Department of Mathematics

Maths 260 Assignment 3

September 17, 2010 Due: 4pm, Tuesday 28th September, 2010

- Put your completed assignment in the appropriate box on the ground floor of the Maths/Physics building **before** 4pm on the date due.
- Late assignments or assignments placed in the wrong box will not be marked.
- Your assignment must be accompanied by a blue Mathematics Department coversheet.
- You must show all your working.

1. (24 marks) Consider the following linear system of differential equations

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\mathbf{A}$ is given below.

(a) For each of the following matrices $\mathbf{A}$,

- classify the equilibrium point at $\mathbf{Y} = \mathbf{0}$, ie say whether the equilibrium is a sink, source, etc;
- sketch the phase portrait near $\mathbf{Y} = \mathbf{0}$.

Make sure your phase portrait contains a selection of representative solutions as well as the straight line solutions, and show the direction solutions travel in time with arrows. *You may check your solutions using pplane, but you will not receive full marks unless you show all your workings to back up your sketches.*

i. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$

ii. $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

iii. $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix}$

iv. $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

(b) Using the matrix $\mathbf{A}$ from (iv) of part (a).

i. Write the general solution to this linear system of differential equations.

ii. Solve the initial value problem with $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

iii. Add your solution to the IVP from (b) (ii) to the phase portrait you sketched in part (iv) of (a). Be sure to draw your solution for values of $t > 0$ and $t < 0$. Mark where $t = 0$.

2. (9 marks)

(a) By direct computation show that the vector

$$\mathbf{v} = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$$

is an eigenvector of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix},$$

and find the corresponding eigenvalue.

(b) Hence find a complex solution to the equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ is given above and $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$.

(c) By writing your complex solution in real and imaginary parts, find the general solution to the DE in terms of real-valued functions.

(d) Solve the IVP with initial conditions $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

(e) Use the Matlab program *analyzer* to plot the solution to the IVP you found in (d), ie plot the functions $x(t)$ and y(t) vs t on the same graph. Print your plot and include this with your answers.

(f) What is the long term behaviour of the solution to the IVP that you found in (d)?

3. (11 marks) Consider the following system of differential equations:

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= 3y + z \\ \dot{z} &= -2y + z. \end{aligned}$$

(a) Write the equations in matrix form.

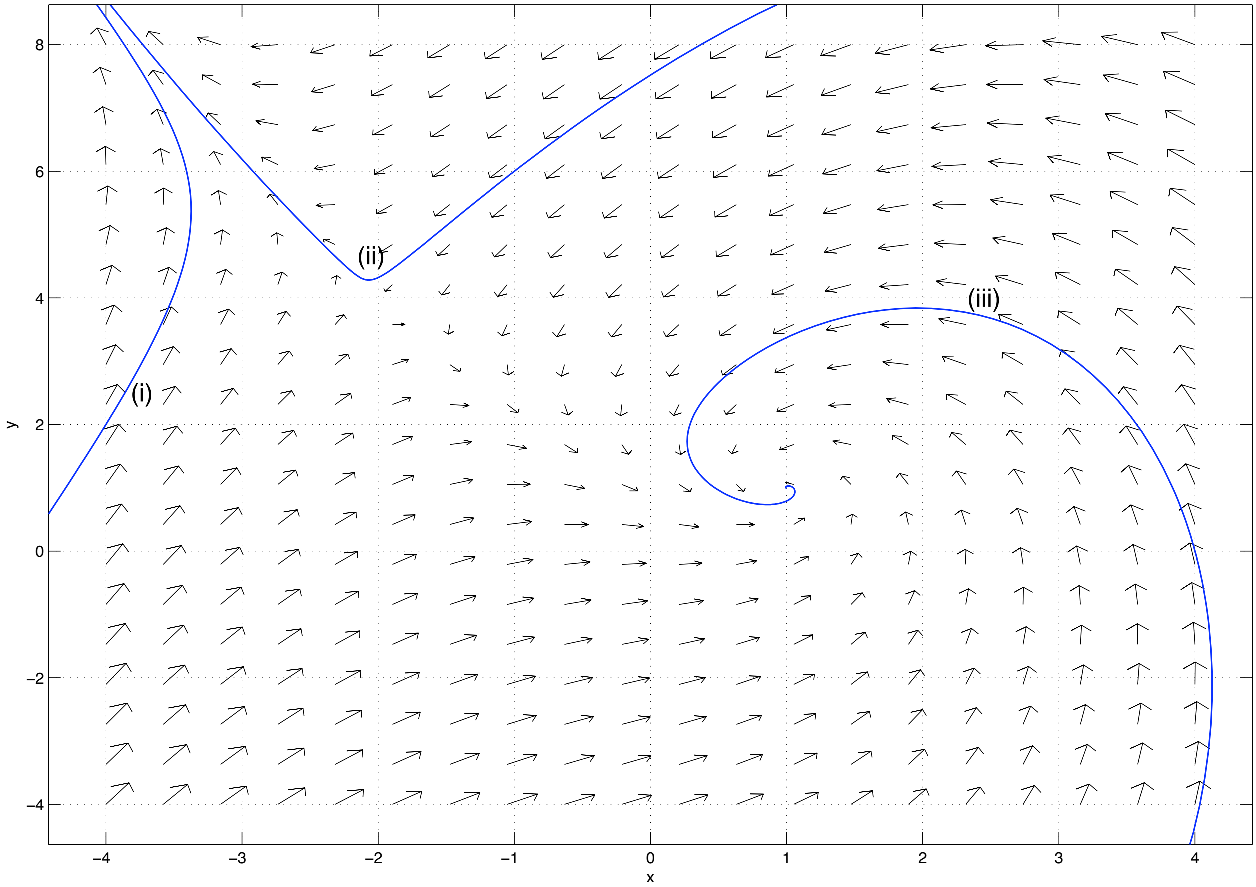
(b) Using the matrix form you found in (a), find the general solution in terms of real-valued functions.

