Invariant differential operators in conformal geometry

Josef Šilhan
Department of Mathematics
The University of Auckland

A thesis submitted for the degree of Doctor of Philosophy
at The University of Auckland, March 2006

This thesis is for examination purposes only and may not be consulted or referred to by any persons other than the examiner.

Invariant differential operators in conformal geometry

Josef Šilhan
Department of Mathematics
The University of Auckland

A thesis submitted for the degree of Doctor of Philosophy
at The University of Auckland, 2006

Abstract

We develop a universal and algorithmic construction of invariant differential operators between irreducible bundles in conformal geometry. The classification of such operators in the flat case is well–known in terms of representation theory. The main result of the thesis is a construction of curved analogues of these. We obtain curved analogues in every case save for an exception which exists in every pattern in every even dimension. The operators are described via explicit formulae in tractor calculus. These are closely related to the usual "∇–formulae" for invariant operators in Riemannian geometry. The construction follows Eastwood's curved translation principle which we implement in the conformal tractor calculus. We work in both real and complex setting and for all signatures.

Further, we use the developed calculus to study one class of these operators – the conformal Killing operator on forms – in detail. We construct invariant prolongations of the corresponding systems of partial differential equations. Using these, we obtain information about the solution space. In particular, we develop a helicity raising and lowering construction in the general setting, and also on conformally Einstein manifolds.

Acknowledgements

First of all I would like to thank my supervisor A/Prof. Rod Gover who initiated and arranged my PhD study at the University of Auckland. He helped me significantly with my study and research.

This study was supported from many resources, in particular by the Postgraduate scholarship of the New Zealand Institute for Mathematics and its Applications and also the Marsden Fund of Rod Gover. I wish to acknowledge further support from the organizers of the workshop "Conformal structure in Geometry, Analysis and Physics", Palo Alto, California (August 2003) and also from the Graduate Research Fund of the University of Auckland contributed to my research. Beside these, I would like to thank to the Department of Mathematics at the University of Auckland for their support during my study.

My study in Auckland would not have been possible without Prof. Jan Slovák from the Masaryk University in Brno (Czech Republic) who arrange my first contacts with Rod Gover. Also, I would like to thank Jan for encouraging me to study differential geometry.

Contents

List of Tables

Chapter 0

Introduction

Riemannian geometry is a basic structure studied in differential geometry. Tangent spaces are equipped with an inner product therefore we have a notion of length. Less is known about conformal structures. There are many ways to approach these. A conformal structure on a manifold M is a class $[g]$ of conformally equivalent (pseudo)metrics on M. The equivalence is given by conformal rescaling i.e. multiplication of the metric by a positive smooth function on M. An alternative description of such structures exists. Conformal structures may be viewed as Cartan geometries of the parabolic type and it will be occasionally useful to have this point of view.

Invariant differential operators are those differential operators which are well–defined on a given class of structures without needing any additional information. They have been studied for more than one hundred of years. For example the famous Maxwell and Dirac operators from theoretical physics are conformally invariant. Invariant operators are well-understood in the Riemannian case, all of them can be expressed via polynomial formulae in the Levi-Civita connection ∇ (corresponding to the metric g), its curvature R and various algebraic operations. The conformal case is much more involved. Conformal structure is less rigid than the Riemannian one hence we can expect fewer invariant operators. Beginning with operators invariant for a Riemannian structure g from the conformal class, conformally invariant are simply those among them which do not depend on the choice $g \in [g]$. However, this characterization is of limited value for higher order operators. A universal and algorithmic construction of all conformally invariant operators, with a single exception in every even dimension, expressed via their formulae, is the main result of this thesis. (The exception corresponds to the operator L_0 in the pattern on page 63.) Moreover, one of these operators, the conformal Killing operator on forms, is treated in detail. We will obtain information about solutions of the corresponding system of partial differential equations (PDE's).

Let us emphasize we shall construct *formulae* for conformal operators in terms of basic and compact form provided by tractor calculus (see below). Actually, there are many related results in the field. In particular, in the literature one finds a complete classification in the conformally flat case and a wealth of existence results in the curved setting. We give a construction which recovers almost all known existence results while also giving, for the first time, an explicit computable and universal algorithm for the construction of formulae. Explicit formulae are important in the study of corresponding systems of PDE's. Solutions of these PDE's have often a straightforward geometrical interpretation. For example, flows corresponding to conformal Killing vectors (conformal Killing forms of tensor rank 1) are automorphisms of conformal manifolds.

In 1980, Eastwood and collaborators devised a curved adaptation [23] of the Jantzen–Zuckerman translation principle [56]. Curved translation is a technique how to build complicated operators from simple ones. Here our construction uses broadly similar ideas. We implement the "curved translation principle" using the conformal tractor calculus. For example, so called standard operators can be "translated" from the exterior derivative d. We will obtain them in the form $S_2^* d S_1$ where S_1 and S_2 are differential (splitting) operators which act between tensor bundles and tractor bundles. Tractor bundles are nondecomposable (but not irreducible) natural conformal vector bundles. It is well–known that there is no invariant connection on the tangent bundle TM. The main point in the tractor calculus is the tractor connection ∇ (actually d is twisted with ∇ in $S_2^* d S_1$) which is conformally invariant and gives rise to an invariant conformal calculus, exploited in this thesis. Tractor bundles are tensor products of the standard tractor bundle E_A. The connection ∇ on E_A is a simple tool well-understood in terms of a Levi–Civita connection from the conformal class. Hence one can rewrite the tractor formula $S_2^* d S_1$ as an explicit formula in terms of the underlying conformal structure on M. Such formula may be very long for operators of higher orders and can be computed by computers.

Let us review briefly the content of the thesis. Chapter 1 presents the necessary algebraic and geometric background. We will need the representation theory of semisimple (reductive) Lie algebras and we use both Weyl's construction (generalized to spinors and densities) and the symbolism of Dynkin diagrams [2]. Details are in 1.1.3, see especially Tables 1.2 and 1.3 for a summary and relations between the different approaches. The conformal (and spin conformal) structure and the tractor calculus, both in a form suitable for the thesis, are summarized in Section 1.2. Finally, we review some well-known facts about conformally invariant operators (namely the complete classification in the flat case) in Section 1.3. Also, we sketch our version of the curved translation in more details here.

Splitting operators in the cases discussed above (the standard ones) are known to exist [20]. These, so called gBGG splitting operators (constructed for the whole parabolic class) are formulated in terms of semi–holonomic jet prolongations. Here we give an alternative construction of splitting operators. We obtain the operator denoted by *DSplit*, which is defined by an explicit tractor formula and does not require any sort of Casimir computation used in [20]. (In particular, it does not require inverting operators.) Actually, *DSplit* differs from the splitting constructed in [20] in curved cases. Our construction of standard operators works for any such curved modification.

In the literature there is a less complete treatment of the so–called non-standard operators. In particular, almost nothing has been published about appropriate splittings for these operators. *DSplit* is well-defined in both standard and nonstandard cases and the construction of *DSplit* in Section 2.1 is the core of the thesis. This is algorithmic in the following sense. First we define the *bottom*, *middle* and *top* splittings B, M and T, respectively, for differential forms where the notation indicates position of a form section "put" into a form tractor. Then we decompose an irreducible bundle into the Cartan product of forms corresponding to columns of the Young diagram and define B, M and T for each column. This yields an inductive procedure. However, the construction of T in the general case is rather complicated and we need a significant development of new notation (see 1.2.6) for this. Similar ideas yield splitting operators for spinors. Finally, *DSplit* is defined as an appropriate composition of B, M and T. These operators are computed explicitly in many examples throughout Section 2.1. For example, the form bundle and the tensor bundle corresponding to Young diagram with two columns are treated in details.

