Introduction to

Conformal Geometry

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Lecture 1

- Flat model for conformal geometry

- Conformal group: $O(n+1,1)$

Lecture 2

- Curved conformal geometry

- Weyl and Cotton tensors

- Characterization of conformal flatness

Lecture 3

- Ambient metric

- Tractor bundle and connection

Riemannian geometry:

M: smooth manifold $\qquad$ g: metric on M

Can measure lengths of tangent vectors:

$|v|^2 = g(v, v), \qquad v \in T_pM$

Isometry: preserves lengths

Conformal geometry:

Can only measure angles:

$$\theta(v, w), \qquad \cos\theta = \frac{g(v, w)}{|v|\ |w|}$$

Same as knowing g up to scale at each point

Conformal mapping: preserves angles

Same as preserving g up to scale

Flat Model

Riemannian geometry:

$\mathbb{R}^n$, Euclidean metric: $\sum (dy^i)^2$

Isometries: $E(n) =$ group of Euclidean motions

$y \to Ay + b, \qquad A \in O(n), \quad b \in \mathbb{R}^n$

Can view $E(n) \subset GL(n+1, \mathbb{R})$ by:

$$(A, b) \leftrightarrow \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

Embed $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ by: $y \leftrightarrow \begin{pmatrix} y \\ 1 \end{pmatrix}$

Then $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} Ay + b \\ 1 \end{pmatrix}$

Conformal Transformations of $\mathbb{R}^n$

Euclidean motions

Dilations $y \to sy, \quad s \in \mathbb{R}_+$

Inversions in spheres

For unit sphere, center at origin, get $y \to \frac{y}{|y|^2}$

Definition: Möb($\mathbb{R}^n$) = Group generated by Euclidean motions, dilations, inversions

Möb($\mathbb{R}^n$) provides one approach to the study of the conformal group.

But inversions don't map $\mathbb{R}^n$ to $\mathbb{R}^n$:
center of sphere goes to ∞

Suggests compactifying $\mathbb{R}^n$ by appending ∞

Unnecessary and inappropriate for Euclidean motions

Compactify $\mathbb{R}^n$ to $S^n = \mathbb{R}^n \cup \infty$.
Metric on $\mathbb{R}^{n+1}$ induces metric on S^n;
hence a conformal structure on S^n.
$S^n \setminus \infty$ is conformally equivalent to $\mathbb{R}^n$ via stereographic projection.

Fruitful point of view: describe conformal geometry of S^n in terms of Minkowski geometry of $\mathbb{R}^{n+2}$: quadratic form Q of signature $(n+1, 1)$

$$x = (x^0, x^1, \cdots, x^{n+1}) \in \mathbb{R}^{n+2}$$

$$Q(x) = \sum_{\alpha=0}^{n} (x^\alpha)^2 - (x^{n+1})^2$$

$\mathcal{N} = \{x : Q(x) = 0\} \subset \mathbb{R}^{n+2} \setminus \{0\}$ Null cone

$\mathbb{P}^{n+1} = \left\{\ell = [x] : x \in \mathbb{R}^{n+2} \setminus \{0\}\right\}$ lines in $\mathbb{R}^{n+2}$

$\mathcal{Q} = \{\ell = [x] : x \in \mathcal{N}\} \subset \mathbb{P}^{n+1}$ Quadric

$\mathcal{Q} \cong S^n :$ Let $y \in S^n$, so $y \in \mathbb{R}^{n+1}$, $|y| = 1$.
The map $S^n \ni y \to \left[\begin{pmatrix} y \\ 1 \end{pmatrix}\right] \in \mathcal{Q}$ is a bijection.

$\pi : \mathcal{N} \to \mathcal{Q}$ projection

$$\tilde{g} = \sum_{\alpha=0}^{n} (dx^\alpha)^2 - (dx^{n+1})^2 \qquad \text{Minkowski metric}$$

$$= \tilde{g}_{IJ} dx^I dx^J, \qquad \text{where} \qquad \tilde{g}_{IJ} = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}$$

$$X = \sum_{I=0}^{n+1} x^I \partial_{x^I} \qquad \text{position vector field on } \mathbb{R}^{n+2}$$

If $x \in \mathcal{N}$, then $X(x) \in T_x\mathcal{N}$

Have $\tilde{g}(X, X) = \tilde{g}_{IJ} x^I x^J = Q(x) = 0$ on $\mathcal{N}$.

