The Journal of Geometric Analysis
Volume 13, Number 4, 2003

Rigidity of Riemannian Foliations with Complex Leaves on Kähler Manifolds

By *Paul-Andi Nagy*

ABSTRACT. We study Riemannian foliations with complex leaves on Kähler manifolds. The tensor T, the obstruction to the foliation be totally geodesic, is interpreted as a holomorphic section of a certain vector bundle. This enables us to give classification results when the manifold is compact.

1. Introduction

Riemannian foliations with totally geodesic leaves and in particular Riemannian submersions with totally geodesic fibers are now quite well understood. Many general structure results in the theory of Riemannian submersions are known (see [4], Chapter 9). For particular symmetric spaces—as spheres or complex and quaternionic projective spaces—classification results are available [7, 6] under some geometric hypothesis on the fibers. For real hyperbolic spaces, or more generally locally symmetric spaces with negative sectional curvature complete classification results are available in the compact case [12, 19]. In the less explored case of pseudo-Riemannian submersions similar results are known to hold under some additional conditions [14, 2, 1]. In the case of Riemannian foliations transversal geometric assumptions were used in order to obtain classification theorems [16].

In a complex setting, a notion of almost Hermitian submersions was proposed in [20] but it turns out that for many classes the horizontal distribution has to be integrable [20, 8, 10]. One might suspect that a less rigid situation, even in the case of a submersion, could arise from the study of Riemannian submersions from an almost-Hermitian manifold. The geometric condition we need here is that the fibers (or the leaves) be almost complex. This is of interest when searching geometric structures admitting a (Riemannian) twistor construction as explained in [3].

In this article we study Riemannian foliations with complex leaves on Kähler manifolds. The totally geodesic case was completely described in [15]—as a byproduct of the classification of nearly Kähler manifolds—where it is shown that under the simple connectivity and completeness assumptions such an object is a Riemannian product of twistor spaces over positive quaternionic Kähler manifolds, Kähler manifolds and homogeneous spaces belonging to three main classes (see [18] for basic quaternionic-Kähler geometry). Note that for the case of the complex projective space this was already known in [7].

Math Subject Classifications. 53C12, 53C24, 53C55.
Key Words and Phrases. Riemannian foliation, Kähler manifold, rigidity.

ISSN 1050-6926

It is then natural to investigate the non-totally geodesic case. It turns out the ambient Kähler geometry is sufficiently strong to force, at least in the compact case, the foliation to be of very special type. More precisely, our main result is the following rigidity theorem.

Theorem 1.1. *Let (M, g, J) be a compact Kähler manifold. If M carries a Riemannian foliation $\mathcal{F}$ with complex leaves then M is locally isometric and biholomorphic with a Riemannian product $M_1 \times M_2$ of Kähler manifolds where M_1 carries a totally geodesic, Riemannian foliation with complex leaves and M_2 carries a Riemannian foliation with complex leaves which is transversally integrable. Moreover, the foliation $\mathcal{F}$ is the Riemannian product of the latter.*

As it is well known, the decomposition theorem of deRham ensures that at least locally one can restrict attention to holonomy irreducible Riemannian manifolds. For the case of the latter, Theorem 1.1 gives:

Corollary 1.2. *On a compact, simply connected, irreducible Kähler manifold any Riemannian foliation with complex leaves is either totally geodesic, or transversally integrable.*

Note that for these rigidity results no assumption on the curvature of the metric g is necessary. In a standard fashion, conditions ensuring total geodesicity of a given foliation are based on bounds on, say, Ricci curvature (see [11] for examples of results of this type). Note also that when studying holomorphic distributions on Kähler manifolds conditions on the metric are necessary even in the case of (real) codimension 2 [13].

The article is organized as follows. In Section 2 we collect some classical facts about Riemannian foliations and then specialize to the case of Kähler manifolds. We are basically starting from O'Neill's equations for the curvature tensor and use the Kähler structure to derive differential relations between the basic tensors A and T. In Section 3 we interpret the tensor T, the obstruction to the foliation to be totally geodesic as a holomorphic section of a certain vector bundle and use the compacity assumption in order to obtain the splitting in Theorem 1.1.