It is easy to obtain formulae for formal adjoints of *DSplit*. (Note we have suitable inner products on tensor and tractor bundles.) We need formal adjoints for the final step of the construction of invariant operators on irreducible bundles in Section 3.1. As mentioned above, the standard operators are of the form $DSplit_1^* \circ d^{\nabla} \circ DSplit_2$. Similarly, we obtain the nonstandard ones as $DSplit_1^* \circ \Box \circ DSplit_1$ or $DSplit_1^* \circ \not{D} \circ DSplit_1$ where $\Box$ denotes the conformal Laplacian and $\not{D}$ the Dirac operator. (Exceptions in even dimensions do not admit construction of this type.) This result is demonstrated on the bundle corresponding to the Young diagram with two columns: all strongly invariant differential operators are described in the "BMT"–calculus in Example 3.1.6.

There are many possible variations on $DSplit$. Some of them are exploited in the study of the conformal Killing equation on differential forms in Section 3.2. This corresponds to the null space of the conformal Killing operator on forms $\sigma \mapsto \mathrm{Proj}^{\boxtimes} \nabla \sigma$ where σ is a section of $\bigwedge^k T^*M[k+1]$, $1 \leq k \leq n-1$. Here $\mathrm{Proj}^{\boxtimes}$ denotes projection to the Cartan component of the $O(g)$–decomposition of the tensor product $T^*M \otimes \bigwedge^k T^*M[k+1]$ for a given metric in the conformal class. (One easily verifies that this does not depend on the choice of the metric.) Solutions are called *conformal Killing forms*. Conformal Killing vectors are the special case for $k = 1$. In general, the issue of their global existence in the Riemannian setting has been pursued recently by Semmelmann and others, see [46, 47] and references therein. Our treatment here concerns primarily the local issues.

The conformal Killing equation is an overdetermined system of linear homogeneous PDE's. This system is equivalent to a finite dimensional prolonged system, i.e. "closed" in the following sense. All first partial derivatives of the dependent variables are determined by algebraic formulae in terms of these same variables. From the linearity it follows that the prolongation gives a 1-1 correspondence between sections $\bigwedge^k T^*M[k+1]$ satisfying $\mathrm{Proj}^{\boxtimes} \nabla \sigma = 0$ and sections of a vector bundle V parallel with respect to a connection Γ. In the case of conformal Killing equation, the prolongation is constructed explicitly in [46] (see also [9] where a wider class of differential equations is treated). However, neither of these results addresses the conformal invariance of the conformal Killing equation. Using the operator T discussed above, we will construct a conformally invariant prolongation in Section 3.2, see particularly Theorem 3.2.11. That is, the bundle V will be a tractor bundle and Γ will the normal tractor connection modified (invariantly) by curvature terms. This captures succinctly what conformal invariance means for components of the prolongation. Moreover, the curvature of Γ provides, at least in principle, obstructions to existence of conformal Killing forms.

Another result of Section 3.2 is an explicit realization of conformal helicity raising and lowering along the lines of [43] (see also [13]). This is an idea that two (or more) solutions of conformal equations can be combined to a solution of other equations. Among others, we shall describe explicit formulae and curvature obstructions for this technique applied to conformal Killing forms and (almost) Einstein metrics. (The latter are considered as densities of the weight 1 satisfying the corresponding conformal Killing equation.) This should have important consequences for manifolds where the conformal Killing forms are known to exist [46].

Chapter 1

Preliminaries

We shall introduce various notation and terminology in this chapter. The brief summary below can help the reader during subsequent chapters with symbols he or she is not familiar with.

Basic notation. Roughly speaking, objects studied in this dissertation are a smooth manifold M with a conformal structure, smooth sections of conformal bundles on M and invariant differential operators on M. We will also consider analogous objects in the complex case. We shall work mostly in both global and local setting but if certain tools are not available globally (e.g. the spin structure), we will assume the local setting implicitly. The dimension of M will be denoted by n and we will work under the assumption $n \geq 3$. We will often distinguish between the even dimensional case $n = 2n'$ and the odd dimensional case $n = 2n' + 1$ where $n' = \lfloor \frac{n}{2} \rfloor$. But to the extent possible we formulate the results in a uniform way for both cases.

The calculus developed later requires considerable notation. The details of this are introduced throughout this chapter, especially (but not exclusively) in Sections 1.1.3, 1.2.5 and 1.2.6. A main point is a generalization of the usual abstract index notation to form–indices. That is, we replace a sequence of k indices $a^1, \dots, a^k$ which are skewed over (i.e. $[a^1 \dots a^k]$ in the usual notation) by one form–index $\mathbf{a}$.

Beside the form–indices, we will need many other types of abstract indices. They will be distinguished by various fonts. A section f with one index can be of the form f_a for a (usual) tensor index, f_A for a tractor index, f_λ for a spin index, f_Λ for a tractor spinor index, $f_{\mathbf{a}}$ for a form index and $f_{\mathbf{A}}$ for a form tractor index. We will use the Euler Fraktur font to indicate more complicated systems of indices. If not stated otherwise, $f_{\mathfrak{a}}$ shall denoted "f with any set of tensor or spinor indices" and $f_{\mathfrak{A}}$ shall denote "f with any set of tractor or tractor spinor indices". The number of indices in systems $\mathfrak{a}$ and $\mathfrak{A}$, where every form is considered as a system of several tensor or tractor indices, will be denoted $|\mathfrak{a}|$ and $|\mathfrak{A}|$, respectively.

In general, we denote bundles by V, U, W ..., the spaces of their sections by $\mathcal{V}$, $\mathcal{U}$, $\mathcal{W}$ ..., and the corresponding representation spaces by $\mathbb{V}$, $\mathbb{U}$, $\mathbb{W}$ ..., respectively. Natural bundles are bundles which can be given by systems of indices. We denote by $\bigotimes$ the tensor product of these objects and by $\bigodot$ and $\bigwedge$ the symmetric and skew symmetric tensor product, respectively. The Cartan product will be denoted by $\boxtimes$.

The flooring function (integer part) and the ceiling function will be denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively, for $x \in \mathbb{R}$. Given a sequence of numbers $c_p, \dots, c_q$ where integers p, q satisfy $0 \le p \le q$, we will use the notation

$$c^m = \sum_{i=p}^{m} c_i, \quad \tilde{c}^m = \sum_{i=m}^{q} c_i \tag{1.1}$$

where $p \le m \le q$ and we put $c^k = \tilde{c}^l = 0$ for $k < p$ and $l > q$. Beside $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_\pm$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we shall also use $\frac{1}{2}\mathbb{Z}$, $\frac{1}{2}\mathbb{N}_0$ etc. For example,

$$\frac{1}{2}\mathbb{Z} := \left\{ \frac{a}{2} \mid a \in \mathbb{Z} \right\}.$$

1.1 Algebraic background and notations for representations

The main purpose of this section is to establish notations for bundles relevant for invariant differential operators studied in this dissertation. These linear bundles can be (as associated bundles) described via corresponding representations of appropriate Lie groups or algebras.

