (*) is solvable for a constant $\bar{R}$.

Key ingredient: Positive Mass Theorem (Schoen-Yau).

Yamabe constant: $Y(M^n, g) \equiv \inf_w \frac{\int R_{g_w} dv_{g_w}}{vol(g_w)^{\frac{n-2}{n}}}$

We have $Y(M^n, g) > 0$ *iff* $\lambda_0(L_g) > 0$.

Dimension four

When $n = 4$, Paneitz operator: (1983)

$$P\varphi \equiv \Delta^2\varphi + \delta[\left(\frac{2}{3}Rg - 2Ric\right) d\varphi]$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor.

For example:
On $(R^4, |dx|^2)$, $P = \Delta^2$,
On (S^4, g_c), $P = \Delta^2 - 2\Delta$,
On (M^4, g), g Einstein, $P = (-\Delta) \circ (L)$.

P has bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_w}(\phi) = e^{-4\omega} P_g(\phi) \ \forall \phi \in \mathcal{C}^\infty(M^4)\,.$$

Properties of Paneitz operator on (M^4, g) :

1. $P_{g_w} = e^{-4w} P_g$
2.

$$P_g w + 2Q_g \;=\; 2Q_{g_w} e^{4w}$$

$$Q \;=\; \frac{1}{12}(-\Delta R + R^2 - 3|Ric|^2)$$

Gauss-Bonnet-Chern Formula:

$$8\pi^2 \chi(M^4) = \int (2Q_g + |W_g|^2)\; dv,$$

where W denotes the Weyl tensor.

•$|W_g|^2 dv_g \;=\; |W_{g_w}|^2 dv_{g_w}$ is a pointwise conformal invariant while the curvature integral $\int Q_g dv_g$ is a global conformal invariant.

Q curvature equation:

$$(**) \quad P_g w + 2Q_g = 2Q_{g_w} e^{4w}$$

Theorem: (Gursky, Chang-Yang)
(i) If $\lambda_1(L_g) > 0$ *and* $\int Q_g dv_g > 0$ *then* $P_g \geq 0$ *with* $KerP = \{constants\}$*.*
(ii) Under assumptions in (i), (**) can be solved with Q_{g_w} *given by a constant.*

Remarks:
1. Based on Adams' version of Trudinger's inequality:

$$\int e^{32\pi^2 w^2} dv \leq C \text{ if } \int Pw \cdot w dV \leq 1.$$

2. To minimize the functional

$$II[w] = \int Pw \cdot w + 4Qw dv - (\int Q dv) \log(\int e^{4w} dv).$$

More generally, consider functional determinants: Branson-Orsted Formula on (M^4, g) for

$$F[w] = log\frac{det\ A_g}{det A_{g_w}},$$

for A_g conformally covariant operators.

The associated Euler equations are of the form:

$$\gamma_1|W|^2 + \gamma_2 Q - \gamma_3 \Delta R = \bar{k} \cdot Vol^{-1} \qquad (***)$$

for constants $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_2\gamma_3 > 0$ and $\bar{k}$.

Existence Chang-Y *Such equations have minimizing solutions.*

Regularity: Chang-Gursky-Y.

Positivity, General existence

Criteria for positivity (Gursky-Viaclovsky): *On* (M^4, g) *with positive Yamabe constant* $Y(M, g)$*, if*

$$\int Q dv + Y(M, g)^2 > 0$$

then the operator P *is positive except on constants.*

General Existence (Djadli-Malchiodi) *On* (M^4, g)*, if* $Ker(P) = \{constants\}$*, then the* Q*-curvature equation is solvable.*

This argument is based on delicate blowup analysis of the Q-curvature equation by Malchiodi, Robert.

Constraints on topology

Sphere Theorem (Chang-Gursky-Y)
Suppose (M^4, g) *is compact and satisfies*

$$Y(M,g) > 0 \text{ and } \int |W|^2 dV < \int 2Q dV$$

then M *is diffeomorhic to* S^4 *or* RP^4.