2. Preliminaries

We start by collecting a number of basic facts about Riemannian foliations and next we will specialize to the Kähler case. Let (M, g) be a Riemannian manifold and let $\mathcal{F}$ be a foliation on M. We denote by $\mathcal{V}$ the integrable distribution induced by $\mathcal{F}$. Let H be the orthogonal complement of $\mathcal{V}$. We assume the foliation $\mathcal{F}$ to be Riemannian, that is

$$\mathcal{L}_V g(X, Y) = 0$$

whenever X, Y are in H and V belongs to $\mathcal{V}$. Let ∇ be the Levi–Civita connection of the metric g. Throughout this article we will denote by V, W vector fields in $\mathcal{V}$ and by X, Y, Z etc. vector fields in H. It is easy to verify that the formula [17]

$$\overline{\nabla}_E F = (\nabla_E F_{\mathcal{V}})_{\mathcal{V}} + (\nabla_E F_H)_H$$

defines a metric connection with torsion on M (here the subscript denotes orthogonal projection on the subspace). The main property of this connection is that it preserves the distributions $\mathcal{V}$ and H.

Of fundamental importance for the theory of Riemannian foliations are the O'Neill's tensors T and A which we are going to define now, following [17], p. 49. We have:

$$T_E F = \left(\nabla_{E_{\mathcal{V}}} F_{\mathcal{V}}\right)_H + \left(\nabla_{E_{\mathcal{V}}} F_H\right)_{\mathcal{V}}$$

whenever E, F belong to TM. Then T vanishes on $H \times H$ and $H \times \mathcal{V}$, it is symmetric on $\mathcal{V} \times \mathcal{V}$ (as a consequence of the integrability of $\mathcal{V}$) and moreover, we have that $< T_V X, W >= - < X, T_V W >$.

The second O'Neill tensor A is defined by

$$A_E F = \left(\nabla_{E_H} F_H\right)_{\mathcal{V}} + \left(\nabla_{E_H} F_{\mathcal{V}}\right)_H$$

for all E and F in TM. Concerning its properties, it vanishes on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V} \times H$, it is skew-symmetric on H (because the foliation $\mathcal{F}$ is Riemannian) and furthermore it satisfies $< A_X V, Y >= - < V, A_X Y >$.

Note that the foliation $\mathcal{F}$ is called *transversally integrable* iff the distribution H is integrable. In this situation the tensor A has to be symmetric and therefore it vanishes, making H a totally geodesic distribution.

With this definitions in hand one can easily express the difference $\nabla - \overline{\nabla}$ in terms of the O'Neill's tensors:

$$\begin{aligned} \nabla_X Y &= \overline{\nabla}_X Y + A_X Y, \ \nabla_X V = \overline{\nabla}_X V + A_X V \\ \nabla_V X &= \overline{\nabla}_V X + T_V X, \ \nabla_V W = \overline{\nabla}_V W + T_V W . \end{aligned}$$

In the rest of this article we will assume that (M, g) is a Kähler manifold of dimension $2m$, with complex structure J. Moreover, we suppose that the foliation $\mathcal{F}$ has complex leaves, that is $J\mathcal{V} = \mathcal{V}$ (then of course, $JH = H$). As $\nabla J = 0$, it follows that $\overline{\nabla} J = 0$, hence we obtain information about the complex type of the tensors A and T as follows

$$\begin{aligned} A_X(JY) = J(A_X Y), \quad A_{JX} V = -J(A_X V) = -A_X(JV) \\ T_{JV} W = J(T_V W), \quad T_{JV} X = -J(T_V X) = -T_V(JX) . \end{aligned} \tag{2.1}$$

We also have $A_{JX} JY = -A_X Y$ and $T_{JV} JW = -T_V W$. A consequence of the last identity is that the foliation $\mathcal{F}$ is harmonic, that is the mean curvature vector field vanishes.

We will use now the Kähler structure on M, together with suitable curvature identities to get some geometric information about the tensors A and T.

Lemma 2.1. *Let X, Y, Z be in H and V, W in $\mathcal{V}$. Then we have:*

(i) $(\overline{\nabla}_X A)(Y, Z) = 0$.

(ii) $< A_X Y, T_V Z >= 0$.

(iii) $< (\overline{\nabla}_V A)(X, Y), W >=< (\overline{\nabla}_W A)(X, Y), V >$.