We need especially irreducible representations of the (complex) conformal algebra $\mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C}$. We shall approach them mostly using Weyl's construction for orthogonal groups from [26]. We also introduce form indices in detail and how to use them for complicated bundles. Another notation for representation, developed in [2], is provided by Dynkin diagrams and will be referred as notation or symbolism of Dynkin diagrams. This is briefly recalled in 1.1.1.

1.1.1. Weyl group, weights and parabolic subalgebras. Let us consider a complex semisimple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and the set of simple roots $\Delta \subseteq \mathfrak{h}^*$. Choosing positive roots $\Delta_+ \subseteq \Delta$, we obtain the set of simple roots $\Pi \subseteq \Delta_+$ which forms a basis of $\mathfrak{h}^*$. Weyl group W is generated by *simple reflections*, i.e. the reflections corresponding to the simple roots. The number of positive roots $\alpha \in \Delta_+$ which are transformed to $w(\alpha) \in \Delta_- = -\Delta_+$ is called the *length* of w for which we write $|w|$. Equivalently (see [26]), the length of w is the minimal number of simple reflections in any expression for w in terms of simple reflections.

The weights of $\mathfrak{g}$ can be described by labelling the nodes of the Dynkin diagram by the integer coefficients referring to the linear combination of fundamental weights [2]. The weight is dominant for $\mathfrak{g}$ if and only if all the coefficients are nonnegative. Such a labelled Dynkin diagram describes an irreducible representation of $\mathfrak{g}$.

The *affine action* of the Weyl group is defined by

$$w.\Lambda = w(\Lambda + \mathrm{R}) - \mathrm{R}$$

for the weight Λ where $\mathrm{R} = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ is the lowest strictly dominant weight of $\mathfrak{g}$. It means (in the terms of the Dynkin diagram) to add one over each node, then act with w and finally subtract one over each node.

The standard parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is defined by a set of simple roots $\Sigma \subseteq \Pi$ and it is generated by the Cartan subalgebra, root spaces corresponding to the positive roots and root spaces corresponding to the negative roots which can be expressed as a negative linear combination of roots from $\Pi \setminus \Sigma$. The corresponding Dynkin diagram for $\mathfrak{p}$ is obtained from the Dynkin diagram for $\mathfrak{g}$ by crossing out nodes corresponding to the simple roots from Σ. Using Satake diagrams, a similar notation can be established for the real case. Each parabolic subalgebra is conjugate to some standard parabolic subalgebra so we will deal only with standard parabolics. The set Σ induces the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$. The reductive part $\mathfrak{g}_0$ includes the semisimple part of $\mathfrak{p}$ and the rest of the Cartan subalgebra; $\mathfrak{g}_+$ is the nilradical of $\mathfrak{p}$.

It follows from the standard parabolic theory that irreducible representations of $\mathfrak{p}$ are irreducible representations of $\mathfrak{g}_0$ with the trivial action of $\mathfrak{g}_+$. Weights of finite dimensional representations of $\mathfrak{g}_0$ can be described by a labelled Dynkin diagram, where coefficients over non–crossed nodes are integers. Such a weight is $\mathfrak{p}$–*dominant* if the coefficients over non–crossed nodes are nonnegative integers and $\mathfrak{g}$–*dominant* if all the coefficients are nonnegative integers. A weight with positive (but possibly non–integer coefficients) over the non–crossed nodes will be called $\mathfrak{p}$–*dominant non–integral* weight.

For each set $\Sigma \subseteq \Pi$, and the corresponding parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, we define $W^{\mathfrak{p}} \subseteq W$ as a subset of all elements, which map the weights dominant for $\mathfrak{g}$ into the weights dominant for $\mathfrak{p}$. Equivalently, $W^{\mathfrak{p}}$ is the set of all elements w for which the set $\Phi_w = w(\Delta_-) \cap \Delta_+$ contains only roots corresponding to $\mathfrak{g}_+$ i.e. the positive roots of $\mathfrak{g}$ which are not roots of the semisimple part of $\mathfrak{g}_0$ (see [40]) and also $W^{\mathfrak{p}} = \{w \in W; |S_\alpha w| = |w| + 1 \text{ for all } \alpha \in \Sigma\ \}$. We connect $w, w' \in W^{\mathfrak{p}}$ by an arrow, $w \longrightarrow w'$, if $w' = S_\alpha(w)$ for a root $\alpha \in \Delta$ and $|w'| = |w| + 1$. We say $w \leq w'$ if $w = w'$ or there is a directed path from w to w'. This defines structure of the Hasse diagram on $W^{\mathfrak{p}}$.

Actually, we are more interested in the real case. A real parabolic subalgebra $\mathfrak{p}' \subseteq \mathfrak{g}'$ of a real semisimple algebra is defined via the complexification i.e. by the property that $\mathfrak{p}'(\mathbb{C}) \subseteq \mathfrak{g}'(\mathbb{C})$ is parabolic.

We shall use properties of parabolic subalgebras only in the conformal case. The algebras relevant for the complex conformal geometry can be described via block matrices

$$\begin{aligned} \mathfrak{g}_0 &= \left\{ \left. \begin{pmatrix} a & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & -a \end{pmatrix} \right| M \in \mathfrak{so}_n(\mathbb{C}) \right\} = \mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C} \subseteq \\ \subseteq \mathfrak{p} &= \left\{ \left. \begin{pmatrix} a & 0 & 0 \\ X & M & 0 \\ 0 & -X^t & -a \end{pmatrix} \right| M \in \mathfrak{so}_n(\mathbb{C}), X \in \mathbb{C}^n \right\} \subseteq \\ \subseteq \mathfrak{g} &= \mathfrak{so}_{n+2}(\mathbb{C}), \end{aligned} \tag{1.2}$$

cf. the flat conformal geometry in Section 1.1.2. The Dynkin diagrams for the algebras $\mathfrak{p} \subseteq \mathfrak{g}$ (with numbered nodes) are displayed in Table 1.1. Later

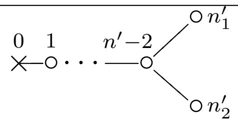
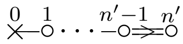
Description of $\mathfrak{p} \subseteq \mathfrak{g} = \mathfrak{so}_{n+2}(\mathbb{C})$ and $W^{\mathfrak{p}} \subseteq W$		
n	Dynkin diagram with numbered nodes	Corresponding simple reflection in W
Even	$\mathfrak{p} =$ 	$S_0, \cdots, S_{n'-2}, S_{n'_1}, S_{n'_2}$
Odd	$\mathfrak{p} =$ 	$S_0, \cdots, S_{n'}$
	$W^{\mathfrak{p}}$ with the Hasse graph structure	
Even	$W^{\mathfrak{p}} = w_0 \to \cdots \to w_{n'-1} \begin{array}{c} \nearrow w_{n'_1} \searrow \\ \searrow w_{n'_2} \nearrow \end{array} w_{n'+1} \to \cdots \to w_n$	
Odd	$W^{\mathfrak{p}} = w_0 \to \cdots \to w_{n'} \to w_{n'+1} \to \cdots \to w_n$	
	$w \in W$ expressed via simple reflections, $i \in \begin{cases} \{1, \dots, n'-1\} & n \text{ even} \\ \{1, \dots, n'\} & n \text{ odd.} \end{cases}$	the length $\|w\|$
Even	$w_0 = \mathrm{id}$	0
	$w_i = S_0 \cdots S_{i-1}$	i
	$w_{n'_1} = S_0 \cdots S_{n'-2} S_{n'_1}$	n'
	$w_{n'_2} = S_0 \cdots S_{n'-2} S_{n'_2}$	n'
	$w_{n'+1} = S_0 \cdots S_{n'-2} S_{n'_1} S_{n'_2}$	$n'+1$
	$w_{n'+i+1} = S_0 \cdots S_{n'-2} S_{n'_1} S_{n'_2} S_{n'-2} \cdots S_{n'-i-1}$	$n'+i+1$
Odd	$w_0 = \mathrm{id}$	0
	$w_i = S_0 \cdots S_{i-1}$	i
	$w_{n'+i} = S_0 \cdots S_{n'-1} S_{n'} S_{n'-1} \cdots S_{n'-i+1}$	$n'+i$
	$w_n = S_0 \cdots S_{n'-1} S_{n'} S_{n'-1} \cdots S_0$	n