Finiteness Theorem (Chang-Qing-Y)
The class of Bach-flat 4-manifolds satisfying
(i) $Y(M,g) > 0$,
(ii) $\int |W|^2 dV \le C_1$,
(iii) $\int Q dV > C_2 > 0$;
contains only a finite number of distinct diffeomorphism classes.

Key Fact: Vanishing of Bach tensor and constant R is an elliptic system. Bach tensor:

$$B_{ij} = \nabla^k \nabla^l W_{kijl} + R^{kl} W_{kijl}$$

Compactification

Embedding Theorem (Schoen-Yau)
(M, g) a locally conformally flat manifold with $Y(M, g) > 0$, then a covering $\bar{M}$ embeds conformally in S^n, and

$$dim(S^n \setminus \bar{M}) \leq (n-2)/2;$$

Thus M is the quotient of a Kleinian group.

Idea of Proof:
1. Development map $\Phi : (\bar{M}, g) \to S^n$.
2. Pull back the Green's function G_p of the sphere, where $LG_p = \delta_p$. It serves as a barrier for construction of actual Green's function $\bar{G}_q \; for \; q \in \bar{M}$ where $\Phi(q) = p$.
3. Study decay of $\bar{G}_q$ at infinity to show $\bar{G}_q = |d\Phi|^{2/(n-2)} G_p \circ \Phi$.

Picture.

Chang-Qing-Y

Suppose $(\Omega \subset R^4, g = e^{2w} g|dx|^2)$ *is a complete metric satisfying*

(i) $0 < C_1 \leq R \leq C_2,\ |\nabla R| \leq C_2,\ Rc \geq -C_3$*;*

(ii) $\int |Q| dv < C_3$*;*

then

$$\bar{M} = S^4 \setminus \{p_1, \ldots, p_N\},$$

and

$$8\pi^2 \chi(\Omega) = \int_\Omega 2Q dv + \sum_{\nu=1}^{N} I_\nu$$

where I_ν *is an isoperimetric constant attached to each end* p_ν*.*

Idea of proof:

1. $e^{w(x)} \approx d(x, \Lambda = R^4 \setminus \Omega)$:

a $e^{w(x)} \geq C d(x, \Lambda)$: Harnack estimate.

b $e^{w(x)} \leq C d(x, \Lambda)$: Blowup analysis:

Picture.

1a) View $|dx|^2 = u^2 g$, hence u is a positive conformal harmonic function on the complete manifold (Ω, g) it follows from the Ricci lower bound the gradient estimate:

$$|\nabla_g log u| \leq C$$

This gives

$$|u(x)| \leq C d(x, \Lambda);$$

and hence the required bound.

1b) Blowup argument: Assume the bound $e^w \leq Cd(.,\Lambda)^{-1}$ fails, i.e. exists a sequence $x_i \in \Omega$ so that $d(x_i,\Lambda)e^{w(x_i)} = A_i \to \infty$ Dilate the space near x_i, and write $x = x_i + (a_i)y$ where $a_i = e^{-w(x_i)}$ and $e^{2w(x)}|dx|^2 = e^{2\bar{w}_i(y)}|dy|^2$. This gives a sequence of conformal metrics defined in increasingly large balls in y coordinate space.

Assumption on R and ∇R means that a subsequence of the metrics $e^{2w_i}|dy|^2$ converges in $C^{2,\alpha}$ uniformly on compact sets in R^4 to a limit metric $e^{2\bar{w}}|dy|^2$. In choosing a subsequence, may assume the balls $B(x_i,(1/2)d(x_i,\Lambda))$ are disjoint, hence we find $\bar{Q}d\bar{V}$ =0 weakly.