(iv) $(\overline{\nabla}_{JX} T)(V, W) = -J(\overline{\nabla}_X T)(V, W)$.

Proof. We will prove (i) and (ii) simultaneously. Let us denote by R the curvature tensor of the Levi–Civita connection of the metric g. We first recall the O'Neill formula (see [17])

$$\begin{aligned} R(X, Y, Z, V) =& < \left(\overline{\nabla}_Z A\right)(X, Y), V > + < A_X Y, T_V Z > \\ & - < A_Y Z, T_V X > - < A_Z X, T_V Y > . \end{aligned} \tag{2.2}$$

Note that the formula in [17] contains ∇ rather then $\overline{\nabla}$. However, after examining the difference $\nabla - \overline{\nabla}$ (which we have already given) it easily turns out that the two expressions are in fact the same.

Since (M, g) is Kähler one has $R(JX, JY, Z, V) = R(X, Y, Z, V)$. Hence by (2.1) we easily arrive at

$$< \left(\overline{\nabla}_Z A\right)(X, Y), V > + < A_X Y, T_V Z >= 0 .$$

Again by (2.1) we see that the first term of the previous equation is J-invariant in Y and V, whilst the second is J-anti-invariant in the same variables. This proves (i) and (ii).

To prove (iii) we use another O'Neill's formula stating that

$$\begin{aligned} R(V, W, X, Y) =& < \left(\overline{\nabla}_V A\right)(X, Y), W > - < \left(\overline{\nabla}_W A\right)(X, Y), V > \\ &+ < A_X V, A_Y W > - < A_X W, A_Y V > \\ &- < T_V X, T_W Y > + < T_V Y, T_W X > . \end{aligned} \tag{2.3}$$

The result follows now by (2.1) and the fact that $R(V, W, JX, JY) = R(V, W, X, Y)$. The identity in (iv) can be proven in the same way, using this time the identity

$$\begin{aligned} R(X, V, Y, W) =& < \left(\overline{\nabla}_X T\right)(V, W), Y > + < \left(\overline{\nabla}_V A\right)(X, Y), W > \\ &+ < A_X V, A_Y W > - < T_V X, T_W Y > \end{aligned}$$

the fact that $R(JX, JV, Y, W) = R(X, V, Y, W)$ and (iii). □

Remark 2.2. (i) By the first two assertions of Lemma 2.1 we obtain that $R(X, Y, Z, V) = 0$, a condition frequently imposed when studying Riemannian foliations (see Chapter 5 of [17] and references therein).

(ii) By (i) and (ii) of the previous lemma it is easy to see that H satisfies the Yang–Mills condition.

(iii) Using (iii) of Proposition 2.1 and [17], p. 52, we get the following relation between the covariant derivatives of A and T

$$2 < \left(\overline{\nabla}_V A\right)(X, Y), W >=< \left(\overline{\nabla}_Y T\right)(V, W), X > - < \left(\overline{\nabla}_X T\right)(V, W), Y > . \tag{2.4}$$

We will make use of this equation in the next section.

Let us denote by $\overline{R}$ the curvature tensor of the connection $\overline{\nabla}$. Another result that will be needed in the next section is the following.

Lemma 2.3. *We have:*

$$\overline{R}(X, Y)V = 2[A_X, A_Y]V + Q(X, Y)V$$

for all X, Y in H and V in $\mathcal{V}$ where we defined $Q(X, Y)V = T_{T_V Y}X - T_{T_V X}Y$.

Proof. Follows from the general formulas in [17], p. 100, and Lemma 2.1, (iii). □

3. The harmonicity of the tensor T

In this section we begin the study of the tensor T. Our main idea is to consider T as an $S^2(\mathcal{V})$-valued 1-form on M and then use Lemma 2.1, (iv) to study differential equations involving T. The analogy we have constantly in mind is the well known fact that on a compact Kähler manifold any holomorphic 1-form is closed. We first develop some preliminary material. We refer the reader to the discussion in Section 4 of [5]. Although our geometric context is different, the guiding principle concerning the Kähler identities and relations between various natural differential operators is the same.