Table 1.1: Description of $\mathfrak{p}$ and $W^{\mathfrak{p}}$ in the conformal case.

we will need the *grading element* defined as

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathfrak{g}_0. \tag{1.3}$$

This has the property that any finite dimensional representation $\mathbb{W}$ of $\mathfrak{g}$ decomposes into $\mathbb{W} = \bigoplus_{i=-k}^{k} \mathbb{W}_i$ where E acts by multiplication by i on $\mathbb{W}_i$. For example, $k = 1$ for the standard and the adjoint representations of $\mathfrak{so}_n(\mathbb{C})$.

The tools mentioned above can be adapted to study of real parabolics but their description is more technical. The main point is to study real cases via their complexifications. In conformal geometry, the semisimple part is the orthogonal algebra $\mathfrak{so}_{p,q}$ with the complexification $\mathfrak{so}_n(\mathbb{C})$ where $n = p + q$. The algebras $\mathfrak{g}_0'$, $\mathfrak{p}'$ and $\mathfrak{g}'$ are real forms of the complex algebras $\mathfrak{g}_0$, $\mathfrak{p}$ and $\mathfrak{g}$, respectively, of the form

$$\mathfrak{so}_{p,q} \oplus \mathbb{R} = \mathfrak{g}_0' \subseteq \mathfrak{p}' \subseteq \mathfrak{g}' = \mathfrak{so}_{p+1,q+1}.$$

The elements of $W^{\mathfrak{p}}$ can be expressed as compositions of simple reflections, see Table 1.1. Here the notation for simple reflections follows the numbering of nodes of the Dynkin diagram. The Hasse graph structure on $W^{\mathfrak{p}}$ yields the corresponding structure on the set of weights $\{w.\Lambda | w \in W^{\mathfrak{p}}\}$ for a weight Λ of $\mathfrak{g}$ which will be referred as the weight of the pattern. Such a pattern is called *regular* if Λ is a $\mathfrak{g}$–dominant weight and *singular* if Λ is not $\mathfrak{g}$–dominant but $\Lambda + R$ is.

Let us note that the algebra $\mathfrak{p}$ is usually represented by upper triangle block matrices (and not lower as shown above). Our version corresponds better to the usual matrix description of the conformal tractor calculus (see 1.2.3) and its development in [16].

1.1.2. Flat model of conformal geometry. Following [27], we briefly recall the construction of the flat model of conformal geometry of signature (p, q) because it is closely related to algebraic structures discussed above. Details can be found e.g. in [49]. Let $\mathbb{T}$ denote $\mathbb{R}^{n+2}$ equipped with a bilinear form of signature $(p+1, q+1)$, given by the block matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{Id}_{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $\mathrm{Id}_{p,q}$ is the diagonal matrix with of signature (p, q) with $+1$'s and -1's on the diagonal. The space of generators of the null cone h is the pseudo-sphere $S^{(p,q)}$. The bilinear form h on $\mathbb{T}$ induces a flat conformal structure on $S^{(p,q)}$. Let us denote the identity component of $O(h)$ by $G := SO^0_{p+1,q+1}$. Then G acts on $S^{(p,q)}$ as the group of all orientation preserving conformal automorphisms. Fixing the point

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{T}$$

on the null cone, the stabilizer of the corresponding point on $S^{(p,q)}$ is the subgroup

$$P = \left\{ \left. \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ x & m & 0 \\ -\lambda x^t x/2 & -\lambda x^t m & \lambda \end{pmatrix} \right| m \in SO_{p,q}, x \in \mathbb{R}^n \right\} \subseteq G$$

where x^t denotes the transpose of x. Therefore we have identification $S^{(p,q)} \simeq G/P$ and $G \longrightarrow S^{(p,q)}$ is a principal P–bundle. Later we will need the subgroup $G_0 \subseteq P$ of matrices above with $x = 0$. Lie algebras $\mathfrak{g}_0$ of G_0 and $\mathfrak{p}$ of P are similar as in (1.2) where the complex case is displayed.

Analogous construction in the curved case leads to the notion of the Cartan geometry on a manifold M. At this point, we need only the fact that the Cartan bundle $\mathcal{G} \longrightarrow M$ is a P–bundle and all conformal bundles are of the form $V = \mathcal{G} \times_P \mathbb{V}$ for a $\mathfrak{p}$–representation $\mathbb{V}$. In particular, $T^*M = \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^* = E_a$ where the latter is the notation for T^*M from [43].

1.1.3. Form index notation for representations. The aim of this section is to introduce a notation for representations derived from the abstract index notation in the sense of [43]. Recall irreducible representations of $\mathfrak{p}$ are just irreducible representations of $\mathfrak{g}_0$. We shall start with the complex algebras (1.2) and then comment briefly upon differences in the real case.

Representations of $\mathfrak{so}_n(\mathbb{C})$ are in bijective correspondence with representations of the complex spin group $Spin_n(\mathbb{C})$, the simply–connected 2–fold cover of the group $SO_n(\mathbb{C})$. Those which factor through the covering map to representations of $SO_n(\mathbb{C})$ will be referred as *tensor representations*, the remaining ones as *spinor representations*. The corresponding terminology will be used on the algebraic level and for $\mathfrak{g}_0$– and $\mathfrak{p}$–representations (and other related cases such as $O_n(\mathbb{C})$) without further mention.