Liouville *The only conformal metric* $e^{2w}|dy|^2$ *with* $Q = 0$ *and* $R \geq 0$ *is the flat metric.*

2. $\int Qdv$ as an 'isoperimetric' quantity:

$$-\int_{e^w \le \lambda} Qdv = (\lambda \frac{d}{d\lambda}) \int_{e^w \le \lambda} dv + \text{positive terms.}$$

3. A covering argument $\Rightarrow$:

$$|\{d(x,\Lambda) = s\}| \ge \begin{cases} Ns^3 \text{ if } dim(\Lambda) = 0 \text{and } |\Lambda| \ge N \\ Cs^{3-3/4\beta} \text{ if } dim(\Lambda) = \beta > 0. \end{cases}$$

4.

$$\begin{aligned} \int_{e^w \le \lambda} e^{4w} dx &\ge \int_{C_2/\lambda}^{C_1} \int_{\{d(x,\Lambda)=s\}} e^{4w} d\sigma ds \\ &\ge \int_{C_2/\lambda}^{C_1} |\{d(x,\Lambda) = s\}| s^{-4} ds. \end{aligned}$$

5. In either case in (3), there is a sequence $\lambda_i \to \infty$ so that

$$(\lambda \frac{d}{d\lambda}) \int_{e^w \le \lambda} dv|_{\lambda=\lambda_i} \to \infty.$$

A contradiction to the finiteness assumption $\int |Q| dv < \infty$ in view of 2.

Stability of R^4: (Bonk-Heinonen-Saksman).

Let $g = e^{2w}|dx|^2$ *be a normal conformal metric on* R^4 *satisfying*

$$\int |Q| dv < C_4;$$

then, there is a bilipschitz equivalence $(R^4, g) \to (R^4, |dx|^2)$.

Remarks:

1. normality:

$$w(x) = \frac{1}{4\pi^2} \int Q(y) e^{4w(y)} log \frac{|y|}{|x-y|} dy.$$

2. w need not be bounded.
3. This result holds in all even dimension.
4. When $n = 2$, the best constant $C_2 = 2\pi$.

Extensions to higher dimensions

Existence of P and Q:
(Branson-Gover, Fefferman-Graham)
On (M^{2n}, g) there exists

$$P = (-\Delta)^n + lower\ order\ terms$$

which is conformally covariant of bidegree$(0, 2n)$;

$$Q = (-\Delta)^{n-1} + lower\ order\ terms$$

so that when $\bar{g} = e^{2w} g$,

$$(**) Pw + Q_g = Q_{\bar{g}} e^{2nw}.$$

Existence of solution to Q-curvature equation:
(Brendle)
On (M^{2n}, g), if P is positive except for constants and

$$\int Q dv < \int_{S^{2n}} Q dv,$$

then the equation (**) is solvable where $Q_{\bar{g}}$ is a constant.

higher dimensions

Relation with Pfaffian: (Alexakis)

$$c_n \text{ pfaffian} = W + Q$$

where W is a pointwise conformal polynomial invariant in Weyl and its derivatives.

Obstruction Tensor: (Fefferman-Graham)

$$\frac{d}{dt}\int Q(g+th)dV(g+th) = \int < O, h > dv$$

A natural generalization of the Bach tensor.

higher dimensions

Finiteness: (Fan)

The set of conformal structures(M^6, g) satisfying the conditions:

(i) $Y(M, g) > 0$,

(ii) $\int |Rm|^3 < C_1$,

(iii) $O_{ij} = 0$,

(iv) $\int Q dV > C_2 > 0$;

contains only a finite number of diffeomorphism types.

What happens in odd dimensions?

To use the Gauss-Bonnet formula, view (M^3, h) as boundary data of (X^4, g):

Definition: (Chang-Qing)

Given $w \in C^\infty(M)$

1. extend w to X to satisfy: $P_4 w = 0, \partial_\nu w = 0$
2. $P_3 w = \partial_\nu \Delta w +$ lower order terms,
3. $Q_3 = \partial_\nu R +$ lower order terms.

P_3 is conformally covariant of bidegree (0,3), and the associated Q_3 curvature equation:

$$P_3 w + Q_3 = \bar{Q}_3 e^{3w}.$$

Gauss-Bonnet:

$$8\pi^2 \chi(X) = \int_X (|W|^2 + 2Q_4) dv + \int_M L + Q_3 d\sigma$$

where $L d\sigma$ is a pointwise conformal invariant.