(c) What type of equilibrium point is at the origin?

4. (8 marks) The following picture shows a vector field for a two-dimensional autonomous systems of ODEs along with some solution curves for the system.

(a) For each solution shown ((i), (ii), (iii)) sketch a graph of $x(t)$ vs t and $y(t)$ vs t. In your sketch show both $x(t)$ and $y(t)$ on the same axes.

(b) The two-dimesensional autonomous system shown in the picture has two equilibrium points. Use the picture to locate the two equilibrium points and classify their type.



5. (13 Marks) Consider the IVP

$$\frac{dx}{dt} = \frac{y}{2} - xt$$
$$\frac{dy}{dt} = xy - y$$

with initial condition $x(0) = 1$, $y(0) = 1$.

(a) Use Euler's method with $h = 1$ to estimate the solution at final time $t = 3$.

(b) Plot (by hand) the graphs of $x(t)$ vs t and $y(t)$ vs t for your approximate solution.

(c) A different fixed step size numerical method is used to find the solution at final time $t = 3$ using a variety of different step sizes. The following results are obtained

Number of steps	Approximation of $x(3)$	Approximation of $y(3)$
3	0.05445862	0.48107356
6	0.16968206	0.40469542
12	0.13010396	0.39290032
24	0.12434939	0.38837586
48	0.12318772	0.38707859
96	0.12291991	0.38673459
192	0.12285522	0.38664614
384	0.12283930	0.38662372

i. Use these results to estimate $x(3)$ and $y(3)$ accurate to 4 decimal places.

ii. Use your answers to (i) to estimate the errors in the approximations to $x(3)$ and $y(3)$ using 12 and 24 steps.

iii. Use your answers to (ii) to get two estimates for the effective order of the method that was used, one using $x(3)$ and one using $y(3)$.

iv. What numerical method do you think was used? Give a reason for your answer.

6. (5 marks) **Challenge question** This question is harder, and is approximately 7% of the total assignment mark. Only attempt this question when you are happy with your answers to the other questions.

This question is about the IVP

$$\frac{dx}{dt} = 2$$
$$\frac{dy}{dt} = x - \frac{y}{t}$$

with initial condition $x(1) = 1$, $y(1) = 1$.

Euler's method is used to compute the solution to the IVP at $t = 3$ for different step sizes. The following results are obtained.

Number of steps	Approximation of $x(3)$	Approximation of $y(3)$
2	5.00000000000000	3.50000000000000
4	5.00000000000000	4.20000000000000
8	5.00000000000000	4.50000000000000
16	5.00000000000000	4.64130434782609
32	5.00000000000000	4.71010638297872
64	5.00000000000000	4.74407894736842
128	5.00000000000000	4.76096204188482
256	5.00000000000000	4.76937826370757
512	5.00000000000000	4.77358010104302
1024	5.00000000000000	4.77567945846905
2048	5.00000000000000	4.77672874776131

(a) Solve the initial value problem explicitly. Give the exact values of $x(3)$ and $y(3)$.

(b) Can you explain why the Euler's method approximation to $x(3)$ doesn't change as the number of steps is increased? What properties must a differential equation have for Euler's method to be exact?