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FOUR-DIMENSIONAL CONFORMAL C-SPACES

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Abstract

We investigate the structure of conformal C-spaces, a class of Riemannian manifolds which naturally arises as a conformal generalization of the Einstein condition. A basic question is when such a structure is closed, or equivalently locally conformally Cotton. In dimension four, we obtain a full answer to this question and also investigate the incidence of the Bach condition on this class of metrics. This is related to earlier results obtained in the Einstein–Weyl context.

1. Introduction

Let (M^n, g) be Riemannian manifold. The metric g is said to be Einstein if and only if

$$\mathrm{Ric} = \lambda g$$

for some real constant λ, where Ric is the so-called Ricci curvature tensor of the metric g. Einstein metrics have long had a privileged role in geometry. Toward the study of Einstein structures, and also because Einstein metrics may be obstructed (for example topologically), various generalizations of the Einstein condition are important [**6**]. One consists in requiring the so-called Cotton tensor C of the metric g to vanish and weakening further we might simply require that g be conformal to a Cotton metric. This is achieved if there is a gradient field ζ solving the equation

$$\iota_\zeta W + C = 0, \tag{1}$$

where ι_ζ indicates insertion (or interior multiplication) of ζ and W denotes the Weyl tensor of the metric g.

The requirement that ζ should be exact makes the condition (1) awkward to deal with directly. It is not obvious, for example, how to give local conformal invariants which characterize metrics which are locally conformally Cotton. This suggests that, in the first instance one might consider a 'Weyl analogue' of the conformal Cotton equation (1) as follows. We will say (following [**18**]) that a Riemannian manifold is a *conformal C-space*, if there is a solution to (1), where ζ is any section of TM. Via a suitable (and natural) interpretation of ζ this is a conformally invariant condition. In this context, we term (1) the conformal C-space equation.

This move is also motivated by the natural tractor/Cartan structures of Cartan [**10**] and Thomas [**24**] (see [**5**, **9**, **15**] for modern treatments) and the corresponding conformal holonomy [**2**, **22**]. Dimension n conformal Riemannian manifolds are naturally equipped with a rank $n+2$ vector bundle with

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a Lorentzian signature metric and compatible canonical connection. This is the tractor bundle and connection and the equivalent [9] principal bundle structure is the Cartan connection. An Einstein structure determines a parallel section I of the standard tractor bundle and, conversely, if a conformal structure admits a parallel standard tractor field I then this (parallel section) determines an Einstein metric on an open dense set (and we say the manifold is *almost Einstein* [17]). This is the set with $X(I)$ non-vanishing, where X is a canonical homomorphism which takes sections of the tractor bundle to conformal densities. Writing Ω for the curvature of the tractor connection, I parallel clearly implies $\Omega I = 0$. An obvious weakening of the almost Einstein condition is to require that there is a section I of the standard tractor bundle (now not necessarily parallel) satisfying $\Omega I = 0$. On the open set where $X(I)$ is non-vanishing, $\Omega I = 0$ is exactly (1), the conformal C-space equation [18]. Thus in the sense of infinitesimal conformal holonomy the conformal C-space equation is a vastly weaker requirement than the Einstein condition (which would have I annihilate the full jet of the tractor curvature).

Using related ideas it is straightforward to manufacture conformal obstructions to conformal C-space metrics. For example on Riemannian four-manifolds with non-vanishing Weyl tensor the conformal invariant

$$|W|^2 C_{abc} + 4W^{dijk}C_{ijk}W_{dabc}$$

(where we have used an obvious abstract index notation) vanishes if and only if the manifold is a conformal C-space; this result is easily recovered, or see [18, Proposition 2.5].

Taking the conformal C-space equation as our basic generalization of the Cotton and Einstein conditions, the fundamental question then is how far we are from these 'integrable cases'. In the case that the Weyl curvature is suitably non-degenerate, answers to this may be found in [18], and see also [21]. Here we initiate a study of this issue on Riemannian manifolds, but where the aim is to remove the assumption of local conditions on the Weyl curvature. One of the main results is the following.

THEOREM 1.1 *If (M^4, g) is a compact conformal C-space then it is locally conformally Cotton.*

Recall that on four-manifolds the Bach tensor B (of conformal relativity [4]) is a conformal invariant with leading term a divergence of the Cotton tensor. This vanishes on half-flat manifolds, and also on locally conformally Einstein structures [6]. Bringing this into the picture leads (see Section 6) to a stronger result.

THEOREM 1.2 *If (M^4, g) is a compact conformal C-space which is Bach-flat then it is locally conformally Einstein.*

We note that in [21] the authors obtained a result that Bach flat conformal C-spaces are locally conformally Einstein in four dimensions, provided that the Weyl tensor satisfies non-degeneracy conditions. So in the compact Riemannian setting the theorem improves their result by removing the need for a non-degeneracy assumption.

Another generalization of the Einstein condition which has been studied extensively (for example [13, 14, 16]) are the Einstein–Weyl equations. A conformal manifold is said to be Einstein–Weyl if it admits a compatible torsion-free connection that has vanishing trace-free symmetrized Ricci curvature. Writing h for the trace-adjusted ('reduced') Ricci tensor and ∇ for the Levi–Civita connection, this problem is equivalent to finding a one-form field ζ and a metric g so that the symmetric part of

$h - \nabla\zeta + \zeta \otimes \zeta$ is pure trace. There is a close connection with conformal C-spaces and this plays a role in the proofs of the main theorems.

Our paper is organized as follows. In Section 2, we collect a number of basic facts of relevance for the study of conformal C-spaces. In Section 3, we study suitably defined symmetries of an algebraic Weyl curvature tensor and show how these can be fully understood in dimension 4. To prove Theorem 1.1 we first show in Section 4 that it holds locally, that is, on the open subset where the Weyl tensor does not vanish. We use a detailed analysis of the properties of the Weyl curvature of a four-dimensional conformal C-space combined with some results from Hermitian geometry [1]. In Section 5, we establish that the unique continuation property holds for the class of conformal C-spaces and this eventually leads to the proof in the compact case. Finally, the last section of the paper is devoted to the proof of Theorem 1.2.

2. Conformal C-spaces and related structures

In this section, we review a number of elementary facts concerning the objects we shall subsequently use. Let (M^n, g), $n \geq 3$, be a Riemannian manifold, ∇ the Levi-Civita connection associated with the metric g and R the Riemannian curvature tensor, given by $R(X, Y)Z = -\nabla^2_{X,Y}Z + \nabla^2_{Y,X}Z$, whenever X, Y, Z are vector fields on M. The Weyl tensor W is defined by the decomposition

$$R = W + S, \tag{2}$$

where the Schouten tensor S is given by $S = h \bullet g$. Here $h = (1/(n-2))(\mathrm{Ric} - (s/2(n-1))g)$ is the reduced Ricci tensor of the metric g, while the Kulkarni–Nomizu product of two symmetric tensors h and k is defined by

$$(h \bullet k)(x, y, z, t) = h(x, z)k(y, t) + h(y, t)k(x, z) - h(x, t)k(y, z) - h(y, z)k(x, t).$$