Our notation for representations is motivated by the abstract indices for bundles and their sections in the sense of [43] (recall $T^*M = E_a$) and by the relation between conformal bundles and $\mathfrak{p}$–representations described in 1.2.2. Following [43], we define representations $\mathbb{E}_a$, $\mathbb{E}[w]$ and $\mathbb{E}_\lambda$ as displayed in Table 1.2. That is, spinor indices shall be denoted by Greek letters. Now all irreducible representations of $\mathfrak{g}_0$ can be found in tensor products of $\mathbb{E}[w]$, $\mathbb{E}_a$ and $\mathbb{E}_\lambda$. Actually, we do not necessary need all of them – it is sufficient to consider only tensor products $(\bigotimes \mathbb{E}_\lambda) \otimes E[w]$ – but we prefer to work primarily with tensor powers of $\mathbb{E}_a$. We shall use the spinor representation $\mathbb{E}_\lambda$ only if it is necessary. Using the highest weights in Table 1.2, the representations

Representations of $\mathfrak{g}_0(\mathbb{C}) = \mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C}$ on densities, forms and spinors	
Even dimension	Odd dimension
$\mathbb{E}[w] \cong \overset{w}{\times}\!-\!\overset{0}{\circ}\cdots\!-\!\overset{0}{\circ}\!<\!\begin{smallmatrix}\circ\, 0\\ \circ\, 0\end{smallmatrix}$	$\mathbb{E}[w] \cong \overset{w}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{0}{\circ}$
$\mathbb{E}^i \cong \overset{-i-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\cdots\!-\!\overset{0}{\circ}\!<\!\begin{smallmatrix}\circ\, 0\\ \circ\, 0\end{smallmatrix}$ $1 \le i \le n'-2$ 1 is over the $(i+1)$th node $\mathbb{E}^{n'-1} \cong \overset{-n'}{\times}\!-\!\overset{0}{\circ}\cdots\!-\!\overset{0}{\circ}\!<\!\begin{smallmatrix}\circ\, 1\\ \circ\, 1\end{smallmatrix}$ $\mathbb{E}^{n'} \cong \mathbb{E}^{n'}_+ \oplus \mathbb{E}^{n'}_- \cong \overset{-n'-1}{\times}\!-\!\overset{0}{\circ}\cdots\!-\!\overset{0}{\circ}\!<\!\begin{smallmatrix}\circ\, 2\\ \circ\, 0\end{smallmatrix} \oplus \overset{-n'-1}{\times}\!-\!\overset{0}{\circ}\cdots\!-\!\overset{0}{\circ}\!<\!\begin{smallmatrix}\circ\, 0\\ \circ\, 2\end{smallmatrix}$	$\mathbb{E}^i \cong \overset{-i-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!-\!\overset{1}{\circ}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{0}{\circ}$ $1 \le i \le n'-1$ 1 is over the $(i+1)$th node $\mathbb{E}^{n'} \cong \overset{-n'-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{2}{\circ}$
Notation: $\mathbb{E}^i = \bigwedge^i \mathbb{E}_a = \mathbb{E}_{\mathbf{a}^i} = \mathbb{E}(i) = \mathbb{E}\{0,\dots,0,1,0,\dots,0\}$ where 1 is on the ith position	
$\mathbb{E}_\lambda \cong \mathbb{E}_{\lambda'} \oplus \mathbb{E}_{\lambda''} \cong \overset{-1}{\times}\!-\!\overset{0}{\circ}\cdots\!-\!\overset{0}{\circ}\!<\!\begin{smallmatrix}\circ\, 1\\ \circ\, 0\end{smallmatrix} \oplus \overset{-1}{\times}\!-\!\overset{0}{\circ}\cdots\!-\!\overset{0}{\circ}\!<\!\begin{smallmatrix}\circ\, 0\\ \circ\, 1\end{smallmatrix}$	$\mathbb{E}_\lambda \cong \overset{-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{1}{\circ}$
Notation: $\mathbb{E}_\lambda = \mathbb{E}(\tfrac{1}{2}) = \mathbb{E}\{0,\dots,0,\tfrac{1}{2}\}$ $\left.\begin{array}{l}\mathbb{E}_{\lambda'} = (\mathbb{E}_\lambda)_+ = \mathbb{E}_+(\tfrac{1}{2}) = \mathbb{E}_+\{0,\dots,0,\tfrac{1}{2}\}\\ \mathbb{E}_{\lambda''} = (\mathbb{E}_\lambda)_- = \mathbb{E}_-(\tfrac{1}{2}) = \mathbb{E}_-\{0,\dots,0,\tfrac{1}{2}\}\end{array}\right\}$ for n even	

Table 1.2: Notation for basic representations

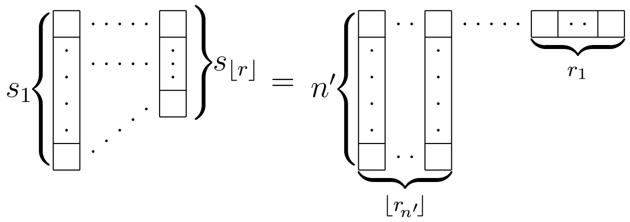
Representations of $\mathfrak{g}_0 = \mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C}$: Weyl's construction and Dynkin diagrams	
Notation for space:	$\mathbb{V}_{(\pm)} = \mathbb{E}_{(\pm)}(l; s_1, \dots, s_{\lfloor r \rfloor})_0[w] = \mathbb{E}_{(\pm)}\{r_1, \dots, r_{n'}\}_0[w]$ • $l \in \{0, \frac{1}{2}\}$, $\lfloor r \rfloor \in \mathbb{N}$, $r_{n'} \in \frac{1}{2}\mathbb{N}_0$, $r_{n'} - l \in \mathbb{N}_0$ • $r_{n'}, l \notin \mathbb{N}_0$ if and only if $\mathbb{V}_{(\pm)}$ is a spinor representation • $s_i \in \mathbb{N}$ for $i \in \{1, \cdots, \lfloor r \rfloor\}$ and $r_j \in \mathbb{N}_0$ for $j \in \{1, \cdots, n'-1\}$ • $r_1, \dots, r_{n'-1}, \lfloor r_{n'} \rfloor$ are numbers of occurrences of their subscript $j \in \{1, \cdots, n'-1\}$ in the sequence $s_1, \dots, s_r$ and $r_{n'} = \lfloor r_{n'} \rfloor + l$ • $n' \geq s_1 \geq \cdots \geq s_r \geq 1$ • The sign appears if $r_{n'} > 0$ (equivalently $s_1 = n'$ or $l = \frac{1}{2}$) and n is even.
Young diagram for tensor indices:	
Additional parameters:	• $r = \sum_{j=1}^{n'} r_j$ • $\tilde{r}^j = \sum_{k=j}^{n'} r_k$, $j \in \mathbb{N}_0$ • $s = l + \sum_{i=1}^{\lfloor r \rfloor} s_i = l + (\sum_{i=1}^{n'-1} i r_i) + n' \lfloor r_{n'} \rfloor$ • $s^i = l + \sum_{k=1}^{\lfloor i \rfloor} s_k$, $i \in \frac{1}{2}\mathbb{N}_0$
Notation for elements:	$f = f_{\mathbf{a}_1^{s_1} \cdots \mathbf{a}_r^{s_r}} = f_{\mathbf{a}_1 \cdots \mathbf{a}_r} \in \mathbb{V}_{(\pm)}$ for $l = 0$ $f = f_{\lambda \mathbf{a}_1^{s_1} \cdots \mathbf{a}_r^{s_r}} = f_{\lambda \mathbf{a}_1 \cdots \mathbf{a}_r} \in \mathbb{V}_{(\pm)}$ for $l = \frac{1}{2}$ (Here we consider implicitly the integer part $\lfloor r \rfloor$ in place of r.)

