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Talk 2: Conformal Killing equations via prolongation

From last time $\mathcal{A}_n$ = algebra of symmetries of Δ in $\mathbb{R}^n$ (Witten)

Recall 0^{th} order: $f \mapsto Cf$ or $f \mapsto Vf$ where $\nabla_a V = 0$.

1^{st} order: $\mathcal{D}_V f = V^a \nabla_a f + \frac{n-2}{2n}(\nabla_a V^a) f$ where

$$\nabla_{(a}V_{b)} - \frac{1}{n} g_{ab} \nabla^c V_c = 0 \quad \text{i.e. } \overset{\text{stet}}{\nabla}_{(a}V_{b)} = \nu g_{ab}$$

2^{nd} order: $\mathcal{D}_V f = \underset{\text{symm trace-free}}{V^{ab}} \nabla_a \nabla_b f + \frac{n}{n+2}(\nabla_a V^{ab}) \nabla_b f + \frac{n(n-2)}{4(n+1)(n+2)}(\nabla_a \nabla_b V^{ab}) f$

$$\text{where } \nabla_{(a}V_{bc)} - \frac{2}{n+2} g_{(ab} \nabla^d V_{c)d} = 0$$

more and more complicated formulae. HOWEVER, claim that

0^{th} order $\longleftrightarrow$ constants 1-dim'l

1^{st} order $\longleftrightarrow$ conformal Killing vectors $(n+1)(n+2)/2$ - dim'l (e.g. 10, $n=3$)

2^{nd} order $\longleftrightarrow$ conformal Killing 2-tensors of dimension

$$\frac{(n-1)(n+2)(n+3)(n+4)}{12} \quad \text{e.g. } n=3: \frac{2 \times 5 \times 6 \times 7}{12} = \underline{\underline{35}}$$

3^{rd} order etc. etc. $\dfrac{(n-1)n(n+1)(n+4)(n+5)(n+6)}{144}$

Anyway: Today's problem: how to solve

$$\nabla_{(a}V_{b)} - \frac{1}{n} g_{ab} \nabla^c V_c = 0 \,? \qquad \text{on } \mathbb{R}^n$$

Warm up problem: how to solve

$$\nabla_{(a}V_{b)} = 0 \qquad \text{Killing equation} \qquad \text{on } \mathbb{R}^n$$

Idea (prolongation): introduce variables for derivatives that you don't know with a view to finding their derivatives and (repeat, repeat, repeat, ...) hope to obtain a closed system.

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<u>Preliminary warm-up example</u>

trace-free part of $\nabla_a \nabla_b \sigma = 0$.

i.e. $\nabla_a \nabla_b \sigma = g_{ab} \rho$.

$$\nabla_a \sigma = \mu_a$$
$$\nabla_a \mu_b = g_{ab} \rho$$

not yet closed.

$$\nabla_b \nabla_a \mu^b = \nabla_a \nabla_b \mu^b \quad (+ R_{ab} \mu^b)$$
$$\Downarrow$$
$$\nabla_a \rho = n \nabla_a \rho \quad (+ R_{ab} \mu^b)$$
$$\Rightarrow \nabla_a \rho = 0 \quad \left(- \tfrac{1}{n-1} R_a{}^b \mu_b \right)$$

$$\begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \longmapsto \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho \\ \nabla_a \rho \underbrace{- \tfrac{1}{n-1} R_a{}^b \mu_b}_{\text{omit in flat case.}} \end{pmatrix}$$

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So: $\nabla_{(a}V_{b)} = 0 \iff \nabla_a V_b = F_{ab}$ for F_{ab} skew.

But now $F_{ab} = \nabla_{[a}V_{b]} \Rightarrow \nabla_{[a}F_{bc]} = 0$ $(F = dV \Rightarrow dF = 0)$

so write this out in detail

$$\begin{aligned}\nabla_a F_{bc} &= \nabla_c F_{ba} - \nabla_b F_{ca}\\ &= \nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a\\ &= 0 \text{ on } \mathbb{R}^n \quad (= R_{bc}{}^d{}_a V_d \text{ on Riemannian...})\end{aligned}$$

So $\nabla_{(a}V_{b)} = 0 \iff \begin{cases} \nabla_a V_b = F_{ab} \\ \nabla_a F_{bc} = 0 \end{cases}$

$\iff F_{bc} =$ constant and so

$$V_b = F_{ab} x^a + K_b$$

i.e. as vector field in traditional form

$$V = \underbrace{F_a{}^b x^a \frac{\partial}{\partial x^b}}_{\text{inf}^{el} \text{ rotations}} + \underbrace{K^b \frac{\partial}{\partial x^b}}_{\text{inf}^{el} \text{ translations}}$$