The Weyl tensor satisfies the first Bianchi identity

$$W(X, Y)Z + W(Y, Z)X + W(Z, X)Y = 0. \tag{3}$$

The second Bianchi identity for W is slightly more complicated and depends on the Cotton tensor C, an element of $TM \otimes \Lambda^2(M)$. It is defined by

$$C(U, X, Y) = (\nabla_X h)(Y, U) - (\nabla_Y h)(X, U)$$

for all U, X, Y in TM and then

$$\sigma_{X,Y,Z}[(\nabla_X W)(Y, Z, U, T)] + (C_U \wedge T - C_T \wedge U)(X, Y, Z) = 0, \tag{4}$$

where σ stands for the cyclic sum. An appropriate contraction of the differential Bianchi identity (4) relates the Cotton tensor C to the Weyl tensor W by the formula

$$\delta^{\nabla} W = -(n-3)C, \tag{5}$$

where $\delta^{\nabla} W = -\sum_{i=1}^{n} (\nabla_{e_i} W)(e_i, \cdot, \cdot, \cdot)$ for an arbitrary local orthonormal frame $\{e_i, 1 \leq i \leq n\}$ on M. The Bach tensor B of the Riemannian manifold (M^n, g) is defined by

$$\langle BX, Y \rangle = \sum_{i=1}^{n} \nabla_{e_i} (\delta^{\nabla} W)(X, e_i, Y) + W(X, e_i, h(e_i), Y)$$

for all X, Y in TM. As is well known this is symmetric and tracefree, and moreover in dimension 4 it is conformally invariant. In dimension 4 it also vanishes on (anti)self-dual metrics [16]. In this paper, we shall mainly study the following class of Riemannian manifolds.

DEFINITION 2.1 *Let (M^n, g) be a Riemannian manifold and let ζ be a vector field on M. Then (M^n, g, ζ) is a conformal C-space if the equation*

$$W(\zeta, \cdot, \cdot, \cdot) + C = 0 \tag{6}$$

is satisfied. If $\zeta = 0$ then (M^n, g) is called a Cotton space.

It should be noted that conformal C-spaces are conformally invariant in the usual sense. Also, a natural subclass to look at consists of *closed* conformal C-spaces, that is, conformal C-spaces (M^n, g, ζ) such that (identifying vector fields and one-forms via the metric) $d\zeta = 0$, which is again a conformally invariant condition. Obviously, the notion of being closed in the conformal C-space context rephrases globally that a Riemannian metric is locally conformal to that of a Cotton space.

REMARK 2.1 Non-Einstein Cotton spaces are known to have vanishing Pontrjagin classes, a fact used in [7] to show that in four dimensions the non-vanishing of the signature implies that the metric is Einstein, provided the manifold is compact. Further results and examples were obtained in dimension 4 [12] under degeneracy assumptions on the spectrum of the Weyl or the Schouten tensor. It is also known that—in the compact case—the metric has to be Einstein when in the presence of a compatible Kähler [6] or closed G_2 structure [8, 11]. Despite continuing interest, a complete classification seems to be still missing.

Note that in dimensions $n \leq 3$, a conformal C-space is automatically Cotton due to the absence of algebraic Weyl curvature tensors, hence the first interesting dimension in this context is when $n = 4$. Related to conformal C-spaces are Einstein–Weyl structures whose definition we give below.

DEFINITION 2.2 Let (M^n, g) be Riemannian. Then g is Einstein–Weyl if

$$h = \nabla\zeta - \zeta \otimes \zeta - \frac{1}{2} d\zeta + fg$$

for some one-form field ζ on M and some smooth function f on M. Moreover (g, ζ) is said to be a closed Einstein–Weyl structure if $d\zeta = 0$.

A central focus of this paper is to investigate the extent to which a conformal C-space is necessarily closed. To approach this we consider, for a given conformal C-space (M^n, g, ζ), the tensor h_ζ defined by

$$h_\zeta = h - \nabla\zeta + \zeta \otimes \zeta.$$

Recall [18] that we have the following.

PROPOSITION 2.1 *Let (M^n, g, ζ) be a conformal C-space. The tensor h_ζ satisfies the identity*

$$W(x, y, h_\zeta z, \cdot) + W(y, z, h_\zeta x, \cdot) + W(z, x, h_\zeta y, \cdot) = 0$$

for all vector fields x, y, z.

Therefore, one must first understand the algebraic structure of the space of two-tensors satisfying the displayed identity and then explore the geometric consequences. In the next section we gather some general facts in this direction, and explore the four-dimensional case in greater depth.

3. The symmetries of algebraic Weyl curvature tensors

In this section, we shall study various algebraic equations capturing additional symmetries of an algebraic Weyl curvature tensor. These are useful for studying various geometric structures, and the relevant connections will be made clear in the next section.

3.1. *The various equations*

Let (V^n, g), $n \geq 4$, be a Euclidean vector space. In what follows we shall use the metric to identify (without further comment) vectors and one-forms. Similarly we identify $\otimes^2 V$ with $\mathrm{End}(V)$ using the convention $\beta = g(h\cdot, \cdot)$. As a point of notation, we shall use $\langle \cdot, \cdot \rangle$ for the form inner product induced by g. Let $b_1 : \Lambda^2 \otimes \Lambda^2 \to \Lambda^3 \otimes \Lambda^1$ be the Bianchi map given by

$$(b_1 R)(x, y, z) = R(x, y)z + R(y, z)x + R(z, x)y$$

whenever x, y, z belong to V and for any R in $\Lambda^2 \otimes \Lambda^2$, where standard notation applies. Consider now a non-vanishing algebraic Weyl-curvature tensor W on V. That is, W is an element of $\Lambda^2 \otimes \Lambda^2$ satisfying the first Bianchi identity (that is, $b_1(W) = 0$) and which is moreover trace-free in the sense that $\sum_{i=1}^n W(\cdot, e_i, \cdot, e_i) = 0$ for any orthonormal frame $\{e_i, 1 \leq i \leq n\}$. Then we can view W as a map $W : \otimes^2 V \to \otimes^2 V$ by setting:

$$W(h) = \sum_{i=1}^n W(e_i, \cdot, he_i, \cdot)$$

for any h in $\otimes^2 V$ and for some arbitrary orthonormal basis $\{e_i\}$ in V. This extension of W preserves the tensor type, that is, it preserves the splitting $\otimes^2 V = \Lambda^2 \oplus S^2_0 \oplus \mathbb{R}g$. Moreover, the restriction of W to $\Lambda^2(V)$ is given by $\langle W(v \wedge w), u \wedge q \rangle = W(v, w, u, q)$ for all v, w, u, q in V. Using that W satisfies the first Bianchi identity, this can also be rephrased to say that

$$W(\alpha) = \frac{1}{2} \sum_{i=1}^n W(e_i, Fe_i)$$

for an arbitrary orthonormal basis $\{e_i, 1 \leq i \leq n\}$ and for all two-forms α with associated skew-symmetric endomorphism F (that is, $\alpha = g(F\cdot, \cdot)$). Let us now fix an algebraic Weyl tensor W.

