# Future Directions in 3-Manifold Geometry and Topology

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- Overview of Geometry and Topology of 3-Manifolds

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  - Hyperbolic 3-manifolds

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  - Ricci flow

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- **Foliations:** 1 and 2-dimensional foliations are indispensable for studying 3-manifold topology

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- **General Relativity:** minimal surfaces, relation between scalar curvature and topology
- **Non-Commutative Geometry:** Jones polynomial and quantum invariants, $l^2$-invariants

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- **PDEs:** elliptic theory, Ricci flow proof of geometrization

# Milestones in 3-Dimensional Geometric Topology

We'll focus on results which emphasize the interaction between geometry and topology of 3-manifolds. There have been many spectacular results in the past 30+ years. This brief survey will omit many important results, and is influenced by my own tastes.

# Milestones in 3-Dimensional Geometric Topology

## Equivariant loop theorem and sphere theorem

(Meeks-Yau) First introduction of minimal surface techniques into 3-manifold topology. This theorem implies that if there is a finite group $G$ acting on a 3-manifold $M$ with $\pi_2(M) \neq 0$, then there is a collection of homotopically non-trivial 2-spheres $\Sigma \subset M$ such that $G$ preserves $\Sigma$. These spheres may be chosen to be minimal surfaces in a $G$-equivariant Riemannian metric on $M$. Although this theorem is subsumed by the Orbifold Theorem, it still has had an important influence for the study of minimal surfaces in 3-manifolds, including work of Freedman, Hass and Scott for minimal $\pi_1$-injective surfaces and Rubinstein for minimal Heegaard surfaces, as well as an alternative proof of the Poincaré conjecture by Perelman which doesn't require Ricci flow on infinite time intervals.

# Milestones in 3-Dimensional Geometric Topology

## Geometrization of Haken 3-manifolds, Dehn Surgery Theorem, Geometrization Conjecture

Thurston revolutionized 3-manifold topology by introducing the study of geometric structures, especially hyperbolic geometry. He proved that Haken 3-manifolds, manifolds which contain an embedded $\pi_1$-injective surface such as a knot complement, admit a canonical decomposition along surfaces of non-negative Euler characteristic into geometric pieces. We'll mention a bit later what the eight 3-dimensional geometries are. His theorem was foreshadowed by work of Jorgensen, Marden, Riley. It made use of important work of Jaco, Shalen, Johansson, and Waldhausen in 3-manifold topology, as well as many important results from Kleinian groups and Teichmüller theory, especially the work of Ahlfors, Bers and Sullivan.

# Milestones in 3-Dimensional Geometric Topology

## Geometrization of Haken 3-manifolds, Dehn Surgery Theorem, Geometrization Conjecture

The techniques of Thurston's proof probably had a greater impact on the field of Kleinian groups than on 3-manifold topology. Thurston's Dehn Surgery Theorem further elucidated the structure of hyperbolic 3-manifolds, and extended the Geometrization Conjecture to many non-Haken examples, demonstrating that hyperbolic manifolds are "generic".

# Milestones in 3-Dimensional Geometric Topology

## Smith Conjecture, Neuwirth Conjecture, Cyclic Surgery Theorem

Shalen and coworkers introduced the study of Bass-Serre theory to go from group representations into $\mathrm{PSL}(2,\mathbb{C})$ coming from hyperbolic structures to group actions on trees (1-dimensional buildings) and $\mathbb{R}$-trees (Morgan-Shalen). This incorporates algebraic geometry of character varieties and Stallings' method of proof of the Sphere Theorem via actions on trees. This enabled the resolution of some longstanding conjectures (along with other advances of a topological nature), including

# Milestones in 3-Dimensional Geometric Topology

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- the Neuwirth conjecture (Culler-Shalen '84): every non-trivial knot complement has a separating incompressible surface with boundary
- the Cyclic Surgery Theorem (Culler-Gordon-Luecke-Shalen '88): one may obtain a lens space by Dehn filling on a (non-torus) knot for at most two fillings.