For each $p \geq 0$ we define $S^2(\mathcal{V}) \otimes \Lambda^p(H)$ to be the space of symmetric endomorphisms $\alpha : \mathcal{V} \times \mathcal{V} \to \Lambda^p(H)$. We also define $S^2_A(\mathcal{V})$ as the subspace of $S^2(\mathcal{V})$ consisting of tensors which vanish on $\mathcal{V} \times \mathcal{A}$ where $\mathcal{A} = \{A_X Y : X, Y \text{ in } H\}$.

The ordinary exterior derivative d does not preserve $\Lambda^\star(H)$ but d_H, the horizontal component of its restriction to $\Lambda^\star(H)$ does. The latter can be extended to $S^2(\mathcal{V}) \otimes \Lambda^\star(H)$ by setting

$$(d_H\alpha)(V, W)(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^i \left(\overline{\nabla}_{X_i}\alpha\right)(V, W)\left(X_0, \dots \hat{X}_i, \dots X_p\right)$$

for every α in $S^2(\mathcal{V}) \otimes \Lambda^p(H)$. Using Lemma 2.1, (i) it is easy to see that d_H preserves $S^2_A(\mathcal{V}) \otimes \Lambda^\star(H)$. The fact that the almost complex structure J is integrable induces a splitting

$$d_H = \partial_H + \overline{\partial}_H$$

on $S^2(\mathcal{V}) \otimes \Lambda^\star(H)$ where $\partial_H : S^2(\mathcal{V}) \otimes \Lambda^{p,q}(H) \to S^2(\mathcal{V}) \otimes \Lambda^{p,q+1}(H)$ and $\overline{\partial}_H : S^2(\mathcal{V}) \otimes \Lambda^{p,q}(H) \to S^2(\mathcal{V}) \otimes \Lambda^{p+1,q}(H)$ where

$$\partial_H = \tfrac{1}{2}\left(d_H + (-1)^r iJd_HJ\right)$$
$$\overline{\partial}_H = \tfrac{1}{2}\left(d_H + (-1)^{r+1} iJd_HJ\right)$$

on $S^2(\mathcal{V}) \otimes \Lambda^r(H)$ and J acts on an element α of $S^2(\mathcal{V}) \otimes \Lambda^p(H)$ by

$$(J\alpha)(V, W)(X_1, \dots, X_p) = \alpha(V, W)(JX_1, \dots, JX_p)\,.$$

We need now a formula relating to the anticommutator of the operators ∂_H and $\overline{\partial}_H$. Let Q be the tensor defined at the end of Section 2. If s belongs to $S^2_A(\mathcal{V})$ we define the action of Q on s to be $Q.s$, an element of $S^2_A(\mathcal{V}) \otimes \Lambda^2(H)$ defined by $(Q.s)(V, W)(X, Y) = s(Q(X, Y)V, W) + s(V, Q(X, Y)W)$. Obviously, this can be extended to give a linear map

$$\mathcal{P} : S^2_A(\mathcal{V}) \otimes \Lambda^p(H) \to S^2_A(\mathcal{V}) \otimes \Lambda^{p+2}(H), \mathcal{P}\alpha = Q.\alpha$$

having the property that $\mathcal{P}(s\alpha)(V, W) = (Q.s)(V, W) \wedge \alpha$ whenever s is in $S^2_A(\mathcal{V})$ and α belongs to $\Lambda^p(H)$.

Lemma 3.1. *The following holds on $S^2_A(\mathcal{V}) \otimes \Lambda^{p,q}(H)$:*

$$\partial_H\overline{\partial}_H + \overline{\partial}_H\partial_H = \mathcal{P}\,.$$

Proof. Let us first compute $d_H^2 q$ where q belongs to $S^2_A(\mathcal{V})$. An easy manipulation yields

$$\left(d_H^2 q\right)(V, W)(X, Y) = \left(\overline{\nabla}^2_{X,Y} q\right)(V, W) - \left(\overline{\nabla}^2_{Y,X} q\right)(V, W)\,.$$

Using the Ricci identity for the connection with torsion $\overline{\nabla}$ (see [4], p. 26) we get

$$\left(\overline{\nabla}^2_{X,Y} q\right)(V, W) - \left(\overline{\nabla}^2_{Y,X} q\right)(V, W) = q\left(\overline{R}(X, Y)V, W\right) + q\left(V, \overline{R}(X, Y)W\right) + 2\left(\overline{\nabla}_{A_XY} q\right)(V, W).$$