Claim: $\tilde{g}(X, V) = 0$ for all $V \in T\mathcal{N}$.

Proof: $Q = 0$ on $\mathcal{N}$. So $dQ(V) = 0$ if $V \in T\mathcal{N}$.

Now $Q = \tilde{g}_{IJ} x^I x^J$, so $dQ = 2\tilde{g}_{IJ} x^I dx^J$.

Gives $0 = dQ(V) = 2\tilde{g}_{IJ} x^I V^J = 2\tilde{g}(X, V)$. $\square$

Let $x \in \mathcal{N}$. Then $\tilde{g}\Big|_{T_x\mathcal{N}}$ is degenerate: $X \perp T_x\mathcal{N}$

But $\tilde{g}\Big|_{T_x\mathcal{N}}$ induces an inner product $g^{(x)}$ on $T_\ell\mathcal{Q}$,

where $\ell = \pi(x) = [x]$. $g^{(x)}$ is defined as follows:

Have $\pi : \mathcal{N} \to \mathcal{Q}$. Gives $\pi_* : T_x\mathcal{N} \to T_\ell\mathcal{Q}$

Now $\pi_*(X) = 0$, and $\pi_* : T_x\mathcal{N}/\text{span}X \to T_\ell\mathcal{Q}$
is an isomorphism

Given v, $w \in T_\ell\mathcal{Q}$, want to define $g^{(x)}(v, w)$

Choose V, $W \in T_x\mathcal{N}$ such that $\pi_*V = v$, $\pi_*W = w$.
Unique up to $V \to V + cX$, $W \to W + c'X$.

Define $g^{(x)}(v, w) = \tilde{g}(V, W)$.
Independent of choices since $X \perp T_x\mathcal{N}$.

$g^{(x)}$ is a positive definite inner product on $T_\ell\mathcal{Q}$.

If $0 \neq s \in \mathbb{R}$, how are $g^{(sx)}$ and $g^{(x)}$ related?

Have $\delta_s : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$, $\quad \delta_s(x) = sx \quad$ Dilations

$\widetilde{g}$ satisfies $(\delta_s)^* \widetilde{g} = s^2 \widetilde{g}$.

Have $\pi(sx) = \pi(x) = \ell$.

Suppose $v,\ w \in T_\ell \mathcal{Q}$.

Choose $V,\ W \in T_x \mathcal{N}$ so that $\pi_* V = v,\ \pi_* W = w$.

Then $V_s = (\delta_s)_* V,\ W_s = (\delta_s)_* W$ satisfy

$\pi_* V_s = v,\ \pi_* W_s = w, \quad$ since $\pi \circ \delta_s = \pi$.

So

$$g^{(sx)}(v, w) = \widetilde{g}(V_s, W_s) = \widetilde{g}((\delta_s)_* V, (\delta_s)_* W)$$

$$= (\delta_s)^* \widetilde{g}(V, W) = s^2 \widetilde{g}(V, W) = s^2 g^{(x)}(v, w)$$

Thus $g^{(sx)} = s^2 g^{(x)}$.

Conclusion: The Minkowski metric on $\mathbb{R}^{n+2}$ invariantly determines a conformal structure on $\mathcal{Q}$, but not a Riemannian structure.

Conformal Group

Idea: Any map of $\mathbb{R}^{n+2}$ which preserves the linear structure and the Minkowski metric will induce a map of $\mathcal{Q}$ preserving the conformal structure.

Definition: $O(n+1,1) =$ linear tranformations of $\mathbb{R}^{n+2}$ preserving Q

Let $L \in O(n+1,1)$. Then $L|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$.
Preserves lines, so induces $L_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$.