Renormalized volume

$(X^4, M^3 = \partial X, g)$ is Poincare Einstein if
1. there is a smooth defining function x so that $\bar{g} = x^2 g$ is a smooth metric on (X^4, M^3);
2. g is complete Einstein metric in the interior of X.
3. A different choice of x gives rise to a conformally equivalent metric on M^3.

Near M^3, can write $g = x^{-2}(dx^2 + g_x)$ where g_x is a family of metrics on M^3:

$$g_x \approx g_0 + x^2 g_2 + x^3 g_3 + \ ...$$

Here g_2 depends only on g_0.

To define a global invariant consider

$$Vol(\{x \geq \epsilon\}) \approx c_0 \epsilon^{-3} + c_2 \epsilon^{-1} + V + o(1).$$

The finite part V is called the renormalized volume (Graham).

Gauss-Bonnet (Anderson, Chang-Qing-Y)

$$8\pi^2\chi(X) = \int |W|^2 dv \; + \; (1/6)V.$$

Idea of Proof:

1. Solve for $-\Delta v = 3$ on X so that near M^3,

$$v \approx \log x + A + Bx^3,$$

where A and B are even in x.

Consider the compactified metric $\bar{g} = e^{2v}g$.

2. $V = \int_M B d\sigma$. (Fefferman-Graham)
3. $3B = Q_3, Q_4 = 0$.
4. Apply Gauss-Bonnet with boundary.

This argument generalizes to higher dimensions.

Uniqueness (Andersson,Qing)

Suppose (X^{n+1}, S^n, g) *is Poincare Einstein, and* $\bar{g}|S^n$ *gives the standard conformal structure on* S^n*, then* $X^{n+1}, S^n, g)$ *is the hyperbolic space.*

Existence (Graham-Lee)

Let (S^n, h) *be a conformal structure which is a small perturbation from that of the standard one, then there exists a Poincare Einstein metric on* (B^{n+1}, S^n, g) *so that* $g|_{S^n} = h$*.*

Open Questions

1. General Existence: Given (X^4, M^3), and a given conformal structure (M^3, h), does there exist a Poincare Einstein metric on (X, M)?
2. Uniqueness: Is the Poincare Einstein metric on (X^4, M^3) unique for a given conformal structure on M^3?

Pseudo-hermitian Geometry (M^3, J, θ)

θ a contact form: $\theta \wedge d\theta \neq 0$

J complex structure on $ker\theta = \xi$ $J^2 = -I$.

T the Reeb vector field so that

$$\theta(T) = 1, \;\; L_T\theta = 0.$$

Choose $e_1, e_2 = Je_1$ so that $d\theta = e^1 \wedge e^2$.

<u>Webster connection</u>

$$\begin{cases} d\,\theta^1 = \theta^1 \wedge \omega_1^1 + A_{\bar{1}}^1\,\theta \wedge \theta^{\bar{1}} \\ \omega_1^1 + \omega_{\bar{1}}^{\bar{1}} = 0 \end{cases}$$

$$d\,\omega_1^1 \;=\; R\,\theta^1 \wedge \theta^{\bar{1}} + 2i\,Im\,(A_{11\,,\,\bar{1}})\,\theta^{\bar{1}} \wedge \theta$$

$A_{\bar{1}}^1$ – Torsion

W – Webster scalar curvature.

II Equation of Geodesics $\quad X = \dot{\gamma}(t)$

$$\begin{cases} \nabla_X X &= \alpha J X \\ \dot{\alpha} &= \langle T\text{or } (T, X), X \rangle \,. \end{cases}$$

A third order system.

Remark *Under the vanishing torsion assumption the equation of geodesics becomes*

$$\begin{cases} \nabla_X X = \alpha J X \\ \dot{\alpha} = 0 \,, \end{cases}$$

It reduces to a second order system. The curvature α is then a constant along each geodesic. The orbits of the α curvature geodesics gives a foliation in the unit contact bundle.