We shall be interested in what follows in the space $\mathcal{E}_W$ of tensors h in $\mathrm{End}(V)$ such that

$$W(x, y, hz, \cdot) + W(y, z, hx, \cdot) + W(z, x, hy, \cdot) = 0 \tag{7}$$

for all x, y, z in V. We also define the spaces $\mathcal{S}_W = \mathcal{E}_W \cap S^2$, $\mathcal{A}_W = \mathcal{E}_W \cap \Lambda^2$ and point out that, a priori, $\mathcal{E}_W$ is not the direct sum of $\mathcal{S}_W$ and $\mathcal{A}_W$. The space $\mathcal{S}_W$ has been studied in detail in [**7**]. In dimension 4, as we shall recall later on, additional information is available [**12**].

LEMMA 3.1 *Let h be in* $\mathrm{End}(V)$. *The following hold:*

(i) *if $b_1(W(\cdot, \cdot, h\cdot, \cdot))$ belongs to Λ^4 then h satisfies* (7) *and*

$$W(hx, y, z, u) + W(x, hy, z, u) = W(x, y, hz, u) + W(x, y, z, hu) \tag{8}$$

whenever x, y, z, u belong to V;

(ii) *h satisfies* (7) *if and only if it satisfies* (8).

Proof. (i) Let us set $T = b_1(W(\cdot, \cdot, h\cdot, \cdot))$. Then

$$W(x, y, hz, u) + W(y, z, hx, u) + W(z, x, hy, u) = T(x, y, z, u)$$

for all x, y, z, u in V. We anti-symmetrize in z, u hence

$$\begin{aligned} W(x, y, hz, u) + W(x, y, z, hu) + (W(y, z, hx, u) - W(y, u, hx, z)) \\ + (W(z, x, hy, u) - W(u, x, hy, z)) = 2T(x, y, z, u) \end{aligned}$$

and further

$$W(x, y, hz, u) + W(x, y, z, hu) - W(hx, y, z, u) - W(x, hy, z, u) = 2T(x, y, z, u)$$

after making use of the Bianchi identity. Since T is a four-form, it belongs to $S^2(\Lambda^2)$, but since the left-hand side in the equation above belongs to $\Lambda^2(\Lambda^2)$, it must vanish and the claim follows.
(ii) follows from the Bianchi identity when taking the cyclic sum on x, y, z in (8).

LEMMA 3.2 *The following hold.*

(i) *Suppose that h is in $\mathcal{E}_W$. Then*

$$W(hF - Fh^\star) = W(F)h - h^\star W(F) \tag{9}$$

and

$$W(hF + Fh^\star) = W(F)h + h^\star W(F) \tag{10}$$

whenever F is a skew-symmetric endomorphism of V and where $h^\star$ stands for the adjoint of h with respect to the metric g.

(ii) *If h in* $\mathrm{End}(V)$ *satisfies both of* (9) *and* (10) *then it satisfies* (7).

(iii) *The identity* (10) *is equivalent to* (7).

Proof. (i) Let us fix an orthonormal basis $\{e_i\}$ in V. From (7) we obtain

$$W(e_i, Fe_i, hv, w) + W(Fe_i, v, he_i, w) + W(v, e_i, hFe_i, w) = 0$$

whenever v, w belong to V. After summation, we obtain

$$2\langle W(F)hv, w\rangle + \sum_{i=1}^{n} W(Fe_i, v, he_i, w) + W(v, e_i, hFe_i, w) = 0.$$

Since $\sum_{i=1}^{n} W(Fe_i, v, he_i, w) = -\sum_{i=1}^{n} W(e_i, v, hFe_i, w) = -W(hF)(v, w)$ we find

$$W(F)h = W(hF) \tag{11}$$

whenever F belongs to Λ^2. Now the equations in (9), (10) follow when using that W respects the splitting $\otimes^2 V = S^2 \oplus \Lambda^2$.
(ii) follows when rewriting (11) using elements of the form $F = v \wedge w$, where v, w belong to V.
(iii) Again by rewriting (10) by means of decomposable elements of the form $F = z \wedge u$ in Λ^2 we find

$$\begin{aligned} &W(z, x, hu, y) - W(u, x, hz, y) + W(hz, x, u, y) \\ &\quad - W(hu, x, z, y) = W(z, u, hx, y) + W(z, u, x, hy) \end{aligned}$$

whenever x, y, z, u belong to V. The use of the Bianchi identity upon the first and third, respectively, the second and fourth terms in the left-hand side of the equation above shows that h satisfies the identity in Lemma 3.1 and therefore belongs to $\mathcal{E}_W$.

It remains now to understand up to what extent (9) and (10) are equivalent.

LEMMA 3.3 *Let W be an algebraic Weyl curvature tensor and let h in $\otimes^2 V$ satisfy* (9). *Then h equally satisfies* (7).

Proof. As in the proof of (iii) of the proposition above we rewrite (9) for decomposable Fs in Λ^2, of the form $F = x \wedge y$, where x, y in V. Since

$$\langle W(hF)z, u\rangle = W(x, z, hy, u) - W(y, z, hx, u)$$

and

$$\langle W(Fh^\star)z, u\rangle = W(hx, z, y, u) - W(hy, z, x, u)$$

for all z, u in V, we arrive at

$$\begin{aligned} &-\langle W(x, z)u + W(x, u)z, hy\rangle + \langle W(y, z)u + W(y, u)z, hx\rangle \\ &\quad = W(x, y, hz, u) - W(x, y, z, hu) \end{aligned} \tag{12}$$

for all x, y, z, u in V. Let T in $\Lambda^3 \otimes \Lambda^1$ be defined by $T = b_1(W(\cdot, \cdot, h\cdot, \cdot))$. We rewrite then (12) as

$$\begin{aligned} T(x, y, z, u) &= W(x, y, z, hu) - W(x, u, z, hy) + W(y, u, z, hx) \\ &= -T(x, y, u, z) \end{aligned}$$

for all x, y, z, u in V. It follows that T belongs to Λ^4 and we conclude the argument by means of Lemma 3.1.

REMARK 3.1 It is easy to see that (12), and therefore (7), is yet equivalent with

$$W(hS - Sh^{\star}) = W(S)h - h^{\star}W(S) \tag{13}$$

for all S in S^2.

LEMMA 3.4 *We have $\pi^{-}\mathcal{E}_W \subseteq \mathrm{Ker}(W_{|\Lambda^2})$, where $\pi^{-} : \otimes^2 V \to \Lambda^2$ is the orthogonal projection.*

Proof. This follows by taking the trace of (7) in the last two arguments.

Therefore $\mathrm{Ker}(W_{|\Lambda^2})$ appears as a first obstruction to the equality of $\mathcal{E}_W$ and $\mathcal{S}_W$, for when $W_{|\Lambda^2}$ is injective the latter spaces coincide. We shall show now that the algebraic structure of $\mathcal{E}_W$ is related to the symmetry group

$$G_W = \{\gamma \in \mathrm{GL}(V) : W(\gamma\cdot, \gamma\cdot, \gamma\cdot, \gamma\cdot) = W\}$$

of the Weyl tensor W. The Lie algebra $\mathfrak{g}_W$ of G_W consists in the space of tensors h in $\otimes^2 V$ satisfying

$$W(hx, y, z, u) + W(x, hy, z, u) + W(x, y, hz, u) + W(x, y, z, hu) = 0 \tag{14}$$

for all x, y, z, u in V. Before making explicit the relationship between $\mathcal{E}_W$ and $\mathfrak{g}_W$ we need to reinterpret the identity (14), along the lines of the treatment of (7), and establish the analogous equivalences.