# Milestones in 3-Dimensional Geometric Topology

## The Orbifold Theorem

In the mid-eighties, Thurston lectured at Princeton on the Orbifold Theorem, which extended his geometrization theorem of Haken manifolds to get a geometric decomposition of good orbifolds with non-trivial singular locus (this includes the case of a 3-manifold together with a finite group action with non-trivial fixed point set, such as in the case of the Smith Conjecture). His proof was incomplete, though. Encouraged by listing the Orbifold Theorem as a conjecture in the '94 version of Kirby's problem list, two groups of people gave a complete proof of the Orbifold Theorem, announced in '98. Cooper, Hodgson, and Kerckhoff followed Thurston's approach, whereas Boileau, Leeb and Porti incorporated more ideas from differential geometry and Cheeger and Gromov's theory of almost collapsed manifolds.

# Milestones in 3-Dimensional Geometric Topology

## Property (P)

Property (P) was stated as a conjecture in the '70s, that there is no non-trivial Dehn filling on a non-trivial knot complement which yields a homotopy 3-sphere. Kronheimer and Mrowka developed a program to prove this conjecture in the '90s, which came to fruition only in 2003 due to various advances. Although Property (P) follows from the Poincaré Conjecture (proven by Perelman in early 2003), and by the solution of the Knot Complement Problem (resolved in 1989 by Gordon and Luecke), when Kronheimer and Mrowka's proof appeared, Perelman's work was not yet generally accepted or digested by the mathematical community. Still, this was a major advance since their techniques proved something much more powerful.

# Milestones in 3-Dimensional Geometric Topology

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Also, their method was quite deep, depending on many of the important advances in low-dimensional topology from the previous 20 years, including:

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- Instanton Floer homology (Donaldson, Floer)
- Seiberg-Witten invariants and SW Floer homology (Seiberg, Witten, Taubes, Kronheimer, Mrowka)

# Milestones in 3-Dimensional Geometric Topology

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- relation between Gromov invariants of symplectic manifolds and Seiberg Witten invariants (Taubes)
- relations between Dondaldson and Seiberg-Witten invariants, conjectured by Witten due to dualities in String Theory, and proven partially by Feehan and Leness.

# Milestones in 3-Dimensional Geometric Topology

## Geometrization Conjecture

In 2002 and 2003, Perelman posted preprints on the Arxiv which appeared to complete a program of Hamilton to prove the Geometrization Conjecture (which has as a well-known corollary the Poincaré Conjecture). After extensive work by several mathematicians, including Kleiner and Lott, Cao and Zhu, and Morgan and Tian, complete proofs filling in the details of Perelman's argument were finished in 2006 (Ye found and corrected an error in Perelman's argument). Part of the argument relied on a proof of Shioya and Yamaguchi classifying 3-manifolds which were collapsed in a certain sense, which replaced a claimed proof by Perelman which didn't appear.

# Milestones in 3-Dimensional Geometric Topology

## Classification of Kleinian groups

A Kleinian group is a finitely generated discrete subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{C})$. Classically, this was a topic in complex analysis when the group $\Gamma$ has a domain $\Omega \subset \mathbb{CP}^1$ on which it acts discontinuously. Kleinian groups were revolutionized in the '70s by the introduction of 3-dimensional manifold and hyperbolic geometric techniques to study $\mathbb{H}^3/\Gamma$ due to Jorgensen, Marden, and Thurston. Thurston formulated a conjectural classification of Kleinian groups called the Ending Lamination Conjecture in terms of the underlying topology of $\mathbb{H}^3/\Gamma$, conformal data associated to $\Omega/\Gamma$, where $\Omega \subset \hat{\mathbb{C}}$ is the domain of discontinuity of $\Gamma$, and end invariants conjectured to exist by Thurston (proven to exist by Bonahon in '86).