Example: The Heisenberg group

On $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, with $(z = x + iy, t)$ as coordinates:

$$\begin{cases} e_1 = \partial_x + y\partial_t,\ e_2 = \partial_y - x\partial_t,\ T = \partial_t \\ \\ \theta = dt + xdy - ydx \\ \\ \omega_1^1 = 0,\ A_{\bar{1}}^1 = 0,\ W = 0. \end{cases}$$

1. The flat model.
2. The blow-up limit of general (M^3, θ, J).
3. The boundary of the upper half space:

$$Re\ w \geq |z|^2.$$

4. It has group structure:

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + xy' - x'y)$$

Geodesics issuing from the origin:

$$\begin{cases} x(s) = 2\frac{1-\cos \alpha s}{\alpha} \\ y(s) = \frac{\sin \alpha s}{\alpha} \\ t(s) = 2\frac{\alpha s - \sin \alpha s}{\alpha^2}. \end{cases}$$

Then geodesics γ issuing from 0 have constant curvature α. The projection of γ to xy plane are planar circle of radius $\frac{1}{\alpha}$

The t-axis is the cut-locus of the origin.

Webster scalar curvature equation

$$\Delta_b u = u_{,1}{}^{1} + u_{,\bar{1}}{}^{\bar{1}};\ Lu = -\Delta_b u + (1/4)Wu,$$

then L is conformally covariant of bidegree $(1,3)$ and under the change of contact form $\bar{\theta} = u^2\theta$,

$$Lu = (1/4)\bar{W}u^3.$$

The analogue of the Yamabe constant:

$$W(M,\theta) = \inf \frac{\int Lu \cdot u\theta \wedge d\theta}{||u||_4^2}$$

Liouville (Jerison-Lee):
The entire solutions of the equation $Lu = (1/4)u^3$ on the Heisenberg are of the form:

$$u_\lambda(z,t) = \lambda(\lambda^4 t^2 + (1+\lambda^2|z|^2)^2)^{-1/2}. \quad (2)$$

Criterion for compactness:
If $W(M,\theta) < W(S^3,\theta_0)$ *then minimizing sequence for* $W(M,\theta)$ *converges.*

Idea of proof: Look for minimizer as limit of minimizers of the subcritical approximations: Let u_ϵ be the minimizer of

$$W_\epsilon(M,\theta) = \inf \frac{\int Lu \cdot u\theta \wedge d\theta}{||u||_{4-\epsilon}^2}.$$

Then

$$Lu_\epsilon = \lambda_\epsilon u_\epsilon^{3-\epsilon}. \qquad (1)$$

Holder inequality gives the convergence $\lambda_\epsilon \to \lambda$ as $\epsilon \to 0$. If the u_ϵ remain bounded, we are done.
If not, a blowup argument gives a convergent sequence of solutions of (1) in the Heisenberg space with λ in place of λ_ϵ.

The Liouville theorem then shows that the solution is the standard one, this contradicts with the assumption $W(M,\theta) < W(S^3,\theta_0)$.

Test function construction:
To attach a standard bubble (2) centered at $q \in M$ to the Green's function G_q satisfying $LG_q = \delta_q$. In the CR normal coordinates (t, z) with center at q, we have

$$G(z,t) = c_0 \rho^{-2} + A + o(1),$$

where $\rho^4 = t^2 + |z|^4$. The constant A is called the CR mass.

If we choose ρ_0 small but $\rho_0 >> 1/\lambda$ and glue the standard bubble u_λ to G_q over the annulus $(1/2)\rho_0 < \rho < \rho_0$ to get $\bar{u}_\lambda$. A careful computation will show that

$$\frac{\int L\bar{u}_\lambda \cdot \bar{u}_\lambda \theta \wedge d\theta}{||u||_4^2} < W(S^3, \theta_0);$$

provided $A > 0$.