LEMMA 3.5 *The following are equivalent:*

(i) *h belongs to $\mathfrak{g}_W$,*
(ii) *$W(hF + h^{\star}F) = -W(F)h - h^{\star}W(F)$ for all F in Λ^2.*

Proof. This follows by using, with minor changes, the same ingredients as in the proof of Lemma 3.2. Details are left to the reader.

PROPOSITION 3.1 *We have $[\mathcal{E}_W, \mathcal{E}_W] \subseteq \mathfrak{g}_W$.*

Proof. We shall make essential use of equation (11) which, recall, all tensors in $\mathcal{E}_W$ must satisfy. Let h_1 and h_2 belong to $\mathcal{E}_W$. Using (11) we have $W(h_1\pi^{-}(h_2F)) = W(\pi^{-}(h_2F))h_1$ for all F in Λ^2. Since $\pi^{-}(h_2F) = \frac{1}{2}(h_2F + Fh_2^{\star})$ we get

$$W(h_1h_2F) + W(h_1Fh_2^{\star}) = W(h_2F + Fh_2^{\star})h_1 = [W(F)h_2 + h_2^{\star}W(F)]h_1$$

after making use of (11) for h_2. Similarly, $W(h_2h_1F) + W(h_2Fh_1^{\star}) = \big[W(F)h_1 + h_1^{\star}W(F)\big]h_2$ when

$$W([h_1, h_2]F) + W(h_1Fh_2^{\star} - h_2Fh_1^{\star}) = -W(F)[h_1, h_2] + h_2^{\star}W(F)h_1 - h_1^{\star}W(F)h_2$$

for all F in Λ^2. But $h_1Fh_2^{\star} - h_2Fh_1^{\star}$ and $h_2^{\star}W(F)h_1 - h_1^{\star}W(F)h_2$ are symmetric tensors therefore $\pi^{-}W([h_1, h_2]F) = -\pi^{-}W(F)[h_1, h_2]$ and this leads to

$$W\left([h_1, h_2]F + F[h_1, h_2]^{\star}\right) = -W(F)[h_1, h_2] - [h_1, h_2]^{\star}W(F)$$

for all F in Λ^2. Therefore, by using the equivalence of (ii) and (i) in Lemma 3.5 we find that $[h_1, h_2]$ belongs to $\mathfrak{g}_W$.

The obstruction for $\mathfrak{g}_W$ to be contained in $\mathfrak{so}(V)$ is measured as follows.

LEMMA 3.6 *We have* $\pi^+\mathfrak{g}_W \subseteq \mathrm{Ker}(W_{|S^2})$ *where* $\pi^+ : \otimes^2 V \to S^2$ *denotes the orthogonal projection.*

Proof. This follows by taking the trace of the identity (14) in the variables x and z.

The space $\mathcal{E}_W$ has also an algebraic structure, though different from that of $\mathfrak{g}_W$. Let $\{\cdot,\cdot\} : \otimes^2 V \times \otimes^2 V \to \otimes^2$ denote the anti-commutator.

PROPOSITION 3.2 *Let W be an algebraic Weyl curvature tensor. Then* $\{\mathcal{E}_W, \mathcal{E}_W\} \subseteq \mathcal{E}_W$.

Proof. Given h in $\mathcal{E}_W$ it is enough to show that h^2 still belongs to $\mathcal{E}_W$. Now if F belongs to Λ^2 and since h satisfies (9) we have $W(hF - Fh^\star) = W(F)h - h^\star W(F)$ and using this for $\pi^-(hF) = \frac{1}{2}(hF + Fh^\star)$ we obtain

$$\begin{aligned} W(h(hF+Fh^\star) - (hF+Fh^\star)h^\star) =& W(hF+Fh^\star)h - h^\star W(hF+Fh^\star) \\ =&(W(F)h + h^\star W(F))h - h^\star(W(F)h + h^\star W(F)) \\ =&W(F)h^2 - (h^2)^\star W(F) \end{aligned}$$

for all F in Λ^2, where we have used that h satisfies (10). It follows that h^2 satisfies (9) hence the claim follows by making use of Lemma 3.3.

COROLLARY 3.1 *Let h belong to $\mathcal{E}_W$. The following hold:*

(i) $W(hFh^\star) = h^\star W(F)h$ *for all F in Λ^2;*
(ii) $W(hx, hy, z, u) = W(x, y, hz, hu)$ *whenever x, y, z, u belong to V.*

Proof. (i) By Lemma 3.2, we know that h satisfies (10), that is, $W(hF + Fh^\star) = W(F)h + h^\star W(F)$ for all F in Λ^2. Using this for $\pi^-(hF) = \frac{1}{2}(hF + Fh^\star)$ we compute

$$\begin{aligned} W(h(hF+Fh^\star) + (hF+Fh^\star)h^\star) =& W(hF+Fh^\star)h + h^\star W(hF+Fh^\star) \\ =&(W(F)h + h^\star W(F))h + h^\star(W(F)h + h^\star W(F)) \\ =&W(F)h^2 + (h^2)^\star W(F) + 2h^\star W(F)h \end{aligned}$$

for all F in Λ^2. Since h^2 belongs to $\mathcal{E}_W$ by Proposition 3.2 and therefore satisfies (10) the claim follows.

(ii) This follows when rewriting (i) by means of decomposable elements of Λ^2.

3.2. *The four-dimensional case*

Let (V^4, g) be a four-dimensional, oriented, Euclidean vector space together with an algebraic Weyl tensor W. We consider the splitting $\Lambda^2(V) = \Lambda^+ \oplus \Lambda^-$ in its self-dual resp. anti-self-dual components. Accordingly, we have the splitting of the algebraic Weyl tensor as

$$W = W^+ + W^-$$

into its self-dual, resp. anti-self-dual parts. Then $W^{\pm}$ belong to $S^2_0(\Lambda^{\pm})$ and let us denote by $\Sigma^{\pm} = \{\lambda_k^{\pm}, 1 \leq k \leq 3\}$ their spectra. Of course $\lambda_1^{\pm} + \lambda_2^{\pm} + \lambda_3^{\pm} = 0$. Consider now the corresponding (normalized) system of eigenforms $W^{\pm}\omega_k^{\pm} = \lambda_k^{\pm}\omega_k^{\pm}$, $k = 1, 2, 3$. These forms are associated to g-compatible almost complex structures $J_k^{\pm}$, $1 \leq k \leq 3$, that is, $\omega_k^{\pm} = g(J_k^{\pm}\cdot, \cdot)$. The almost complex structures satisfy the quaternion identities, that is, $J_1^{\pm}J_2^{\pm} + J_2^{\pm}J_1^{\pm} = 0$, $J_3^{\pm} = J_1^{\pm}J_2^{\pm}$ and moreover $[J_k^+, J_p^-] = 0$ for all $1 \leq k, p \leq 3$. Note that for h in $S^2_0(V)$, $\{h, J_k^{\pm}\}$ belongs to $\Lambda^{\mp}$ for all $1 \leq k \leq 3$.