Using now Lemma 2.3 and the fact that q vanishes on vectors of the form $A_X Y$ with X, Y in H we obtain that

$$\left(d_H^2 q\right)(V, W)(X, Y) = 2\left(\overline{\nabla}_{A_XY} q\right)(V, W) + q(Q(X, Y)V, W) + q(V, Q(X, Y)W)\,. \quad (3.1)$$

We consider now α in $S^2_A(\mathcal{V}) \otimes \Lambda^{p,q}(H)$ and let $\{e^I\}$ be a local basis of closed basic (p, q)-forms in $\Lambda^{p,q}(H)$. We write $\alpha = \sum\limits_I q_I e^I$ with q_I in $S^2_A(\mathcal{V})$ and use the multiplicative property of d_H to obtain $d^2_H \alpha = \sum\limits_I d^2_H q_I \wedge e^I$. But $\partial_H \overline{\partial}_H \alpha + \overline{\partial}_H \partial_H \alpha = (d^2_H \alpha)^{p+1,q+1} = \sum\limits_I (d^2_H q_I)^{1,1} \wedge e^I$. But $A_{JX}JY = -A_X Y$ and $Q(JX, JY) = Q(X, Y)$ hence $(d^2_H q_I)^{1,1} = \mathcal{P} q_I$ and the proof is finished. □

At this stage let us recall another particular feature of Kähler geometry, namely the Kähler identities. We state them on $\Lambda^p(H)$ as follows:

$$[\partial_H, L^\star] = -i\overline{\partial}^\star_H, [\overline{\partial}_H, L^\star] = i\partial^\star_H$$
$$[\partial^\star_H, L] = -i\overline{\partial}_H, \left[\overline{\partial}^\star_H, L\right] = i\partial_H$$

where L is multiplication with ω^H in $\Lambda^2(H)$ defined by $\omega^H(X, JY) =< X, JY >$. Of course these are projection of the Kähler identities of M and, furthermore, it is easy to see that they hold on $S^2_A(\mathcal{V}) \otimes \Lambda^\star(H)$ too.

Let us now define α_T in $S^2(\mathcal{V}) \otimes \Lambda^1(H)$ by setting $\alpha_T(V, W)X =< T_V W, X >$. In virtue of Lemma 2.1, (ii) we have that α_T belongs to $S^2_A(\mathcal{V}) \otimes \Lambda^1(H)$. Moreover, we define ζ in $S^2_A(\mathcal{V}) \otimes \Lambda^{0,1}(H)$ by $\zeta = \alpha_T + iJ\alpha_T$. Then

Lemma 3.2. (i) $\overline{\nabla}_{JX}\zeta = -i\overline{\nabla}_X \zeta$ *for all* X *in* H.

(ii) $\overline{\partial}_H \zeta = 0$.

(iii) $\partial^\star_H \zeta = \overline{\partial}^\star_H \zeta = 0$.

(iv) $\partial^\star_H \partial_H \zeta = -i\mathcal{P}^\star L\zeta$.

Proof. (i) is a straightforward consequence of Lemma 2.1, (iv), while (ii) comes immediately by (i) and the fact that $2\overline{\partial}_H = d_H + iJd_H J$ on $S^2_A(\mathcal{V}) \otimes \Lambda^{0,1}(H)$. (iii) We notice first that for any α in $S^2(\mathcal{V}) \otimes \Lambda^1(H)$ we have

$$-\left(d^\star_H \alpha\right)(V, W) = \sum_i \left(\overline{\nabla}_{e_i}\alpha\right)(V, W)e_i$$

whenever V, W are $\mathcal{V}$ and $\{e_i\}$ is an arbitrary local orthonormal basis in H. Therefore, when V and W in $\mathcal{V}$ are fixed, $-(d^\star_H \alpha_T)(V, W)$ equals the trace over H of $< (\overline{\nabla}.T)(V, W), \cdot >$. Since by Lemma 2.1, (iv) this last tensor is J-anti-invariant over H, it has no trace and we deduce that $d^\star_H \alpha_T = 0$. In the same way one proves that $d^\star_H J\alpha_T = 0$, thus $d^\star_H \zeta = 0$ and the proof of (iii) is clearly finished. (iv) We use (in the classical way) the Kähler identities and the previous lemma. We have

