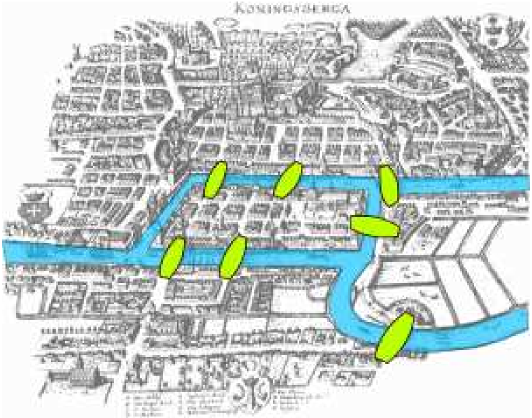
Maths 190 Lecture 21

- **Topic for today**: Graph theory
- **Question of the day**: Can one walk across every bridge in Königsberg in 1736 exactly once?



The Königsberg bridges

Königsberg is now called Kaliningrad and is a town in Russia (that little disconnected bit between Poland and Lithuania on the coast).

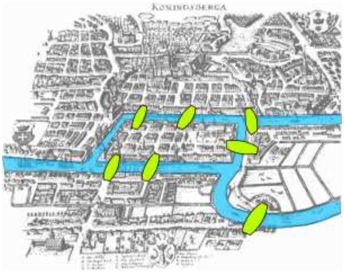
In 1736, the mathematician Leonhard Euler was asked whether or not it is possible to stroll from ones house, crossing every bridge exactly once, and returning to ones house.

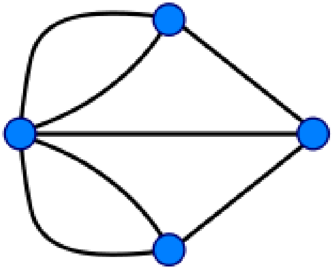
Today's big idea: Abstraction

Euler simplified the description of the problem so that all irrelevant aspects were removed.

This is one of the great contributions of mathematics: It provides a language in which one can express problems, and a set of tools for solving them.

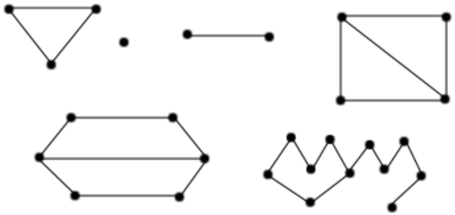
Replace the landmasses with points and the bridges with lines.





Graphs

A **graph** is a picture made from dots (called vertices) and lines between dots (called edges). Some examples are:



A **path** is a list of edges, each starting from the end point of the previous one.

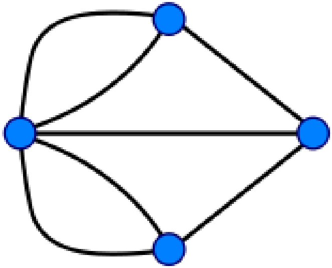
A graph is **connected** if there is a path between any two vertices.

Euler circuit

A **circuit** is a path which returns to its starting point.

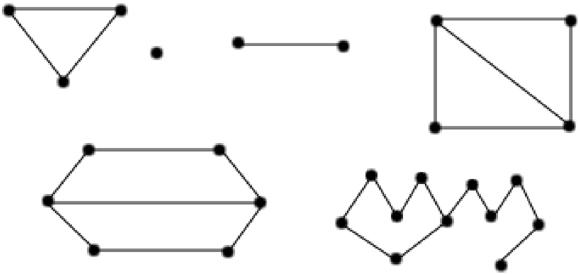
An **Euler circuit** in a connected graph is a path which starts and ends at a vertex and which crosses every edge once.

The Königsberg bridges problem is precisely whether there exists an Euler circuit in the graph:



Euler circuits

Which of these graphs have an Euler circuit?



Draw another graph which has an Euler circuit. Draw another graph which does not have an Euler circuit.

Euler circuits

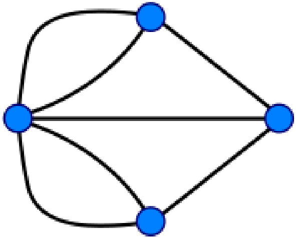
What is the crucial property of those graphs which have an Euler circuit?

When can an Euler circuit exist?

The **degree** of a vertex in a graph is the number of edges which meet that vertex.

Theorem: If a connected graph has an Euler circuit then the degree of every vertex is even.

Hence, one now knows that the Königsberg bridges problem does not have a solution.



Euler's theorem

We have seen that it is necessary for every vertex in a connected graph to have even degree if there is an Euler circuit.

Amazingly, this condition is also sufficient. In other words, every connected graph for which all vertices have even degree, has an Euler circuit.

Proof of Euler's theorem

- Suppose we have a connected graph for which all vertices have even degree.
- Start at any vertex v and make a path where no edge is used twice.
- Since the degree of every vertex is even, every time you enter a vertex there is a way out. Hence, there is no way to get stuck.
- Except, for the vertex you started with. It is possible to get stuck here.
- Since the graph is finite, at some point one must get stuck.
- In other words, we have proved that just taking a random un-used edge at each step leads to a circuit in the graph.

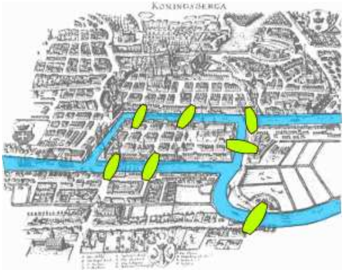
Proof of Euler's theorem continued

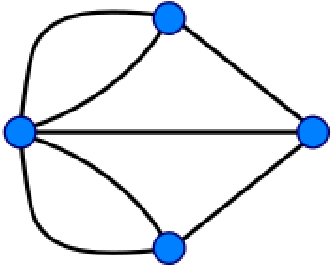
- If our circuit covers every edge then we are finished.
- Suppose there are some edges not used.
  Take one of these edges which meets the current circuit at a vertex v.
- Repeat the process again, starting with that vertex v and that edge and avoiding all edges used so far.
- By the same reason as above, one must form a circuit which returns to the vertex v.
- Joining the "new" circuit with the "old" one leads to a bigger circuit.
- Repeating the process a finite number of times proves the theorem.

Examples

Bridge building

How many bridges do you need to build in Königsberg before the citizens can enjoy an Euler circuit?





Graph theory

Graph theory is a major topic in mathematics, with applications anywhere there are networks: infrastructure, communications, optimisation, the internet.

The idea of Euler circuits and even degree is important for many scheduling problems: airline flight scheduling, movement of trucks and trains, etc.

Important ideas from today

Abstraction is a powerful tool for solving problems.

Read Sections 5.3 of the book.