It will be important for subsequent computations to note that $|\omega_k^{\pm}| = 2$, $1 \leq k \leq 3$ (here we use the norm on forms). Define now the endomorphisms $\sigma_{i,j} = J_i^+ J_j^-$, $1 \leq i, j \leq 3$; then the $\sigma_{i,j}$ are orthogonal involutions of V, producing an orthogonal basis in $S^2_0(V)$. Note also that $|\sigma_{i,j}| = 2$, $1 \leq i, j \leq 3$, where the inner product on $S^2(V)$ is defined as usual: $\langle S_1, S_2 \rangle = \sum_{i=1}^4 \langle S_1 e_i, S_2 e_i \rangle$, for some orthonormal basis $\{e_i, 1 \leq i \leq 4\}$ in V.

LEMMA 3.7 *We have* $W(\sigma_{i,j}) = (\lambda_i^+ + \lambda_j^-)\sigma_{i,j}$ *for all* $1 \leq i, j \leq 3$.

Proof. Let S be in $S^2_0(V)$. Let $\{e_i, 1 \leq i \leq 4\}$ be an orthonormal basis in V and let v, w be arbitrary vectors in V. We compute by expanding $e^i \wedge v$ in the basis $\omega_k^{\pm}$, $1 \leq k \leq 3$,

$$\begin{aligned} W(e_i, v, Se_i, w) &= \frac{1}{2}\sum_{k=1}^{3} \omega_k^+(e_i, v)\langle W(\omega_k^+)Se_i, w\rangle + \omega_k^-(e_i, v)\langle W(\omega_k^-)Se_i, w\rangle \\ &= \frac{1}{2}\sum_{k=1}^{3} \lambda_k^+\omega_k^+(e_i, v)\omega_k^+(Se_i, w) + \lambda_k^-\omega_k^-(e_i, v)\omega_k^-(Se_i, w). \end{aligned}$$

Summing now over i we obtain that

$$\begin{aligned} W(S) &= \sum_{i=1}^{4} W(e_i, \cdot, Se_i, \cdot) \\ &= -\frac{1}{2}\sum_{k=1}^{3} \lambda_k^+ J_k^+ S J_k^+ + \lambda_k^- J_k^- S J_k^-. \end{aligned}$$

We now take $S = \sigma_{i,j}$, $1 \leq i, j \leq 3$, to arrive after a short computation to the proof of the lemma.

COROLLARY 3.2 *Any algebraic Weyl tensor is, in four dimensions, subject to the algebraic identities*

$$W(\{F, G\}) = \{W(F), G\}_0 + \{F, W(G)\}_0$$

and

$$-W([F, G]) = [W(F), G] + [F, W(G)]$$

whenever F, G *are in* $\Lambda^2(V)$.

Proof. Both claims follow when diagonalizing W on Λ^2 and S^2 as mentioned above.

PROPOSITION 3.3 *The following hold:*

(i) $\mathcal{A}_W = \{\alpha \in \Lambda^2 : W(\alpha) = 0\}$,
(ii) $\mathcal{E}_W = \mathcal{S}_W \oplus \mathcal{A}_W$.

Proof. (i) By Lemma 3.4 we need only see that $\mathrm{Ker}(W_{|\Lambda^2}) \subseteq \mathcal{A}_W$. Indeed, if h in Λ^2 satisfies $W(h) = 0$, from the last equation in Corollary 3.2 we get $-W[F, h] = [W(F), h]$ for all F in Λ^2, in other words h satisfies (10). We conclude now by Lemma 3.2(iii).

(ii) Pick h in $\mathcal{E}_W$ and split it as $h = h_s + h_a$ along $\mathrm{End}(V) = S^2 \oplus \Lambda^2$. Then $W(h_a) = 0$ by Lemma 3.4 hence h_a belongs to $\mathcal{E}_W$ by (i) whence so does h_s and the proof is finished.

REMARK 3.2 When starting from the assumption that h is a Codazzi tensor (that is, ∇h is completely symmetric) on a Riemannian four-manifold a description of the space $\mathcal{S}_W$, together with the constraints implied on the Weyl tensor has been obtained by Derdzinski in [**12**] by geometric means.

We finish this section with the following fact, to be used extensively in the next section.

COROLLARY 3.3 *Let h be in $\mathcal{S}_W$ such that* $\mathrm{Tr}(h)$*=0.*

(i) *If $W^+ = 0$ and* $\det_{\Lambda^-} W^- \neq 0$ *then we must have $h = 0$.*
(ii) *If $W(h) = 0$ and h does not vanish then* $\mathrm{Ker}(W^{\pm})$ *are each one-dimensional, provided that $W^{\pm} \neq 0$, respectively.*

Proof. (i) It is easy to see (see [**18**] for instance) that h in $\otimes^2 V$ belongs to the space $\mathcal{E}_W$ if and only if $W^{\star}(h) = 0$. By making use of Lemma 3.7, applied to the Weyl curvature tensor $W^{\star}$, it follows that $\sum_{1 \le i,j \le 3} \lambda_j^- h_{ij}\sigma_{ij} = 0$, where $h = \sum_{1 \le i,j \le 3} h_{ij}\sigma_{ij}$ and the claim follows eventually.

(ii) In this case, from $W^{\star}(h) = W(h) = 0$ one obtains by means of Lemma 3.7 the system

$$\sum_{1 \le i,j \le 3} (\lambda_i^+ \mp \lambda_j^-) h_{ij}\sigma_{ij} = 0,$$

where as before $h = \sum_{1 \le i,j \le 3} h_{ij}\sigma_{ij}$. Therefore $\sum_{1 \le i,j \le 3} \lambda_i^+ h_{ij}\sigma_{ij} = \sum_{1 \le i,j \le 3} \lambda_j^- h_{ij}\sigma_{ij} = 0$. The claim follows by taking into account that either $\mathrm{Ker}(W^{\pm})$ is one-dimensional, a situation which clearly leads to $h = 0$, or otherwise $W^{\pm} = 0$.

4. A local classification

We consider in what follows a four-dimensional Riemannian manifold (M^4, g) satisfying the conformal C-space condition (6) for some vector field ζ. To avoid trivial statements we shall assume in what follows that W does not vanish identically, in other words g is not a conformally flat metric. We recall that in four dimensions the well-known formula

$$\langle W_X, W_Y \rangle = \frac{|W|^2}{4} \langle X, Y \rangle \tag{15}$$

holds for all X, Y in TM. Here, W_X stands for the tensor $W(X, \cdot, \cdot, \cdot)$. It follows that Weyl nullity vanishes identically in some open set, that is, $W(K, \cdot, \cdot, \cdot) = 0$ for some vector field K implies that