# Milestones in 3-Dimensional Geometric Topology

## Classification of Kleinian groups

This was resolved in 2004 as the cumulative work of many mathematicians, including A., Ahlfors, Bers, Bonahon, Brock, Canary, Marden, Masur, Minsky, Otal, Sullivan, Thurston and many others. Minsky was the principal architect of the proof, with joint contributions from Brock, Canary and Masur. A. and Calegari-Gabai independently resolved the Tameness conjecture of Marden, which implies that $\mathbb{H}^3/\Gamma$ may be compactified to a compact manifold with boundary, which guaranteed the existence of the end invariants by work of Canary and Bonahon.

# Future Directions in 3-Manifold Geometric Topology

We'll focus on geometric questions about 3-manifolds, and ignore or miss many important questions. Kirby's problem list has many problems in low-dimensional topology contributed by many mathematicians which are still left unresolved.

## Geometric Structures on 3-Manifolds

Thurston's geometrization conjecture resolved the classification of 3-manifolds admitting a geometric structure modelled on one of Thurston's eight geometries. These geometries are associated to Lie groups which act transitively on $\mathbb{R}^3$ or $\mathbb{S}^3$ with compact point stabilizers. The isotropy group is either

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- $\mathrm{O}(2)$ ($\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{E}^1$, Nil, and $\mathbb{S}^2 \times \mathbb{E}^1$ geometries), or

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- $\mathrm{O}(2)$ ($\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{E}^1$, Nil, and $\mathbb{S}^2 \times \mathbb{E}^1$ geometries), or
- finite (Solv geometry).

# Future Directions in 3-Dimensional Geometric Topology

## Definition of Geometric Structures on 3-Manifolds

A more general type of geometry (also introduced by Thurston but studied previously by others) comes from a group acting analytically and transitively on $\mathbb{S}^3$ or $\mathbb{R}^3$ but with non-compact point stabilizers (such as $\mathrm{SO}(1,1), \mathrm{SO}(2,1)$ or $\mathrm{GL}(3)$).

Let $X$ be a manifold, and $G$ a Lie group acting transitively and analytically on $X$. Then a manifold $M$ admits a $(G, X)$ geometry if there is a holonomy map $\rho : \pi_1(M) \to G$ and a developing map $dev : \widetilde{M} \to X$ which is an immersion, such that the following diagram commutes for each covering translation $\varphi \in \pi_1(M)$:

$$\begin{array}{cccc} dev: & \widetilde{M} & \to & X \\ \varphi: & \downarrow & \rho(\varphi): & \downarrow \\ dev: & \widetilde{M} & \to & X \end{array}$$

# Future Directions in 3-Dimensional Geometric Topology

## Geometric Structures on 2-Manifolds

In 2-dimensions, the maximal geometries of this type are $(\mathrm{PSL}(2,\mathbb{C}), \mathbb{CP}^1)$ (complex projective) and $(\mathrm{PGL}(3,\mathbb{R}), \mathbb{RP}^2)$ (real projective) geometries. These geometric structures on surfaces are still being intensively studied by Dumas, Goldman, Kapovich, Marden and others. Complex projective structures are important in Kleinian groups and Teichmüller theory.

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

- Projective geometry
  $(\mathrm{PGL}(4,\mathbb{R}),\mathbb{RP}^3)$, which is the maximal dimensional group acting analytically on a compact 3-manifold.
  Cooper, Porti, and others have studied these recently. For example, $\mathbb{RP}^3\#\mathbb{RP}^3$ admits no projective structure. Any hyperbolic 3-manifold admits a projective structure via the Klein or Minkowski models of hyperbolic space.
  Convex projective structures, where the developing image is convex, have many analogues to the structure of negatively curved manifolds.

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

- Conformal geometry
  $(PO(4,1;\mathbb{R}),\mathbb{S}^3)$, where $PO(4,1;\mathbb{R})$ acts on $\mathbb{S}^3$ by Möbius transformations, which are the maximal group of conformal transformations of $\mathbb{S}^3$. This is also the action of $\mathrm{Isom}(\mathbb{H}^4)$ on $\partial_\infty \mathbb{H}^4$. A manifold has a conformal structure if and only if it has a conformally flat Riemannian metric.
  Five of Thurston's eight geometries are conformally flat.
  Connect sums of conformally flat manifolds are also conformally flat.