Asymptotically Heisenberg:
The CR normal coordinates near $q \in M$: there exists $\widehat{\theta}$ and coordinates $(\widehat{z}, \widehat{t})$ so that

$$\begin{cases} \widehat{\theta} = (1 + O(\widehat{\rho}^4))\widehat{\Theta} + O(\widehat{\rho}^5)d\widehat{z}^1 + O(\widehat{\rho}^5)d\widehat{z}^{\bar{1}}, \\ \widehat{\theta}^1 = (1 + O(\widehat{\rho}^4))d\widehat{z}^1 + O(\widehat{\rho}^4)d\widehat{z}^{\bar{1}} + O(\widehat{\rho}^3)d\widehat{\Theta}; \end{cases}$$

where

$$\begin{cases} \widehat{\Theta} = d\widehat{t} + i\widehat{z}^1 d\widehat{z}^{\bar{1}} - i\widehat{z}^{\bar{1}} d\widehat{z}^1 \\ \widehat{\rho}^4 = \widehat{t}^2 + |\widehat{z}^1|^4. \end{cases}$$

Consider the blowup of $(M^3, J, \widehat{\theta})$ at $q \in M^3$:

$$\begin{cases} \theta = G_q \widehat{\theta} \\ \theta^1 = G_q(\widehat{\theta}^1 + i(\log G_q)_{,\bar{1}}\widehat{\theta}). \end{cases}$$

In "inverted CR normal coordinates":

$$z = \frac{\widehat{z}^1}{\widehat{w}},\ t = -\frac{\widehat{t}}{|\widehat{w}|^2},\ \widehat{w} = \widehat{t} + i|\widehat{z}^1|^2;$$

we let $\widehat{\Theta} = \rho^{-4}\Theta$, $\widehat{\rho} = 1/\rho$, then near infinity,

$$\begin{cases} \theta = (1 + A\rho^{-2} + O(\rho^{-3}))\Theta + O(\rho^{-3})dz + O(\rho^{-3})d\bar{z} \\ \theta^1 = (-\rho^2 \frac{\bar{w}}{w^2} + O(\rho^{-2}))dz + O(\rho^{-4})d\bar{z} + O(\rho^{-3})\Theta. \end{cases}$$

The CR Mass:
For an asymptotically Heisenberg 3-manifold, let

$$m(J,\theta) = \lim_{\rho \to \infty} i \int_{\partial B_\rho} \omega_1{}^1 \wedge \theta.$$

The definition is motivated by the requirement that under deformation $\frac{d}{dt} J_t = E$ we have:

$$\frac{d}{dt}|_{t=0} \{ \int W_t \theta \wedge d\theta - m(J_t, \theta) \} = 2Re \int A_{11} E_{\bar{1}\bar{1}} \theta \wedge d\theta.$$

The CR Hirachi operator:

$$Pu = \Delta_b^2 u + T^2 u - 2i\{u_{11}A_{\bar{1}\bar{1}} - u_{\bar{1}\bar{1}} + u_1 A_{\bar{1}\bar{1},1} - u_{\bar{1}} A_{11,\bar{1}}.$$

Under the conformal change of contact form $\bar{\theta} = e^{2u}\theta$ we have

$$\bar{P} = e^{-4u}P.$$

The associated Q-curvature is given by:

$$Q = -\Delta_b R - 2ImA_{11}{}^{11},$$

The Q-curvature equation:

$$Pu + Q = \bar{Q}e^{4u}.$$

An important connection of Q to the Szego kernel:

$$K(z,z) = \phi(z)x(z) + \psi(z)\log x(z),$$

where x is the defining function at the boundary, and

$$8\pi^2\psi = Q.$$

A criterion for positivity of the CR mass:
Cheng-Chiu-Malchiodi-Y(WIP):
If the Hirachi operator is strictly positive, then the CR mass is non-negative, and equals zero only if (M, J, θ) *is the standard 3-sphere.*

Remarks:
1. Strict positivity means $P \geq 0$, and zero eigenvalue is isolated.
2. Relation to the Kohn operator:

$$\bar{\partial}_b^* \bar{\partial}_b u = -\Delta_b u + iTu.$$

3. Isolation of the zero eigenvalue follows from closedness of $\bar{\partial}_b$, and is equivalent to the embeddability of the CR- structure.