In particular dim of sol^{n} space is $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$

On Riemannian manifold end up with

$$\begin{pmatrix} V_b \\ F_{bc} \end{pmatrix} \mapsto \begin{pmatrix} \nabla_a V_b - F_{ab} \\ \nabla_a F_{bc} - R_{bc}{}^d{}_a V_d \end{pmatrix}$$

itself a connection

(flat iff

$$R_{abcd} = \Lambda(g_{ac}g_{bd} - g_{bc}g_{ad}))$$

<u>Conformal case</u> $\quad \underline{\nabla_a V_b = F_{ab} + \nu g_{ab}}$

same rigmarole: $\nabla_a F_{bc} = \nabla_c(\nabla_b V_a - \nu g_{ba}) - \nabla_b(\nabla_c V_a - \nu g_{ca})$

$$= -g_{ab}\nabla_c \nu + g_{ac}\nabla_b \nu$$

$\Rightarrow \nabla^b F_{bc} = -(n-1)\nabla_c \nu$

so let $\rho_c = \frac{1}{n-1}\nabla^b F_{bc}$ so this reads $\underline{\nabla_c \nu = -\rho_c}$

and substituting back

$$\underline{\nabla_a F_{bc} = g_{ab}\rho_c - g_{ac}\rho_b}$$

So $\nabla_d \nabla_a F_{bc} = g_{ab} \nabla_d \rho_c - g_{ac} \nabla_d \rho_b$

So $g_{ab}\nabla_d \rho_c - g_{ac}\nabla_d\rho_b = g_{db}\nabla_a\rho_c - g_{dc}\nabla_a\rho_b$

$\Rightarrow$ (tracing over a and b)

$$n\nabla_d\rho_c - \nabla_d\rho_c = \nabla_d\rho_c - g_{dc}\nabla_a\rho_a$$

9-10 Robin
$10\tfrac{1}{2}$-$11\tfrac{1}{2}$ Me
$11\tfrac{3}{4}$-$12\tfrac{3}{4}$ Alice
13-14 Lunch
$18\tfrac{1}{2}$-$19\tfrac{1}{2}$ Dinner
$19\tfrac{3}{4}$-$20\tfrac{3}{4}$ Willard

i.e. $(n-2)\nabla_d\rho_c = -g_{cd}\nabla^a\rho_a$

tracing again over d and c

$$(n-2)\nabla^d\rho_d = -n\nabla^a\rho_a$$

$$\Rightarrow \nabla^a\rho_a = 0$$

$$\Rightarrow (n\neq 2)\ \nabla_d\rho_c = 0$$

So $\nabla_a V_b = F_{ab} + \nu g_{ab}$

$\nabla_a F_{bc} = g_{ab}\rho_c - g_{ac}\rho_b \qquad \nabla_a \nu = -\rho_a$

$\nabla_a \rho_b = 0$

$$\begin{pmatrix} & V_b & \\ F_{bc} & & \nu \\ & \rho_b & \end{pmatrix} \mapsto \begin{pmatrix} & \nabla_a V_b - F_{ab} - \nu g_{ab} & \\ \nabla_a F_{bc} - g_{ab}\rho_c + g_{ac}\rho_b & & \nabla_a\nu + \rho_a \\ & \nabla_a\rho_b & \end{pmatrix}$$

anyway ρ_b is constant and work back...

$$F_{ab} = m_{ab} - \rho_a x_b + \rho_b x_a \qquad \nu = \lambda - \rho_a x^a$$

$$\Rightarrow V_a = s_a - m_{ab}x^b + \lambda x_a - \rho_b x^b x_a + \tfrac{1}{2} x_b x^b \rho_a$$

In particular, dimension of solution space is

$$n + \frac{n(n-1)}{2} + 1 + n = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}$$

as predicted

$$\begin{bmatrix} \lambda & -\rho^t & 0 \\ -s & m & \rho \\ 0 & s^t & -\lambda \end{bmatrix}$$ s.t. Lie brackets agree!

$= \dim O(n+1,1)$ as Lie group.

Cool down problem. $\nabla_a\nabla_b\sigma = 0$ on $\mathbb{R}^n$ or $\nabla_a\nabla_b\sigma = (\nabla_a\sigma)(\nabla_b\sigma) + \tfrac{1}{2}(\nabla_c\sigma)(\nabla^c\sigma)g_{ab}$

$\sigma = \log\Omega$ for flat-to-flat conformal factor.