Table 1.3: Notation for representations: the general case

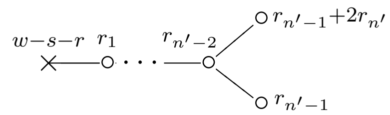
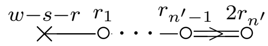
Trace–freeness:	$f = f_{\mathbf{a}_1^{s_1}\cdots\mathbf{a}_r^{s_r}} \in \mathbb{V}_{(\pm)} \Longrightarrow \boldsymbol{g}^{a^i_j a^{\bar\imath}_{\bar\jmath}} f = 0$ and $f = f_{\lambda\mathbf{a}_1^{s_1}\cdots\mathbf{a}_r^{s_r}} \in \mathbb{E}_{(\pm)} \Longrightarrow \boldsymbol{g}^{a^i_j a^{\bar\imath}_{\bar\jmath}} f = \boldsymbol{\beta}^{a^i_j}{}_{\lambda'}{}^{\lambda} f = 0$ where $a^i_j \in \mathbf{a}_j^{s_j}$, $a^{\bar\imath}_{\bar\jmath} \in \mathbf{a}_{\bar\jmath}^{s_{\bar\jmath}}$ and $1 \leq j, \bar\jmath \leq \lfloor r \rfloor$, $j \neq \bar\jmath$
Signs for n even and $r_{n'} > 0$:	$f = f_{\mathbf{a}_1^{s_1}\cdots\mathbf{a}_r^{s_r}} \in \mathbb{V}_{\pm} \Longrightarrow \tilde{\boldsymbol{\epsilon}}_{\mathbf{b}_j}{}^{\mathbf{a}_j} f = \pm f$ $f = f_{\lambda\mathbf{a}_1^{s_1}\cdots\mathbf{a}_r^{s_r}} \in \mathbb{V}_{\pm} \Longrightarrow \tilde{\boldsymbol{\epsilon}}_{\mathbf{b}_j}{}^{\mathbf{a}_j} f = \bar{\boldsymbol{\epsilon}}_{\lambda'}{}^{\lambda} f = \pm f$ where $s_j = \lvert\mathbf{a}_j\rvert = \frac{n}{2}$
Dynkin diagrams:	$\mathbb{V}_{(+)} \supseteq$  $\mathbb{V} \supseteq$  The inclusion $\supseteq$ is isomorphism for $r_{n'} \in \mathbb{N}_0$.

Table 1.3: Notation for representations: the general case continued

living in $\mathbb{E}_{a_1\ldots a_s}[w] := (\bigotimes^s \mathbb{E}_a) \otimes \mathbb{E}[w]$ are tensor representations and the spinor ones can be found in $\mathbb{E}_{\lambda a_1\ldots a_s}[w] := \mathbb{E}_\lambda \otimes (\bigotimes^s \mathbb{E}_a) \otimes \mathbb{E}[w]$, see [26]. That is, we need at most one spinor index in the general case.

We shall construct representation we need from $\mathbb{E}_{a_1\ldots a_s}[w]$ or $\mathbb{E}_{\lambda a_1\ldots a_s}[w]$ using various symmetrizations of indices, the metric, the volume form and the Clifford matrices. The results described below in detail are summarized in Table 1.3 for $\mathfrak{g}_0$–representations.

Tensor representations of $\mathfrak{so}_n(\mathbb{C})$: Weyl's construction

The algebra $\mathfrak{so}_n(\mathbb{C})$ is the semisimple part of $\mathfrak{g}_0$. We will consider representations of $\mathfrak{sl}_n(\mathbb{C}) \supseteq \mathfrak{so}_n(\mathbb{C})$ first. Following the abstract index notation we shall denote the trivial representation by $\mathbb{E}$, the standard representation by $\mathbb{E}^a$ and its dual by $\mathbb{E}_a = (\mathbb{E}^a)^*$. All irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ can be extracted from $\mathbb{E}_{a_1\cdots a_s}$ for an appropriate $s \geq 1$ using certain symmetrizations of the indices $a_1, \ldots, a_s$. The simplest example is the decomposition $\mathbb{E}_{ab} = \mathbb{E}_{(ab)} \oplus \mathbb{E}_{[ab]}$ to $\mathfrak{sl}_n(\mathbb{C})$–irreducibles. That is, we use $[\ldots]$ for skew–

symmetrization and $(\ldots)$ for symmetrization of the enclosed indices.

Henceforth we follow Weyl's construction from [26]. The representations $\mathbb{E}_{[a^1\cdots a^k]}$ and $\mathbb{E}_{(a^1\cdots a^k)}$ are $\mathfrak{sl}_n(\mathbb{C})$–irreducible. In general, we can consider (skew)–symmetrizations of various indices of $\mathbb{E}_{a_1\cdots a_s}$, $s \geq 0$. Proper compositions of these operations leading to all $\mathfrak{sl}_n(\mathbb{C})$–irreducibles are described by *Young diagrams*. We shall use the following two notations for them:

$$\text{Young}(s_1, \ldots, s_r) = \text{Young}\{r_1, \ldots, r_n\}$$

where $n \geq s_1 \geq \cdots \geq s_r \geq 1$ are lengths of columns and $r_j \in \mathbb{N}_0$ is the number of columns of the length $j \in \{1, \ldots, n\}$. We shall denote the number of columns by $r := \sum_{j=1}^{n} r_j$ and the number of boxes by $s = \sum_{i=1}^{r} s_i$. See Table 1.3 for displayed diagrams. Let us note we also admit the "empty" Young diagram i.e. $r = 0$.

Here, following [43], we only briefly review this result. We shall apply *Young symmetries* or *Young projection* (for a given Young diagram) to the indices of $\mathbb{E}_{a_1\cdots a_s}$, which now correspond to the boxes, in the two following steps. First, for each row, we symmetry over all indices therein. Second, we each column, we skew over all indices therein. The result is an $\mathfrak{sl}_n(\mathbb{C})$–irreducible representation denoted by

$$\mathbb{E}(s_1, \ldots, s_r) = \mathbb{E}\{r_1, \ldots, r_n\} = \mathbb{E}(0; s_1, \ldots, s_r) \tag{1.4}$$

(the last of these is used for the sake of compatibility with the spinor case, see below) and every irreducible representation can be obtained in this way. The irreducibility means any further (skew)–symmetrization either loses no information or vanishes. For example, skew–symmetrization over any set of $s_1 + 1$ or more indices is zero. Subdiagrams consisting from columns $i, \cdots, r$ (of lengths $s_i, \ldots, s_r$), $1 \leq i \leq r$ satisfy the similar property: skew–symmetrization over any set of $s_i + 1$ or more indices from columns $i, \cdots, r$

is zero. (Analogously, symmetrization over $r+1$ or more indices is zero etc.) Also, whole columns of indices are mutually symmetric, if they are of the same length. This follows from the structure of Young projection, in particular from the first step where we symmetry over all indices in every row. Example 1.1.1 below demonstrates some of these properties.

We will often deal with representations $\mathbb{E}_{[a^1 \cdots a^k]} = \mathbb{E}(k)$, $0 \leq k \leq n$. Also we need r-tuples of skew–symmetric indices in the general case because every column of Young diagrams is skewed over. To simplify the notation, we will abbreviate $[a^1 \cdots a^k]$ via multiindices. That is, we will use the *form indices*

$$\mathbf{a}^k = [a^1 \cdots a^k], \ k \geq 0$$

where $\mathbf{a}^0$ simply means the index is absent i.e. $\mathbb{E}_{\mathbf{a}^0} = \mathbb{E}$. In the other words, $\mathbb{E}_{\mathbf{a}^k} = \bigwedge^k \mathbb{E}_a$.

Irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ we have constructed are (in general reducible) representations of $\mathfrak{so}_n(\mathbb{C}) \subseteq \mathfrak{sl}_n(\mathbb{C})$. Representations of $\mathfrak{so}_n(\mathbb{C})$ correspond to bundles on complex oriented (pseudo)riemannian manifolds. These geometric structures are defined by two distinguished sections - the metric g and the volume form ϵ. We shall use the same notation on the representation level to obtain $\mathfrak{so}_n(\mathbb{C})$–irreducibles. The first step is to use the metric $g_{ab} \in \mathbb{E}_{(ab)}$. For example, this yields the decomposition $\mathbb{E}_{ab} = \mathbb{E}_{(ab)} \oplus \mathbb{E}_{(ab)_0} \oplus \mathbb{E}_{[ab]}$ where the index 0 indicates the trace–free part. The metric provides an isomorphism $\mathbb{E}_a \cong \mathbb{E}^a$ i.e. we can raise and lower indices. Now in the general case, we can apply $g^{a_i a_j}$ to $\mathbb{E}(s_1, \ldots, s_r)$ for any $1 \leq i < j \leq s$. The intersection of kernels of all these mappings i.e. the *trace–free part* will be denoted by attaching the index 0 to (1.4) i.e. by $\mathbb{E}(s_1, \ldots, s_r)_0$ etc.

The next step is to employ the volume form $\epsilon \in \mathbb{E}_{\mathbf{a}^n}$. This (together with the metric) yields the Hodge isomorphism $\tilde{\epsilon} : \mathbb{E}_{\mathbf{a}^k} \xrightarrow{\cong} \mathbb{E}_{\mathbf{a}^{n-k}}$, $0 \leq k \leq n$. Hence it is sufficient to consider Young diagrams with $s_1 \leq n' = \lfloor \frac{n}{2} \rfloor$ or equivalently $r_j = 0$ for $j > n'$. In the even dimensional case $n = 2n'$, we can assume $\tilde{\epsilon} : \mathbb{E}_{\mathbf{a}^{n'}} \longrightarrow \mathbb{E}_{\mathbf{a}^{n'}}$ satisfies $\tilde{\epsilon}^2 = \mathrm{id}$ after normalization of ϵ by a complex scalar. Thus we have the decomposition $\mathbb{E}_{\mathbf{a}^{n'}} = (\mathbb{E}_{\mathbf{a}^{n'}})_+ \oplus (\mathbb{E}_{\mathbf{a}^{n'}})_-$ of to the eigenspaces of $\tilde{\epsilon} : \mathbb{E}_{\mathbf{a}^{n'}} \longrightarrow \mathbb{E}_{\mathbf{a}^{n'}}$ with the eigenvalues $+1$ and -1, respectively. In the general case $\mathbb{E}_{(\pm)}(s_1, \ldots, s_r)$, $s_1 = n'$, n even, we have many columns of the length n' and we can apply $\tilde{\epsilon}$ to any of them. However, they are mutually symmetric hence an eigenvalue $v \in \mathbb{E}_{(\pm)}(s_1, \ldots, s_r)$ satisfies $\tilde{\epsilon}(v) = \pm v$ and this sign called *sign of the representation* does not depend on the choice of the form index $\mathbf{a}_i^{n'}$ of v. (We use the term sign to cover both orientation in the tensor case and chirality of spinors, see below.) We will denote these representation by

$$\mathbb{E}_{(\pm)}(s_1, \ldots, s_r)_0 = \mathbb{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0 = \mathbb{E}_{(\pm)}(0; s_1, \ldots, s_r)_0 \tag{1.5}$$

where $n' \geq s_1 \geq \cdots \geq s_r \geq 1$ and the sign applies only in even dimensions, then in the case $s_1 = n'$.

Theorem (Weyl, see e.g. [26]). *The representations of $\mathfrak{so}_n(\mathbb{C})$ of the form (1.5) are irreducible. That is, if the sign applies, the representation corresponding to both signs are irreducible. On the contrary, every irreducible tensor representation of $\mathfrak{so}_n(\mathbb{C})$ can be obtained in this way.*

The form (multi)indices provide a manageable way to deal with elements of $v \in \mathbb{E}(s_1, \ldots, s_r)_0$. We shall indicate the structure via (form) indices attached to v i.e.

$$v = v_{\mathbf{a}_1 \cdots \mathbf{a}_r} = v_{\mathbf{a}_1^{s_1} \cdots \mathbf{a}_r^{s_r}} = v_{[a_1^1 \cdots a_1^{s_1}] \cdots [a_1^1 \cdots a_r^{s_r}]} \in \mathbb{E}(s_1, \ldots, s_r)_0 \tag{1.6}$$

where $\mathbf{a}_i = \mathbf{a}_i^{s_i} = [a_i^1 \cdots a_i^{s_i}]$. We will usually omit the superscript indicating the valence s_i on indices of v, as they will be known from the context. Obviously, this notation can be used also for elements of the whole (reducible) space $\mathbb{E}_{\mathbf{a}_1 \cdots \mathbf{a}_r}$ or possibly for valences $s_i > n'$.

To able to deal effectively with various traces in formulae for differential operators, we will use the following abbreviations

$$\begin{aligned}
\mathbf{a}^k &:= a^1 \cdots a^k = [a^1 \cdots a^k], \quad k \geq 0, \\
\dot{\mathbf{a}}^k &:= a^2 \cdots a^k = [a^2 \cdots a^k], \quad k \geq 1, \\
\ddot{\mathbf{a}}^k &:= a^3 \cdots a^k = [a^3 \cdots a^k], \quad k \geq 2, \\
\dddot{\mathbf{a}}^k &:= a^4 \cdots a^k = [a^4 \cdots a^k], \quad k \geq 3,
\end{aligned}$$

where, in an obvious way, if for example $k = 1$ then $\dot{\mathbf{a}}^k$ simply means the index is absent. Also if, for example, $k = 1$ then $\ddot{\mathbf{a}}$ means the term containing the index $\ddot{\mathbf{a}}$ is absent.

For example, the following possible structures of indices are equivalent:

$$v_{\mathbf{a}^3} = v_{[a^1a^2a^3]} = v_{[a^1\dot{\mathbf{a}}^3]} = v_{[a^1a^2\ddot{\mathbf{a}}^3]} \in \mathbb{E}_{\mathbf{a}^3} = \mathbb{E}(3) = \mathbb{E}\{0, 0, 1, 0, \ldots, 0\}.$$

Example 1.1.1. Let us consider $v_{\mathbf{abc}} \in \mathbb{E}(k, l, m)$, $k \geq l \geq m \geq 1$. This means $\mathbf{a} = \mathbf{a}^k$, $\mathbf{b} = \mathbf{a}^l$ and $\mathbf{c} = \mathbf{a}^m$. The structure of $\mathbb{E}(k, l, m)$ yields

$$v_{[\mathbf{a}b^1]\dot{\mathbf{b}}c} = v_{[\mathbf{a}|\mathbf{b}|c^1]\dot{\mathbf{c}}} = v_{\mathbf{a}[\mathbf{b}c^1]\dot{\mathbf{c}}} = 0.$$

where, as usually, $|..|$ denotes indices excluded from the skew–symmetrization $[\mathbf{a}c^1]$. From this, it is easy to show the first two equalities of

$$v_{b^1[\dot{\mathbf{a}}a^1]\dot{\mathbf{b}}\mathbf{c}} = \frac{1}{k} v_{\mathbf{abc}}, \quad v_{c^1[\dot{\mathbf{a}}|\mathbf{b}|a^1]\dot{\mathbf{c}}} = \frac{1}{k} v_{\mathbf{abc}} \quad \text{and} \quad v_{\mathbf{a}c^1[\dot{\mathbf{b}}b^1]\dot{\mathbf{c}}} = \frac{1}{l} v_{\mathbf{abc}}. \tag{1.7}$$