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

- Conformal geometry
  **Problem:** Classify hyperbolic 3-manifolds which have non-trivial conformal deformations.
  Examples due to Apanosov-Tetanov, Johnson-Millson, Kapovich, Bart-Scannell, and Tan.
  A closed conformally flat 3-manifold has a canonical volume associated to it (boundary of the 4-D hyperbolic "convex hull"). It would be interesting to see how this volume relates to the simplicial volume.

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

CR geometry
$(PU(2,1), \mathbb{S}^3)$, where $PU(2,1)$ acts on $\partial_\infty \mathbb{CH}^2$, complex hyperbolic space (holomorphic automorphisms of the unit ball in $\mathbb{C}^2$). This preserves a contact plane field, *i.e.* a nowhere integrable 2-plane field.
The Nil and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ geometries admit CR structures.
There are examples due to Schwartz of complex hyperbolic structures on the Whitehead link complement and some Dehn fillings on it, but little is known in general.

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

$(\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{C}), \mathbb{S}^1 \times \mathbb{CP}^1)$, where the Lie group acts preserving the product structure. Manifolds modelled on this structure would have a natural foliation induced by the $\mathbb{CP}^1$ factors.

$\mathbb{H}^2 \times \mathbb{R}$ geometry admit this structure.

There are hyperbolic 3-manifolds which have holonomy in this Lie group, but it is unknown if there is a corresponding developing map and geometric structure.

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

- Affine structures
  $(\mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R}), \mathbb{R}^3)$, which induces a projective structure via the embedding

  $$(\mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R}), \mathbb{R}^3) \subset (\mathrm{GL}(4,\mathbb{R}), \mathbb{RP}^3)$$

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  Euclidean, Nil, and Solv 3-manifolds admit affine structures.
- Lorentz structures
  $(\mathbb{R}^3 \rtimes \mathrm{O}(2,1), \mathbb{R}^3)$, which are important in relativity and also induces an affine structure.
  There are examples due to Mess, Drumm, Goldman, Margulis, Scannell and others, but relatively little is known. Mess classified Lorentz manifolds whose $\mathrm{O}(2,1)$ holonomy is

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

The principal problem in the subject is:

**Problem:** For a geometry $(G, X)$, classify 3-manifolds with a geometric structure modelled on $(G, X)$.

There are several difficulties for studying geometric structures. The holonomy might not be faithful and might not determine the geometric structure uniquely, the developing map might not be an embedding, and there may be deformations of the structure. Since this problem is probably too difficult to solve in full generality, a refinement is the question of whether one can promote a weaker structure to a stronger structure?

**Problem:** If $M$ admits a symplectically fillable contact structure, then does it admit a CR structure?

# Future Directions in 3-Dimensional Geometric Topology

## Examples of 3-Dimensional Geometric Structures

**Question:** Given $M^3$, is there an algorithm to classify all $(G, X)$ structures on $M$?

To start, for each $G$, one may compute the real algebraic variety

$$\{\rho : \pi_1(M) \to G\}.$$

But the difficulty is to determine which holonomies correspond to geometric structures. Moreover, there may be multiple developing maps for a given holonomy.

# Future Directions in 3-Dimensional Geometric Topology

## Other Geometric Structures

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- pseudo-Anosov flows. These arise naturally for fibered 3-manifolds
- quasi-geodesic flows
- taut foliations and essential laminations
- tight contact structures

# Future Directions in 3-Dimensional Geometric Topology

## Other Geometric Structures

- pseudo-Anosov flows. These arise naturally for fibered 3-manifolds
- quasi-geodesic flows
- taut foliations and essential laminations
- tight contact structures
- $CAT(0)$ cubed structures: non-positively curved

# Future Directions in 3-Dimensional Geometric Topology

## Other Geometric Structures

**Question:** Which aspherical 3-manifolds are homotopy equivalent to a locally $CAT(0)$ cube complex? The dimension must be $\geq 3$. This is related to the Virtual Haken Conjecture and the Virtual Fibration Conjecture (via LERF).

