Maths 260 Lecture 23

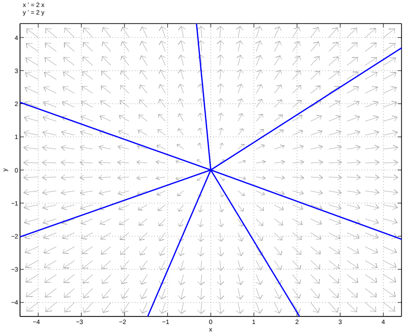
- **Topics for today:** Linear systems with repeated eigenvalues
  Linear systems with zero eigenvalues
- **Reading for this lecture:** BDH Section 3.5
- **Suggested exercises:** BDH Section 3.5; 1, 3, 5, 7, 11, 21
- **Reading for next lecture:** BDH Section 3.7

Linear systems with repeated eigenvalues

Example 1: Find the general solution for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{Y}$$

Phase portrait



Every non-zero solution is a straight-line solution.

Example 1 illustrates a general situation:

If matrix **A** has a repeated eigenvalue λ with two *linearly independent* eigenvectors v_1 and v_2, then

$$\mathbf{Y_1} = \mathrm{e}^{\lambda t}\mathbf{v_1} \quad \text{and} \quad \mathbf{Y_2} = \mathrm{e}^{\lambda t}\mathbf{v_2}$$

are linearly independent straight line solutions.

If **A** is a 2 by 2 matrix, we construct a general solution from a linear combination of these two solutions as usual:

$$\mathbf{Y}(t) = c_1\mathrm{e}^{\lambda t}\mathbf{v_1} + c_2\mathrm{e}^{\lambda t}\mathbf{v_2}$$

Then every solution except the equilibrium at the origin is a straight line solution.

If $\lambda > 0$ then every non-zero solution tends to ∞ as $t \to \infty$, and the origin is a source.

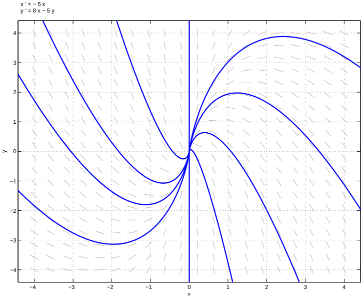
If $\lambda < 0$ then every non-zero solution tends to the origin as $t \to \infty$, and the origin is a sink.

What happens if we cannot find two linearly independent eigenvectors?

Example 2: Investigate solutions to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 8 & -5 \end{pmatrix} \mathbf{Y}$$

Phase portrait:



- We see that the system has only one straight line solution.
- We cannot write the general solution as a linear combination of solutions of the form $\mathrm{e}^{\lambda t}\mathbf{v}$ because we do not have enough such solutions.

Finding a second solution

Theorem: Consider the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ has a repeated eigenvalue λ with just one linearly independent eigenvector. Pick a specific eigenvector $\mathbf{v_1}$ for λ. Then

$$\mathbf{Y_1} = \mathrm{e}^{\lambda t}\mathbf{v_1}$$

is a straight-line solution and

$$\mathbf{Y_2} = \mathrm{e}^{\lambda t}(t\mathbf{v_1} + \mathbf{v_2})$$

is a second, linearly independent solution of the system, where $\mathbf{v_2}$ is a vector satisfying

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v_2} = \mathbf{v_1}$$

Vector $\mathbf{v_2}$ is called a **generalised eigenvector**.

Proof:

We can use this second solution $\mathbf{Y_2}(t)$ to construct the general solution for the previous example.

Example 2 again: Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 8 & -5 \end{pmatrix} \mathbf{Y}$$

In the phase portrait shown earlier, we see that all solutions are tangent at the origin to the direction of the straight-line solution (but tangent from one side only).

This is always the case in a 2×2 system: when there is a non-zero repeated eigenvalue with only one corresponding linearly independent eigenvector, all solution curves in the phase plane are tangent (from one side) to the straight-line solution.

Exercise: prove this.

Important note:

There is some freedom when choosing a generalised eigenvector.

For example, in Example 2

$$\mathbf{v_2} = \begin{pmatrix} \frac{1}{8} \\ y \end{pmatrix}$$

is a generalised eigenvector for any choice of y.

However, a multiple of a generalised eigenvector *is not* usually a generalised eigenvector.

For example, in Example 2

$$\mathbf{v_2} = k \begin{pmatrix} \frac{1}{8} \\ y \end{pmatrix}$$

is not a generalised eigenvector unless $k = 1$.

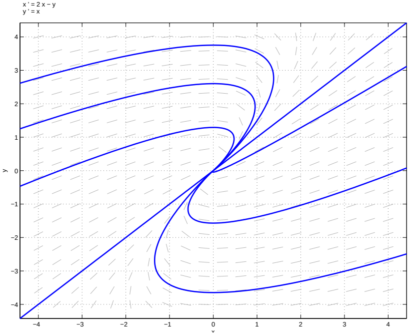
Different choices of the generalised eigenvector all lead to the same general solution.

Example 3

Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}$$

Phase portrait:



Linear systems with zero eigenvalues

Example 4: Find the general solution to the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \mathbf{Y}$$

The general solution is

$$\mathbf{Y}(t) = c_1 \mathrm{e}^{-5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

If $c_1 = 0$, then

$$\mathbf{Y}(t) = c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

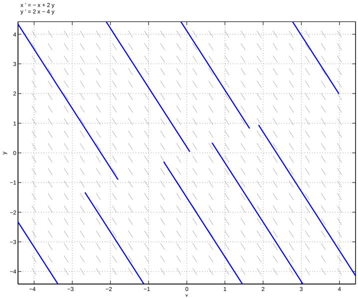
which is constant, so this is an equilibrium solution for all choices of c_2.

This is a general result: all points on a line of eigenvectors corresponding to a zero eigenvalue are equilibrium solutions.

If $c_1 \neq 0$, the first term in the general solution tends to zero as $t \to \infty$, i.e., the solution tends to the equilibrium

$$\mathbf{Y}(t) = c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

as $t \to \infty$, along a line parallel to the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.



We get similar behaviour in other linear systems with a zero eigenvalue, but details of the general solution and the phase portrait may vary depending on the specific example.

Example 5: Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix} \mathbf{Y}$$

Important ideas from today

In linear systems with repeated non-zero eigenvalues, the behaviour of solutions depends on the number of linearly independent eigenvectors corresponding to the repeated eigenvalue.

For a 2×2 system, there are two possibilities:

- If there are two linearly independent eigenvectors, then every solution except the equilibrium is a straight line solution.
- If there is only one independent eigenvector, then there is only one straight line solution, and all non-equilibrium solutions are tangent to that solution.

In both cases the equilibrium is a sink if the eigenvalue is negative and is a source if the eigenvalue is positive.

In a linear system with a zero eigenvalue, all points on the line(s) of eigenvectors corresponding to the zero eigenvalue are equilibrium solutions. Other details of the phase portrait depend on the specific system.