$K = 0$ in the open set of points where W does not vanish. Consider now the splitting of two-forms

$$\Lambda^2(M) = \Lambda^-(M) \oplus \Lambda^+(M)$$

in anti-self-dual resp. self-dual parts. With respect to this splitting, the Weyl tensor decomposes as $W = W^- + W^+$. We start by investigating the case when

$$W(F) = 0 \tag{16}$$

for some two form F on M. We split $F = F^+ + F^-$ into its self-dual resp. anti-self-dual components and we consider the open sets $D_F^{\pm} = \{m \in M : F_m \neq 0\}$ together with $D_F = D_F^+ \cup D_F^-$. Obviously

$$W^+(F^+) = 0, \tag{17}$$

$$W^-(F^-) = 0. \tag{18}$$

To make statements precise it is also necessary to consider $\mathcal{W}^{\pm} = \{m \in M : W_m^{\pm} \neq 0\}$ as well as $\mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^- = \{m \in M : W_m \neq 0\}$. We work on the open set $D_F^+ \cap \mathcal{W}^+$ which we assume to be non-empty (actually we will show that this leads to a contradiction so it will turn out that $W^+ = 0$ on D_F^+). Since $W^+ : \Lambda^+ \to \Lambda^+$ is symmetric and trace-free it follows that there are (locally defined, that is, in some open region around each point in $D_F^+ \cap \mathcal{W}^+$) g-compatible almost complex structures I, J, K satisfying the quaternion identities, that is, $IJ + JI = 0$, $K = IJ$, and such that

$$W^+(\omega_J) = 0, \quad W^+(\omega_I) = \lambda\omega_I \quad \text{and} \quad W^+(\omega_K) = -\lambda\omega_K \tag{19}$$

for some nowhere vanishing function λ, locally defined on $D_F^+ \cap \mathcal{W}^+$. Here $\omega_J = g(J\cdot, \cdot)$, etc. in Λ^+ are the so-called Kähler forms of the almost complex structures above.

Let us recall now that the Nijenhuis tensor of the almost complex (g, J) is defined by

$$N_J(X, Y) = [X, Y] - [JX, JY] + J[X, JY] + J[JX, Y]$$

whenever X, Y belong to $\Gamma(TM)$. When $N_J = 0$ the almost complex J is said to be integrable and actually gives rise to a complex structure. In this case it is customary to call (g, J) a Hermitian structure on M. Now is a good moment to recall the following important result.

THEOREM 4.1 [1] *Let (M^4, g, J) be a Hermitian surface. Then the self-dual Weyl tensor W^+ is given by*

$$W^+ = \frac{k}{4}\left(\frac{1}{2}\omega \otimes \omega - \frac{1}{3}Id_{|\Lambda^+M}\right) + \hat{F} \otimes \omega + \omega \otimes \hat{F},$$

where k is the so-called conformal scalar curvature and $\hat{F} = \frac{1}{2}d^+\theta$. Here θ is the Lee form of (g, J), defined by $d\omega = \theta \wedge \omega$, where $\omega = g(J\cdot, \cdot)$ is the so-called Kähler form.

Note that directly from the definition of the Lee form θ we get after differentiation and using that ω belongs to $\Lambda^+(M)$, that $\langle \hat{F}, \omega \rangle = 0$. Theorem 4.1, giving the structure of the self-dual Weyl tensor of a Hermitian surface, is one of the main ingredients in the proof of the following.

PROPOSITION 4.1 *Let F in Λ^2 satisfy (16). Then $D_F^{\pm} \cap \mathcal{W}^{\pm} = \emptyset$, in other words $W^{\pm}$ vanishes identically on $D_F^{\pm}$.*

Proof. We will only prove that $D_F^{+} \cap \mathcal{W}^{+} = \emptyset$ the proof of the second part of the claim being completely analogous. Therefore, let us assume that $D_F^{+} \cap \mathcal{W}^{+}$ is not empty and work towards getting a contradiction. The main idea is to show that the almost complex structure J defined in (19) is actually complex.

Indeed, it is well known (see [6] for instance) that in dimension 4 integrability (that is the vanishing of the Nijenhuis tensor) is equivalent to

$$(\nabla_{JX} J)JY = (\nabla_X J)Y \tag{20}$$

whenever X, Y belong to TM, where ∇ denotes the Levi-Civita connection associated with the metric g. On the other hand, since $(\nabla_X J)J + J(\nabla_X J) = 0$ for all X in TM we can write

$$\nabla J = a_1 \otimes I + a_2 \otimes K$$

for some one-forms a_1, a_2 on M. Using that ω_J annihilates W^+ we obtain

$$\sum_{k=1}^{4} W(e_k, Je_k, X, Y) = 0$$

whenever X, Y belong to TM and for some local orthonormal frame $\{e_k, 1 \leq k \leq 4\}$. Taking $Y = e_i$ and differentiating in the direction of e_i we get

$$\sum_{1 \leq i,k \leq 4} (\nabla_{e_i} W)(e_k, Je_k, X, e_i) + W(e_k, (\nabla_{e_i} J)e_k, X, e_i) = 0.$$

Since $\delta^{\nabla} W = -C$ we get further

$$-\sum_{k=1}^{4} C(X, e_k, Je_k) + \sum_{k=1}^{4} W(\nabla_{e_i} J)(X, e_i) = 0.$$

By the conformal C-space equation, the first sum equals

$$-W(\omega_J)(\zeta, X) = -W^{+}(\omega_J)(\zeta, X) = 0.$$

We compute

$$\begin{aligned} W(\nabla_{e_i} J)(X, e_i) &= a_1(e_i) W(\omega_I)(X, e_i) + a_2(e_i) W(\omega_K)(X, e_i) \\ &= \lambda a_1(e_i)\langle IX, e_i\rangle - \lambda a_2(e_i)\langle KX, e_i\rangle. \end{aligned}$$

Summing over i it follows that $a_1(I\cdot) = a_2(K\cdot)$ or further $a_2 = a_1(J\cdot)$, a fact which is clearly equivalent with (20).

We shall now relate the result of Proposition 4.1 to properties of the form ζ, part of the defining data of our conformal C-space (M^4, g, ζ). This is done by means of the following observation.

LEMMA 4.1 *We have* $W(d\zeta) = 0$.

Proof. This follows from Proposition 2.1 and Lemma 3.4.

If the open subsets $D^{\pm}$ of M are defined as $D^{\pm} = D^{\pm}_{d\zeta}$ we find immediately from Proposition 4.1 the following.

COROLLARY 4.1 *Let* (M^4, g, ζ) *be a conformal* C*-space. Then* $D^{\pm} \cap \mathcal{W}^{\pm} = \emptyset$*, that is,* $W^{\pm}$ *vanishes identically on* $D^{\pm}$.

What remains to be dealt with is the behaviour of the Weyl tensor W^- on D^+. In other words we shall work on $\mathcal{W}^- \cap D^+$, which, as before, is to be assumed non-empty. Localizing further let us define $U = \{m \in \mathcal{W}^- \cap D^+ : \det(W^-_m) \neq 0\}$.

LEMMA 4.2 *The metric* g *is Einstein–Weyl on* U.

Proof. By Proposition 2.1 the trace-free part of the tensor $h_\zeta = h - \nabla\zeta + \zeta \otimes \zeta$ belongs to the space $\mathcal{E}_W$. Since in four dimensions the splitting $\mathcal{E}_W = \mathcal{S}_W \oplus \mathcal{A}_W$ holds by Proposition 3.3, it follows that the symmetric, trace-free part of h_ζ belongs to $\mathcal{S}_W$. But on U we have that $W^+ = 0$ since W^+ vanishes on D^+ by Corollary 4.1 and also that $d^-\zeta = 0$ since, again by Corollary 4.1, we know that $d^-\zeta$ vanishes on $\mathcal{W}^-$. The claim follows now by applying Corollary 3.3(i) to the symmetric, trace-free part of the tensor h_ζ.