$$\begin{aligned}\partial^\star_H \partial_H \zeta &= -i\partial^\star_H \left[\overline{\partial}^\star_H, L\right]\zeta = -i\left(\partial^\star_H \overline{\partial}^\star_H\right)L\zeta = -i\left\{\partial^\star_H, \overline{\partial}^\star_H\right\}L\zeta + i\overline{\partial}^\star_H\left[\partial^\star_H, L\right]\zeta \\ &= -i\left\{\partial^\star_H, \overline{\partial}^\star_H\right\}L\zeta + \overline{\partial}^\star_H \overline{\partial}_H \zeta = -i\left\{\partial^\star_H, \overline{\partial}^\star_H\right\}L\zeta\end{aligned}$$

where $\{\cdot, \cdot\}$ denotes the anti-commutator. It suffices now to dualize the equation in Lemma 3.1. □

Before proceeding to the proof of the Theorem 1.1 we need one more preliminary result.

Lemma 3.3. *We have* $\mathcal{P}\zeta = 0$.

Proof. Obviously it suffices to show that $\mathcal{P}\alpha_T = 0$. But it is straightforward to see that

$$\begin{aligned}(\mathcal{P}\alpha)(V,W)(X,Y,Z) = {} & \alpha(Q(X,Y)V,W)(Z) + \alpha(V,Q(X,Y)W)(Z) \\ & - \alpha(Q(X,Z)V,W)(Y) - \alpha(V,Q(X,Z)W)(Y) \\ & + \alpha(Q(Y,Z)V,W)(X) + \alpha(V,Q(Y,Z)W)(X)\end{aligned}$$

whenever α belongs to $S^2_A(\mathcal{V}) \otimes \Lambda^1(H)$. We have:

$$\begin{aligned}< T_V Q(X,Y)W, Z > &= - < Q(X,Y)W, T_V Z >= - < T_{T_W Y}X - T_{T_W X}Y, T_V Z > \\ &=< X, T_{T_V Z}(T_W Y) > - < Y, T_{T_W X} T_V Z > \ .\end{aligned}$$

But $< X, T_{T_V Z}(T_W Y) >= - < T_{T_V Z}X, T_W Y >=< T_W T_{T_V Z}X, Y >$ hence

$$< T_V Q(X,Y)W, Z >=< T_W T_{T_V Z}X, Y > - < T_{T_W X} T_V Z, Y > \ .$$

Taking the alternate sum on X, Y, Z of this formula gives now easily the result. □

Let us assume, in the rest of this section, that the manifold M is compact and then prove Theorem 1.1. At first, taking the scalar product with ζ in Lemma 3.2, (iv) we get

$$< \partial_H^\star \partial_H \zeta, \zeta >= -i < \mathcal{P}^\star L\zeta, \zeta >= -i < L\zeta, \mathcal{P}\zeta >= 0$$

where for the last step we used Lemma 3.3. Now integrating over M we obtain that $\partial_H \zeta = 0$ and since $\overline{\partial}_H \zeta$ vanishes [cf. Lemma 3.2, (ii)] it follows that $d_H \zeta = 0$ and further $d_H \alpha_T = 0$. Using now (2.4) we obtain that $(\overline{\nabla}_V A)(X,Y) = 0$ and we conclude by invoking Lemma 2.1, (i) that

$$\left(\overline{\nabla}_E A\right)(X,Y) = 0 \tag{3.2}$$

for all E in TM. To extract the remaining information encoded in the fact that $d_H \alpha_T = 0$ we will compute, in the lemma below, the square of d_H.

Lemma 3.4. *The following holds on* $S^2_A(\mathcal{V}) \otimes \Lambda^1(H)$

$$d_H^2 = 2\mathcal{L} + \mathcal{P} \tag{3.3}$$

where

$$\begin{aligned}(\mathcal{L}\alpha)(V,W)(X,Y,Z) = {} & \left(\overline{\nabla}_{A_Y Z}\alpha\right)(V,W)(X) - \left(\overline{\nabla}_{A_X Z}\alpha\right)(V,W)(Y) + \left(\overline{\nabla}_{A_X Y}\alpha\right)(V,W)(Z) \\ & + \alpha(V,W)(A_X A_Y Z - A_Y A_X Z + A_Z A_X Y)\,.\end{aligned}$$