Claim: $L_{\mathcal{Q}}$ is a conformal transformation of $\mathcal{Q}$.

Proof: Let $x \in \mathcal{N}$, so $\pi(x) = [x] = \ell \in \mathcal{Q}$.
Then $L_{\mathcal{Q}}([x]) = [Lx]$. Must show that $(L_{\mathcal{Q}})^* g^{(Lx)}$ is a multiple of $g^{(x)}$.
Will actually show they are equal.
Let v, $w \in T_\ell \mathcal{Q}$.
Choose V, $W \in T_x \mathcal{N}$ so that $\pi_* V = v$, $\pi_* W = w$.
Then $L_* V$ satisfies $\pi_* L_* V = (L_{\mathcal{Q}})_*(v)$, sim. for W.
So

$$(L_{\mathcal{Q}})^* g^{(Lx)}(v,w) = g^{(Lx)}\left((L_{\mathcal{Q}})_*(v), (L_{\mathcal{Q}})_*(w)\right)$$
$$= \widetilde{g}(L_* V, L_* W) = (L^* \widetilde{g})(V,W) = \widetilde{g}(V,W)$$
$$= g^{(x)}(v,w) \qquad \square$$

Thus $O(n+1,1)$ acts on $\mathcal{Q}$ by conformal transformations. $-I \in O(n+1,1)$ acts trivially.

Definition: The conformal group of $\mathcal{Q} = S^n$ is $O(n+1,1)/\{\pm I\}$.

Metrics in the conformal class on $\mathcal{Q}$ correspond to sections of $\mathcal{N} \to \mathcal{Q}$ (modulo ± 1):

Let $s : \mathcal{Q} \to \mathcal{N}$ be a section of π. If $\ell \in \mathcal{Q}$, then $x = s(\ell)$ is a point on ℓ. So $g^{(x)}$ is a metric in the conformal class at ℓ. Thus $\ell \to g^{(s(\ell))}$ is a metric in the conformal class on $\mathcal{Q}$. $g^{(s(\ell))}$ determines $s(\ell)$ up to ± 1.

Explicit realization on S^n

Realize $S^n \cong \mathcal{Q}$ by $x^{n+1} = 1$. This section s determines the usual metric on S^n, since we can choose V, W so that $dx^{n+1}(V) = dx^{n+1}(W) = 0$, in which case $\tilde{g}$ becomes $\sum_{\alpha=0}^{n} (dx^\alpha)^2$.

Recall $L_{\mathcal{Q}}$ is the map induced on $\mathcal{Q}$ by L.

Write $L_{S^n} : S^n \to S^n$ for the corresponding conformal transformation of S^n, when $\mathcal{Q}$ is identified with S^n this way.

How does $L \in O(n+1,1)$ act on S^n?

Let $L \in O(n+1,1)$. Write $L = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix}$.

Let $y \in S^n$. So $y \in \mathbb{R}^{n+1}$ and $|y| = 1$.

Set $x = \begin{pmatrix} y \\ 1 \end{pmatrix} \in \mathcal{N}$. Then

$$Lx = \begin{pmatrix} A & b \\ c^t & d \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} Ay + b \\ c^t y + d \end{pmatrix} \in \mathcal{N}.$$

Now $\begin{pmatrix} Ay + b \\ c^t y + d \end{pmatrix} = (c^t y + d) \begin{pmatrix} \frac{Ay+b}{c^t y+d} \\ 1 \end{pmatrix}$

and $\left| \frac{Ay+b}{c^t y+d} \right| = 1$.

So obtain

$$L_{S^n}(y) = \frac{Ay+b}{c^t y+d}$$

Conclusion: The conformal tranformations of S^n are fractional linear transformations induced by $L \in O(n+1,1)$.

This argument also gives the conformal factor associated to L_{S^n}.

Recall $(L_{\mathcal{Q}})^* g^{(Lx)} = g^{(x)}$

If $x^{n+1} = 1$, then $g^{(x)} = g_{S^n}$ is usual metric on S^n.