To study the geometry of (U, g), we start from the following lemma, part of which summarizes the information that has already been obtained.

LEMMA 4.3 *On the open subset* U *of* M *the following hold:*

(i) $W^+ = 0$,
(ii) $d^-\zeta = 0$*, that is, the two-form* $F = d\zeta$ *belongs to* Λ^+,
(iii) $\nabla_X F = 2\alpha \wedge X - X \lrcorner (F \wedge \zeta) + 3g(\zeta, X)F$ *for all* X *in* TU*. Here the one-form* α *is given by* $\alpha = h(\zeta, \cdot) + fg(\zeta, \cdot) - df$ *for some smooth function* f *on* U.

Proof. (i), (ii) these follow directly from Corollary 4.1.
(iii) By Lemma 4.2 the conformal C-space (U, g) is also Einstein–Weyl, that is,

$$h = \nabla\zeta - \zeta \otimes \zeta - \frac{1}{2} d\zeta + f \cdot g$$

for some smooth function f. By differentiation, we get

$$(\nabla_X h)Y = \nabla^2_{X,Y}\zeta - (\nabla_X\zeta)Y\zeta - g(\zeta, Y)\nabla_X\zeta - \frac{1}{2} Y \lrcorner \nabla_X F + (Xf)g(Y, \cdot)$$

for all X, Y in TU. We now skew-symmetrize in X, Y and obtain that

$$\begin{aligned}(h \bullet g)(\zeta, Z, X, Y) = \frac{1}{2}(\nabla_Z F)(X, Y) - F(X, Y)g(\zeta, Z) \\ - g(\zeta, Y)g(\nabla_X \zeta, Z) + g(\zeta, X)g(\nabla_Y \zeta, Z) \\ + (X.f)g(Y, Z) - (Y.f)g(X, Z)\end{aligned}$$

whenever X, Y, Z belong to TU, when using that $R = W + h \bullet g$ and taking into account the conformal C-space equation. It suffices now to make use of the Einstein–Weyl equation in the second and fourth terms in the expansion of $(h \bullet g)(\zeta, Z, X, Y)$ to obtain, after computing to some extent, the claimed result.

LEMMA 4.4 *U is the empty set.*

Proof. Arguing by contradiction, let us suppose that U is not empty. We write $F \wedge \zeta = \star\beta$ for some one-form β on U and notice that $X \lrcorner (F \wedge \zeta) = \star(\beta \wedge X)$ for all X in TU, since $\lrcorner$ and $\wedge$ are dual operators with respect to the Riemannian metric g. Therefore (iii) of Lemma 4.3 becomes

$$-\nabla_X F = 2\alpha \wedge X + \star(X \wedge \beta) + g(\zeta, X)F$$

for all X in TU. Since F belongs to Λ^+ by Lemma 4.3(ii), it follows that

$$-\nabla_X F = (X \wedge \gamma)^+ + g(\zeta, X)F \tag{21}$$

whenever X belongs to TU, where the one-form γ is given by $\gamma = \frac{1}{2}(2\alpha + \beta)$. From the definition of U, the two-form F is nowhere vanishing on U. Therefore, we can write $F = \lambda g(J\cdot, \cdot)$ on U, for some smooth, nowhere vanishing, function λ and g-compatible almost complex structure J. Localizing even further if necessary, we also choose a g-compatible almost complex I such that $IJ + JI = 0$ and set $K = IJ$. Then equation (21) becomes

$$\nabla_X(\lambda J) = \frac{1}{4}[\gamma(JX)J + \gamma(IX)I + \gamma(KX)K] + \lambda g(\zeta, X)J$$

for all X in TU. Identifying the J-invariant, resp. J-anti-invariant components in the equation above yields

$$d\lambda = \frac{1}{4}J\gamma + \lambda\zeta \tag{22}$$

and

$$\lambda\nabla_X J = \frac{1}{4}[\gamma(IX)I + \gamma(KX)K]$$

for all X in TU. As usual, let us now write $\omega_J, \omega_I, \omega_K$ for the self-dual two-forms associated with J, I, K. Also, we let an endomorphism G of TU act on a one-form η in $\Lambda^1(U)$ by $G\eta = \eta(G\cdot)$.

The last equation above leads to $4\lambda d\omega_J = (I\gamma) \wedge \omega_I + (K\gamma) \wedge \omega_K$ and further to $d\omega_J = (\lambda^{-1}/2)(J\gamma) \wedge \omega_J$, after making use of the simple algebraic fact that

$$J\gamma \wedge \omega_J = I\gamma \wedge \omega_I = K\gamma \wedge \omega_K.$$

But since $F = \lambda\omega_J$ is a closed two-form we get $d\omega_J = -\lambda^{-1} d\lambda \wedge \omega_J$ hence $J\gamma = -2d\lambda$. Inferring this in (22) gives $\zeta = \frac{3}{2}\lambda^{-1} d\lambda$, therefore $d\zeta = 0$, so that $F = 0$. This contradicts the non-vanishing of F on U and hence the proof is complete.

We are now in position to clarify, locally, up to what extent a four-dimensional conformal C-space is closed.

THEOREM 4.2 *Let (M^4, g, ζ) be a conformal C-space. On the open set $\mathcal{W}$ where the Weyl tensor W does not vanish we have that $d\zeta = 0$. In other words, the metric g is locally conformal to a Cotton metric on $\mathcal{W}$.*

Proof. Lemma 4.4 implies that $\det(W^-) = 0$ on $\mathcal{W}^- \cap D^+$. From the supposition that $\mathcal{W}^- \cap D^+$ is not empty it follows that $\mathrm{Ker}(\mathcal{W}^-)$ is one-dimensional. Proposition 4.1 ensures then the vanishing of W^-, a contradiction. Therefore $\mathcal{W}^- \cap D^+$ is empty and similarly one shows that $\mathcal{W}^+ \cap D^-$ is empty as well, whence the proof of the claim.

5. A Weitzenböck formula and the unique continuation of the Weyl tensor

Let $(M^n, g, \zeta), n \geq 3$, be a conformal C-space, where M is supposed to be connected. We will produce a Weitzenböck type formula for the Weyl tensor which will enable us to prove the unique continuation property for W (note that this is well known for Cotton spaces [6]). Recall that a smooth section of some vector bundle $E \to M$ has the weak unique continuation property if it vanishes over M as soon as it vanishes over some non-empty subset of M. The strong unique continuation property is said to hold if the section vanishes identically as soon as it has, at some point, an infinite-order contact with the zero section. We shall compute first the action of Laplacian on W. Note that, as usual, $\Delta = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$ when W is considered as a two-form with values in $\Lambda^2(M)$.

PROPOSITION 5.1 *For any conformal C-space (M^n, g, ζ), where $n \geq 3$ and M is compact, the following estimate holds pointwise:*

$$\|\nabla^\star\nabla W\|^2 \leq k(\|\nabla W\|^2 + \|W\|^2)$$

for some positive constant k.