Proof. Let $\{e^i\}$ be a local basis of closed basic 1-forms on H. If α belongs to $S^2_A(\mathcal{V}) \otimes \Lambda^1(H)$ we write $\alpha = \sum_i q_i \cdot e^i$ where the q_i's belong to $S^2_A(\mathcal{V})$. Then $d_H^2 \alpha = \sum_i d_H^2 q_i \wedge e^i$ and furthermore, by (3.1), we have that $(d_H^2 q_i - \mathcal{P})(V,W)(X,Y) = 2(\overline{\nabla}_{A_X Y} q_i)(V,W)$. Hence, a short computation gives

$$\begin{aligned}\tfrac{1}{2}\left(d_H^2 \alpha - \mathcal{P}\right)(V,W)(X,Y,Z) = \sum_i \Big[& \left(\overline{\nabla}_{A_Y Z} q_i\right)(V,W) \cdot e^i(X) - \left(\overline{\nabla}_{A_X Z} q_i\right)(V,W) \cdot e^i(Y) \\ & + \left(\overline{\nabla}_{A_X Y} q_i\right)(V,W) \cdot e^i(Z)\Big]\,.\end{aligned}$$

We have now to convince that the right hand side of the previous equation equals the claimed expression for $\mathcal{L}\alpha$ in the statement. To this aim, recall that each of the (horizontal) forms e^i is basic, that is (see [17]) $de^i(V, \cdot) = 0$ whenever V belongs to $\mathcal{V}$. This leads easily to $(\overline{\nabla}_V e^i)X = -e^i(A_X V)$ for all X in H and V in $\mathcal{V}$. It follows by a routine computation that

$$\overline{\nabla}_{A_Y Z}\left(q_i \cdot e^i\right)(V, W)(X) + \left(q_i \cdot e^i\right)(V, W)(A_X A_Y Z) = \left(\overline{\nabla}_{A_Y Z} q_i\right)(V, W) \cdot e^i(X).$$

The proof is finished by taking the symmetric sum in X, Y, Z in the last equation and then doing summation over i. □

Remark 3.5. Formulas of type (3.3) can be proven for forms of any degree and the operator $\mathcal{L}$ can be given a more concise form. Since only the case of 1-forms is needed for our purposes this presentation makes more clear subsequent computations.

Lemma 3.6. $A_X(T_V W) = 0$.

Proof. Let us recall first the following O'Neill formula:

$$R(V_1, V_2, V_3, Z) =< \left(\overline{\nabla}_{V_2} T\right)(V_1, V_3), Z > - < \left(\overline{\nabla}_{V_1} T\right)(V_2, V_3), Z > .$$

Now, by Lemma 2.1, (ii) and (3.2) we get $< (\overline{\nabla}_{V_1} T)(A_X Y, V_3), Z >= 0$ and it follows that $R(V_1, A_X Y, V_3, Z) =< (\overline{\nabla}_{A_X Y} T)(V_1, V_3), Z >$. Since (M, g, J) is Kähler $R(JV_1, A_X(JY), V_3, Z) = R(V_1, A_X Y, V_3, Z)$ which yields further to

$$\left(\overline{\nabla}_{A_{JX} Y} T\right)(V_1, V_3) = -J\left(\overline{\nabla}_{A_X Y} T\right)(V_1, V_3).$$

Using this and relations (2.1) for the tensor A we obtain after some computations that

$$(\mathcal{L}\alpha_T)(V, W)(JX, JY, Z) + (\mathcal{L}\alpha_T)(V, W)(X, Y, Z) = 2 < T_V W, A_X A_Y Z - A_Y A_X Z > .$$

Or the vanishing of $d_H \alpha_T = 0$ and $\mathcal{P}\alpha_T$ implies (cf. Lemma 3.4) that of $\mathcal{L}\alpha_T$ hence

$$< T_V W, A_X A_Y Z - A_Y A_X Z >= 0$$

for all X, Y, Z in H and V, W in $\mathcal{V}$. Taking in this last equation $Y = JX$ we arrive at $< A_X(T_V W), A_X(JZ) >= 0$ and the conclusion is straightforward. □