Take $y \in S^n$ and apply above with $x = \begin{pmatrix} y \\ 1 \end{pmatrix}$.

Then $Lx = (c^t y + d) \begin{pmatrix} L_{S^n}(y) \\ 1 \end{pmatrix} = sx'$

with $s = (c^t y + d)$, $x' = \begin{pmatrix} L_{S^n}(y) \\ 1 \end{pmatrix}$, so $(x')^{n+1} = 1$.

Use $g^{(sx')} = s^2 g^{(x')}$; get

$(L_{\mathcal{Q}})^* g^{(Lx)} = (L_{\mathcal{Q}})^* g^{(sx')} = s^2 (L_{\mathcal{Q}})^* g^{(x')}$

So $(L_{S^n})^* g_{S^n} = (c^t y + d)^{-2} g_{S^n}$.

Conclusion: The conformal factor associated to the fractional linear transformation

$L_{S^n}(y) = \frac{Ay+b}{c^t y+d}$ is $(c^t y + d)^{-2}$.

Can conjugate L_{S^n} by stereographic projection to obtain formulae for the conformal transformations on $\mathbb{R}^n$.

Why are all conformal transformations of this form?

Liouville's Theorem: Suppose $n \geq 3$. If $U \subset S^n$ is open and connected and $\varphi : U \to S^n$ is a C^1 conformal transformation, then φ is the restriction to U of L_{S^n} for some $L \in O(n+1, 1)$.

This is false for $n = 2$: holomorphic and conjugate-holomorphic maps are conformal. But the global result is true for $n = 2$: every conformal transformation of S^2 is L_{S^2} for some $L \in O(3,1)$. This follows by complex analysis: conformal transformations are holomorphic or anti-holomorphic; the holomorphic ones are in $PSL(2, \mathbb{C})$, isomorphic to the identity component of $O(3,1)$.

Liouville's Theorem is a result about solutions of overdetermined systems of pde's; cf. talks of M. Eastwood. The condition that φ is conformal is $\varphi^* g = \Omega^2 g$, where $\Omega > 0$ is an unknown function. This is a first order system of pde's for the components of φ. There are $n+1$ unknowns: Ω and the n components of φ. There are $n(n+1)/2$ equations: the components of g. These are equal for $n = 2$, so problem is determined in that case–lots of solutions. Overdetermined for $n > 2$.

The proof proceeds by first analyzing Ω. We know that Ω for any L_{S^n} is of the form

$$\Omega(y) = (c^t y + d)^{-1}.$$

So try to show that Ω^{-1} is a linear function of y. Actually, the Theorem is proved in its R^n realization. Corresponding statement is that the analogous Ω^{-1} on $\mathbb{R}^n$ is a quadratic function of the form $\mu|x - x_0|^2 + \kappa$. The Hessian of Ω^{-1} must be a constant multiple of the identity. Once this is known, one shows that either $\mu = 0$ or $\kappa = 0$. The first case corresponds to a dilation composed with an isometry, the second to an inversion composed with an isometry. The fact about the Hessian of Ω^{-1} can be derived by differentiations and manipulations of the equation $\varphi^* g = \Omega^2 g$.

The "ambient" realization of the flat model for conformal geometry and the conformal group works exactly the same way for other signatures.

For conformal geometry in signature (p, q) with $p+q=n$, the flat model is a quadric $\mathcal{Q}$ of signature $(p+1, q+1)$ in $\mathbb{P}^{n+1}$. The conformal group is $O(p+1, q+1)/\{\pm I\}$, acting as linear transformations on $\mathcal{Q}$.

However, the conformal compactification of $\mathbb{R}^n$ to get the quadric involves adding more than a single point at infinity. The inversion $x \to \frac{x}{|x|^2}$ is still conformal, where now $|x|^2$ is with respect to a mixed signature metric. So a full null-cone gets mapped to infinity and must be included in the compactification. This is all easily analyzed in terms of the geometry of the quadric in $\mathbb{P}^{n+1}$.

In the special case of Lorentz signature one obtains the conformal compactification of Minkowski space. This is important for relativity.