**Question:** Which 3-manifolds are (a component of) the fixed point set of an anti-holomorphic involution action on a Kähler Einstein manifold or Calabi-Yau 3-fold? ("complexified" $M^3$ ?)

**Question:** For which $M^3$ does $M^3 \times S^1$ admit a symplectic structure? It is conjectured to be true by Kronheimer and Taubes if and only if $M$ fibers over $S^1$ (Thurston proved that this is sufficient).

There are many other types of geometric structures on 3-manifolds, and the study of them is just in its infancy!

# Future Directions in 3-Dimensional Geometric Topology

## Hyperbolic 3-manifolds

The recent classification of Kleinian groups has led to the application of the techniques to the study of closed or finite volume hyperbolic 3-manifolds. Kleinian groups may arise as geometric limits of infinite sequences of finite volume hyperbolic 3-manifolds, and therefore may be used to study the topology and geometry of certain infinite classes of finite volume hyperbolic 3-manifolds.

# Future Directions in 3-Dimensional Geometric Topology

## Hyperbolic 3-manifolds

**Question:** Give a topological characterization of hyperbolic manifolds $M$ with infinitely generated fundamental group. There are some obvious necessary conditions, such as covers of $M$ with finitely generated fundamental group must be atoroidal, irreducible and tame, and elements of $\pi_1 M$ must be finitely divisible, but very little is known otherwise.

# Future Directions in 3-Dimensional Geometric Topology

## Hyperbolic 3-manifolds

**Question:** Is there an algorithm to tell if two hyperbolic 3-manifolds have the same volume? If $M_1$ and $M_2$ are hyperbolic, and $\mathrm{Vol}(M_1) \neq \mathrm{Vol}(M_2)$, then one may tell this by computing the two volumes to enough accuracy until they disagree. But if $\mathrm{Vol}(M_1) = \mathrm{Vol}(M_2)$, how do we prove that these are the same? It is conjectured that volumes of hyperbolic 3-manifolds are irrational (and probably transcendental).

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## Hyperbolic 3-manifolds

For example, Catalan's constant

$$K = 0.915965 = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \cdots$$

is the volume of a hyperbolic orbifold, and is not known to be irrational. There is a conjecture of Ramakrishnan which would imply that two hyperbolic manifolds have the same volume if and only if they are scissors congruent. This means that one may be cut up into finitely many polyhedra and reassembled to form the other one. This conjecture would imply the existence of such an algorithm, by reducing it to a homological question. It would also imply that many hyperbolic volumes are linearly independent over $\mathbb{Q}$, so would imply that most hyperbolic volumes are irrational.

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## Volume conjecture

Let $K$ be a knot with hyperbolic complement. The (normalized) $N$th colored Jones polynomial $J'_N$ is a polynomial invariant of link complements which is (roughly) obtained by taking the Jones polynomial of a cabling of a knot. Kashaev made the following conjecture (reinterpreted by Murakami-Murakami):

**Conjecture:**

$$\mathrm{Vol}(\mathbb{S}^3 - K) = 2\pi \lim_{N\to\infty} \frac{1}{N} \log |J'_N(K)(e^{2\pi i/N})|.$$

It has been checked for the figure eight knot and some other examples.

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## Volume conjecture

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$$\mathrm{Vol}(\mathbb{S}^3 - K) = 2\pi \lim_{N\to\infty} \frac{1}{N} \log |J'_N(K)(e^{2\pi i/N})|.$$

This conjecture would be remarkable, since it would give the first hint of a connection between quantum invariants and geometric invariants of 3-manifolds. This conjecture has been generalized in many ways, and it is an active area of research. Most likely, a resolution of the conjecture will lead to insights into both the structure of TQFT's and into hyperbolic manifolds.

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## Small volume and small Margulis constants

Jorgensen and Thurston showed that volumes of hyperbolic 3-manifolds and orbifolds are well-ordered.