For each m in M we define $\mathcal{V}_m^0$ to be the vectorial subspace of $\mathcal{V}_m$ spanned by $\{A_X Y : X, Y \text{in } H_m\}$ and let H_m^0 be the linear span of $\{A_X V : X \text{ in } H_m, V \text{ in } \mathcal{V}_m\}$. By (3.2) and using parallel transport with respect to the connection $\overline{\nabla}$ we see that we obtained smooth distributions $\mathcal{V}^0$ and H^0 of TM which are furthermore $\overline{\nabla}$-parallel. We denote by $\mathcal{V}^1$ resp. H^1 the orthogonal complement of $\mathcal{V}^0$ resp. H^0 in $\mathcal{V}$ resp. H. We moreover, define distributions $D^i = \mathcal{V}^i \oplus H^i, i = 0, 1$ of TM. They are both $\overline{\nabla}$-parallel because D^0 already has this property and D^0 is orthogonal to D^1 (of course $TM = D^0 \oplus D^1$, an orthogonal direct sum). Using Lemma 2.1, (ii) and Lemma 3.6 we find after a straightforward verification involving only the definitions of D^0, D^1 that the tensors T resp. A are vanishing on D^0 resp. D^1 so as these distributions are in fact ∇-parallel. The proof of the Theorem 1.1 is finished by means of the decomposition theorem of DeRham.

References

[1] Baditoiu, G. Semi-Riemannian submersions with totally geodesic fibers, eprint **arXiv:math**. DG/0005254.

[2] Baditoiu, G. and Ianus, S. Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces, *Diff. Geom. Appl.*, **16**, 79–94, (2002).

[3] Bérard Bergery, L. and Ochiai, T. On some generalizations of the construction of twistor spaces, *Proc. Symp.*, Durham/Engl. 1982, 52–59, (1984).

[4] Besse, A. *Einstein Manifolds*, Springer-Verlag, New York, (1987).

[5] Donaldson, S.K. Yang–Mills invariants of four manifolds in geometry of low-dimensional manifolds I, *London Math. Soc. Lect. Notes*, **150**, 5–40, (1992).

[6] Escobales, R. Riemannian submersions with totally geodesic fibers, *J. Diff. Geom.*, **10**, 253–276, (1975).

[7] Escobales, R. Riemannian submersions from complex projective spaces, *J. Diff. Geom.*, **13**, 253–276, (1978).

[8] Falcitelli, M. and Pastore, A.M. A note on almost Kähler and nearly Kähler submersions, *J. Geom.*, **69**, 79–87, (2000).

[9] Gray, A. and Hervella, L.M. The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.*, **123**, 35–58, (1980).

[10] Johnson, D.L. Kähler submersions and holomorphic connections, *J. Diff. Geom.*, **15**, 71–79, (1980).

[11] Kamber, F. and Tondeur, Ph. Curvature properties of harmonic foliations, *Illinois J. Math.*, **18**, 458–471, (1984).

[12] Kim, H. and Walschap, G. Riemannian foliations on hyperbolic manifolds, *Indiana J. Math.*, **41**, 37–41, (1992).

[13] Lebrun, C. Fano manifolds, contact structures and quaternionic geometry, *Intl. J. Math.*, **6**, 419–437, (1995).

[14] Magid, M. Riemannian submersions from anti-de Sitter space with totally geodesic fibers, *J. Diff. Geom.*, **16**, 323–331, (1981).

[15] Nagy, P-A. Nearly Kähler geometry and Riemannian foliations, *Asian J. Math.*, **6**(3), 481–504, (2002).

[16] Tondeur, Ph. and Vanhecke, L. Transversally symmetric Riemannian foliations, *Tôhoku Math. J.*. **42**, 307–317, (1990).

[17] Tondeur, Ph. *Geometry of Foliations*, Birkhäuser, Verlag, (1997).

[18] Salamon, S. Quaternionic Kähler manifolds, *Invent. Math.*, **67**, 143–171, (1982).

[19] Walschap, G. Foliations of symmetric spaces, *Am. J. Math.*, **115**(5), 1189–1195, (1993).

[20] Watson, B. Almost hermitian submersions, *J. Diff. Geom.*, **11**, 147–165, (1976).

Received April 14, 2003
Revision received July 15, 2003

Humboldt-Universität zu Berlin, Institut für Mathematik
Rudower Chaussee 25, D-10099 Berlin
e-mail: nagy@mathematik.hu-berlin.de

Communicated by Peter Li