Proof. Using the second Bianchi identity (see 2.3) for W we obtain

$$\begin{aligned}(\delta^\nabla dW)(Y, Z, U, T) = {} & (\nabla_T C)(U, Y, Z) - (\nabla_U C)(T, Y, Z) \\ & + (\hat{\delta}^\nabla C)(U, Y)g(Z, T) - (\hat{\delta}^\nabla C)(T, Y)g(Z, U) \\ & - (\hat{\delta}^\nabla C)(U, Z)g(Y, T) + (\hat{\delta}^\nabla C)(T, Z)g(Y, U),\end{aligned}$$

where we have set $\hat{\delta}^{\nabla} C = \sum_{i=1}^{n} (\nabla_{e_i} C)(\cdot, e_i, \cdot)$. On the other side, using the conformal C-space equation one shows (see [18]) that

$$\hat{\delta}^{\nabla} C = -W(\nabla \zeta + (n-3)\zeta \otimes \zeta).$$

From the above formula combined with the fact that $W(d\zeta) = 0$ (see Lemma 4.1) it follows that $\hat{\delta}^{\nabla} C$ is a symmetric tensor, hence we end up with

$$(\delta^{\nabla} d^{\nabla} W)(Y, Z, U, T) = (d^{\nabla} C)(T, U, Y, Z) - W(\nabla \zeta + (n-3)\zeta \otimes \zeta) \bullet g.$$

But $d\delta^{\nabla} W = (n-3) d^{\nabla} C$ and thus

$$\begin{aligned}(\Delta W)(Y, Z, U, T) = (n-3)(d^{\nabla} C)(Y, Z, U, T) - (d^{\nabla} C)(U, T, Y, Z)\\ - W(\nabla \zeta + (n-3)\zeta \otimes \zeta) \bullet g.\end{aligned} \tag{23}$$

Now the classical Weitzenböck formula asserts that $\Delta W = \nabla^{\star} \nabla W + q(W)$ for some endomorphism q (which for our purposes is not necessary to make explicit). Since any linear term in the Weyl tensor can be estimated (M is compact) by $\mathrm{const}\|W\|$ we eventually find by making use of the triangular inequality, in the form $|x+y|^2 \leq 2(|x|^2 + |y|^2)$ that

$$\|\nabla^{\star} \nabla W\|^2 \leq \mathrm{const}(\|W\|^2 + \|d^{\nabla} C\|^2).$$

On the other hand, by making use of the conformal C-space equation we compute

$$\begin{aligned}(d^{\nabla} C)(X, Y, Z, U) &= (\nabla_X C)(Y, Z, U) - (\nabla_Y C)(X, Z, U)\\ &= -(\nabla_X W)(\zeta, Y, Z, U) + (\nabla_Y W)(\zeta, X, Z, U)\\ &\quad - W(\nabla_X \zeta, Y, Z, U) + W(\nabla_Y \zeta, X, Z, U).\end{aligned}$$

Making use of the second Bianchi identity in the second line of the above and keeping in mind that C is linear in W (because of the conformal C-space equation) we find

$$d^{\nabla} C = -\nabla_{\zeta} W + \text{linear terms in } W.$$

Hence

$$\begin{aligned}\|d^{\nabla} C\|^2 &\leq \|\nabla_{\zeta} W\|^2 + \mathrm{const}\|W\|^2 \leq \|\zeta\|^2 \|\nabla W\|^2 + \mathrm{const}\|W\|^2\\ &\leq \mathrm{const}(\|\nabla W\|^2 + \|W\|^2)\end{aligned}$$

and the result follows immediately.

Using classical results of Aronszajn [3, 20] yields then.

THEOREM 5.1 *Let (M^n, g, ζ) be a compact, conformal C-space. The Weyl tensor W has the weak unique continuation property.*

Proof of Theorem 1.1 From Theorem 4.2, we know that W vanishes on the open set $D = \{m \in M : (d\zeta)_m \neq 0\}$. If D is empty then trivially $d\zeta = 0$; if not, by using the unique continuation property established above we obtain that $W = 0$, hence the proof is finished.

6. Bach flat manifolds

We are interested here in conformal C-spaces (M^4, g, ζ) which are in addition assumed to be Bach flat, that is $B = 0$. This has been previously studied, under some genericity assumptions in [21] and our objective in this section is to discuss the general case of this setting.

THEOREM 6.1 *Let (M^4, g, ζ) be a conformal C-space such that $B = 0$. The following hold.*

(i) *On the open set where W does not vanish g is a closed Einstein–Weyl metric.*
(ii) *If M is compact and not locally conformally flat then g is a closed Einstein–Weyl metric over all M.*

Proof. (i) Differentiating the C-space equation and using the definition of the Bach tensor one finds [18, 21] that the tensor h_ζ satisfies the additional condition

$$W(h_\zeta) = 0.$$

We also recall that by Theorem 4.2 we have that $d\zeta = 0$ on $\mathcal{W}$, in other words h_ζ is symmetric on $\mathcal{W}$ or on M, if the latter is compact, and in both cases h_ζ belongs to $\mathcal{S}_W$. Now let h_ζ^0 denote the trace-free part of h_ζ. Working now on $\mathcal{W}^+$ we see from Corollary 3.3(ii) that around each point where h_ζ^0 does not vanish $W^+ : \Lambda^+ \to \Lambda^+$ has one-dimensional kernel. But this is a contradiction in view of Proposition 4.1. Therefore h_ζ^0 vanishes on $\mathcal{W}^+$. By a similar argument, h_ζ^0 vanishes on $\mathcal{W}^-$ and this proves the claim in (i).
(ii) We reinterpret (i) to say that the Weyl tensor W vanishes on the open set where h_ζ^0 does not. If the latter is assumed non-empty, the use of the unique continuation property for W (cf. Theorem 5.1) yields the vanishing of W. Hence if g is not locally conformally flat we must have $h_\zeta^0 = 0$ on M, and the claim follows.

Under the assumptions above it is easy to see that the metric g must be locally conformally Einstein. We note that, as shown by Pedersen and Swann [23], it is sufficient to have Bach flat, with the Einstein–Weyl equations also holding, to conclude that a compact four-manifold is locally conformally Einstein. In the compact four-manifold setting, it is also the case that an Einstein–Weyl metric with Cotton tensor zero is necessarily locally conformally Einstein [19]. As a variation of this theme we note the following result.

THEOREM 6.2 *Let (M^4, g, ζ) be a compact conformal C-space. If g is assumed to be Bach flat then either*

(i) *g is conformal to an Einstein metric, or*
(ii) *g is a locally conformally flat metric, that is, $W = 0$.*

Proof. Using the conformal invariance of the conformal C-space equation (see [18] for instance) and that ζ is a Weyl one form, we shall work in a Gauduchon gauge [14], that is the unique metric in the conformal class of g such that $d^\star\zeta = 0$. Then a well-known result of Gauduchon [16] states that ζ is a Killing field on M whence ζ is parallel, given that it is also closed. That g is an Einstein–Weyl metric is thus strengthened to $h = -\zeta \otimes \zeta + \lambda g$ for some smooth function λ on M. Again from $\nabla\zeta = 0$ we get $C_X = d\lambda \wedge X$ for all X in TM, and from the fact that the Cotton tensor is trace-free it follows that $d\lambda = 0$. In other words $C = 0$ leading to $W(\zeta, \cdot, \cdot, \cdot) = 0$. Now if ζ is not zero this yields $W = 0$ by (15), and otherwise if $\zeta = 0$ then from the formula above h is Einstein.

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