**Problem:** Identify the smallest volume hyperbolic 3-orbifolds and manifolds with various topological characteristics.

Recent breakthroughs have been made by Marshall and Martin, who identified the smallest volume hyperbolic 3-orbifolds (with volume $.039\ldots$), and Gabai, Meyerhoff and Milley who showed that the Weeks manifold is the smallest volume orientable manifold with volume $.9427\ldots$.

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## Small volume and small Margulis constants

Recently, A. has shown that the Whitehead link complement and the $(-2,3,8)$ pretzel link complement are the smallest volume two cusped orientable 3-manifolds, with volume $4K = 3.66...$.
It is an interesting problem to identify the smallest volume hyperbolic manifolds with $n$ cusps, or with bounds on the betti numbers (Culler and Shalen have results of this type). Many times, minimal volume manifolds end up being arithmetic. So far, there is no good explanation for this phenomenon.

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## Small volume and small Margulis constants

If $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ is a discrete group, then the Margulis constant of $\Gamma$ is the smallest number $\epsilon$ such that for any 2 elements $\xi_1, \xi_2 \in \Gamma$, we have that either

- the group $\langle \xi_1, \xi_2 \rangle$ is (virtually) abelian, or
- $\max_{i=1,2} dist(\xi_1 z, z) \geq \epsilon, \ \forall z \in \mathbb{H}^3$.

Gehring and Martin have found optimal Margulis constants for Kleinian groups with torsion. It follows from work of Culler and Shalen that for $\Gamma$ torsion-free, there exists $V$ such that if $\mathrm{Vol}(\mathbb{H}^3/\Gamma) > V$, then $\epsilon > \log 3$. Thus, there are only countably many Margulis constants $< \log 3$.

**Problem:** Compute the smallest Margulis constants, and compute the torsion-free groups with Margulis constant $< \log 3$.

# Future Directions in 3-Dimensional Geometric Topology

## Hyperbolic structure and topology

A natural invariant of a manifold $M$ is the minimal number of critical points of a Morse function on $M$. Let's call this minimum $k(M)$. Another natural invariant is $rank(\pi_1 M)$, the minimal number of generators of $\pi_1 M$. Then $2rank(\pi_1 M) + 2 \leq k(M)$. Can one obtain a bound in the other direction?

**Conjecture:** There exists $f : \mathbb{N} \to \mathbb{N}$ such that if $M$ is a hyperbolic 3-manifold with $rank(\pi_1 M) \leq n$, then $k(M) \leq f(n)$.

# Future Directions in 3-Dimensional Geometric Topology

## Hyperbolic structure and topology

**Conjecture:** There exists $f : \mathbb{N} \to \mathbb{N}$ such that if $M$ is a hyperbolic 3-manifold with $rank(\pi_1 M) \leq n$, then $k(M) \leq f(n)$.
Some progress has been made on this conjecture under the hypothesis that $inj(M) > \epsilon$ and $rank \pi_1 M = 2$ by A. and by Biringer and Souto when $rank \pi_1 M > 2$.
**Conjecture:** There are finitely many commensurability classes of $n$-generator arithmetic hyperbolic 3-manifolds.
This conjecture would follow from the previous one.

# Future Directions in 3-Dimensional Geometric Topology

## Hyperbolic structure and topology

An important conjecture which goes back to Waldhausen is the focus of much current research.

**Conjecture:** If $M$ is a hyperbolic manifold, then there exists a finite-sheeted cover $\widetilde{M} \to M$ such that $M$ is Haken, *i.e.* has an embedded $\pi_1$-injective surface. More strongly, we may choose $\widetilde{M} \to M$ such that $\beta_1(M) > 0$.

There are several stronger variations on this conjecture.

If $M$ is arithmetic, then the Virtual Haken Conjecture would follow from the generalized Taniyama-Shimura conjecture for number fields.

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## Hyperbolic structure and topology

Let $\lambda_1(M)$ denote the minimal non-zero eigenvalue of the Laplacian on $M$.

**Question:** For $\epsilon > 0$, does there exist $V > 0$ such that if $M$ is a closed hyperbolic manifold with $\mathrm{Vol}(M) > V$ and $\lambda_1(M) > \epsilon$, then $M$ is Haken?

If this question were answered affirmatively, it would imply the Virtual Haken Conjecture. There is a potential approach to answering this question using minimal surfaces and Heegaard splittings (using methods of Casson-Gordon, Lackenby, and Rubinstein).

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## Hyperbolic structure and topology

### Theorem

*(Long-Lubotzky-Reid) If $M$ is a closed hyperbolic manifold, there exists $\epsilon > 0$ and a cofinal sequence of finite-index covers $M_n \rightarrow M$ such that $\lambda_1(M_n) > \epsilon$.*

Such a sequence of covers is said to have *property ($\tau$)*. The proof makes use of some number-theoretic techniques and a deep theorem of Bourgain and Gamburd about constructions of expander graphs.

# Future Directions in 3-Dimensional Geometric Topology

## Ricci Flow in 3-Dimensions

It is an important problem to simplify Perelman's argument proving the Geometrization Conjecture using Ricci flow. In particular, it is important to understand in more detail the structure and stability of singularities occurring in the 3-D Ricci flow. This might enable one to understand how the Ricci flow with surgery behaves for a parameterized family of Riemannian metrics.

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## Ricci Flow in 3-Dimensions

This might enable the proof of the Generalized Smale Conjecture in full generality. This was proven by Hatcher for $\mathbb{S}^3$ and for Haken manifolds, and by Gabai for hyperbolic 3-manifolds. Even with the resolution of the Geometrization Conjecture, there are still a few cases of the Smale conjecture left unresolved (further cases have been covered by McCullough and Rubinstein).

The Smale conjecture gives a conjectured classification of the homotopy type of $Diff(M)$. For $\mathbb{S}^3$ or for $M$ hyperbolic, the Smale conjecture states that $Diff(M) \simeq \mathrm{Isom}(M)$.

# Future Directions in 3-Dimensional Geometric Topology

## Ricci Flow in 3-Dimensions

In principle, the Geometrization Conjecture gives an algorithm to tell whether two 3-manifolds are homeomorphic. It is an interesting question whether Ricci flow can give any insight into the computational complexity of this problem.

**Question:** Is Ricci flow (with surgery) in 3-D algorithmic? Can we numerically simulate Ricci flow on a 3-manifold in a stable fashion, and use this to find the geometric decomposition of a 3-manifold? A 3-manifold may be described combinatorially as a simplicial complex, for example. If $M$ is hyperbolic, does the Ricci flow and its numerical simulation converge quickly to the hyperbolic metric, to give a polynomial time algorithm to tell if a manifold is hyperbolic?

# Future Directions in 3-Dimensional Geometric Topology

## Ricci Flow in 3-Dimensions

Another important problem in 3-D Ricci flow is to see if it is possible to prove the Orbifold Theorem using Ricci flow. Most likely this will work, but there are some issues that don't occur in the manifold case having to do with "Ricci solitons" on bad oribfolds, that don't exist in the manifold case.

# Future Directions in 3-Dimensional Geometric Topology

## Conclusion

To summarize, some possible future directions for the study of 3-D geometric topology which we believe will be fruitful are:

- continue with the study of geometric structures on 3-manifolds, and connections between these structures

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## Conclusion

To summarize, some possible future directions for the study of 3-D geometric topology which we believe will be fruitful are:

- continue with the study of geometric structures on 3-manifolds, and connections between these structures
- apply the classification of Kleinian groups to the study of hyperbolic 3-manifolds

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## Conclusion

To summarize, some possible future directions for the study of 3-D geometric topology which we believe will be fruitful are:

- continue with the study of geometric structures on 3-manifolds, and connections between these structures
- apply the classification of Kleinian groups to the study of hyperbolic 3-manifolds
- understand the structure of 3-D Ricci flow more precisely in order to refine the classification of 3-manifolds via geometric decompositions