The last of these follows from vanishing of $[\mathbf{b}c^1]$. We shall use these relations (and analogous for the general case) often in examples and without further mention. Further, the form indices of the same valence are symmetric i.e. $v_{\mathbf{abc}} = v_{\mathbf{bac}}$ if $k = l$. Finally, the trace–freeness means

$$v \in \mathbb{E}(k, l, m)_0 \Longleftrightarrow g^{a^1b^1} v_{\mathbf{abc}} = g^{a^1c^1} v_{\mathbf{abc}} = g^{b^1c^1} v_{\mathbf{abc}} = 0.$$

Representations of $\mathfrak{so}_n(\mathbb{C})$: the general case

Now we employ Clifford matrices, an analogue of Clifford section which defines a spin structure on spin manifolds. The representation on Dirac spinors is $\mathbb{E}^\lambda$ and its dual $\mathbb{E}_\lambda$. Details about spinors are postponed to 1.2.1. At this point, we need only that Clifford matrices $\beta_a \in \mathbb{E}_a \otimes \text{End}\, \mathbb{E}_\lambda$ satisfy $2\beta_{(a}\beta_{b)} = -g_{ab}\text{id}$. Also we will need $\mathbb{E}_\lambda \cong \mathbb{E}^\lambda$, see 1.2.1.

The volume form $\epsilon \in \mathbb{E}_{\mathbf{a}^n}$ yields the endomorphism $\bar{\epsilon} := c\epsilon^{\mathbf{a}^n}\beta_{a^1} \cdots \beta_{a^n} \in \text{End}\, \mathbb{E}_\lambda$, $c \in \mathbb{C}$ called *chirality operator*. For an appropriate choice of c, $\bar{\epsilon} = \text{id}$ for n odd and $\bar{\epsilon}^2 = \text{id}$ for n even [44, Appendix]. In the latter case, $\bar{\epsilon}$ has two eigenvalues $+1$ and -1 hence we have the corresponding decomposition $\mathbb{E}_\lambda = (\mathbb{E}_\lambda)_+ \oplus (\mathbb{E}_\lambda)_-$ for n even. The eigenspaces are interchanged upon replacing c by $-c$ therefore we can assume $\mathbb{E}_{\lambda'} = (\mathbb{E}_\lambda)_+$ and $\mathbb{E}_{\lambda''} = (\mathbb{E}_\lambda)_-$ similarly as in Table 1.2 where representation of $\mathfrak{g}_0$ are displayed. In the terminology of [44], $\mathbb{E}_{\lambda'}$ and $\mathbb{E}_{\lambda''}$ are called reduced spinors. It follows from the notation of Dynkin diagrams (cf. Table 1.2) that $(\mathbb{E}_\lambda)_+ \boxtimes (\mathbb{E}_\lambda)_+ \cong (\mathbb{E}_{\mathbf{a}^{n'}})_+$ and similarly for the -1–eigenspace. That is, $(\mathbb{E}_\lambda)_\pm$ has the coefficient 1 and $(\mathbb{E}_{\mathbf{a}^{n'}})_\pm$ the coefficient 2 over the same nod.

Now we describe the general spinor case. We will denote $\mathbb{E}_\lambda \otimes \mathbb{E}(s_1, \ldots, s_r)$ by $\mathbb{E}(\frac{1}{2}; s_1, \ldots, s_r)$. On the other hand, it follows from the notation of the Dynkin diagrams that all spinor representations are the Cartan products $\mathbb{E}_\lambda \boxtimes \mathbb{V}$ (for n odd) and $(\mathbb{E}_\lambda)_+ \boxtimes \mathbb{V}_{(+)}$ and $(\mathbb{E}_\lambda)_- \boxtimes \mathbb{V}_{(-)}$ (for n even) where $\mathbb{V} = \mathbb{E}(s_1, \ldots, s_r)_0$ with appropriate Young symmetries.

We will use β to identify better $(\mathbb{E}_\lambda)_{(\pm)} \boxtimes \mathbb{V}_{(\pm)}$ inside $(\mathbb{E}_\lambda)_{(\pm)} \otimes \mathbb{V}_{(\pm)}$. We can consider β^a as a homomorphism on the latter tensor product by applying β^a to the spinor index and contracting the index a with one of (lower) indices of $\mathbb{V}$. The intersection of kernels of all these homomorphisms will be denoted by

$$\mathbb{E}_{(\pm)}(\frac{1}{2}; s_1, \ldots, s_r)_0 = \mathbb{E}_{(\pm)}\{r_1, \ldots, r_{n'} + \frac{1}{2}\}_0 \tag{1.8}$$

where the sign in the last display called *sign of the representation* applies for n even. If the sign is not attached for n even in last display, this will mean direct sum of both component. Elements shall be denoted similarly as in the tensor case with one additional index λ (or possibly λ' or λ'').

The irreducibility of these representations is a delicate question. This can be proved easily for $\mathbb{E}_{(\pm)}(1, \ldots, 1)$ (spinor valued symmetric tensors) and $\mathbb{E}_{(\pm)}(k)$ (spinor valued forms) by computing their dimensions. But I am not aware of a proof of irreducibility for the general case (1.8) and we do not need this fact for the subsequent constructions.

Representations of $\mathfrak{g}_0 = \mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C}$

These representations correspond to bundles of complex spin oriented conformal manifolds. The key objects for these geometrical structures are the conformal metric $\boldsymbol{g}$, the conformal volume form $\boldsymbol{\epsilon}$ and the conformal Clifford section $\boldsymbol{\beta}$. Corresponding objects on the representation level are now $\boldsymbol{g}_{ab} \in \mathbb{E}_{(ab)}[2]$, $\boldsymbol{\epsilon}_{\mathbf{a}^n} \in \mathbb{E}_{\mathbf{a}^n}[n]$ and $\boldsymbol{\beta}_a \in \mathbb{E}_a \otimes \text{End }(\mathbb{E}_\lambda)[1]$ satisfying $2\boldsymbol{\beta}_{(a}\boldsymbol{\beta}_{b)} = -\boldsymbol{g}\text{id}$. Recall $\mathbb{E}[w]$ is defined in Table 1.2. We will use them analogously as g, ϵ and β above i.e. the representation we will deal with throughout the thesis, is of the form

$$\mathbb{E}_{(\pm)}(l; s_1, \ldots, s_r)_0[w] = \mathbb{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w] \tag{1.9}$$

where $l \in \{0, \frac{1}{2}\}$ and $r_{n'} - l \in \mathbb{N}_0$. This an irreducible tensor representation for $l = 0$ (or $r_{n'} \in \mathbb{N}_0$) and a spinor representation for $l = \frac{1}{2}$ (or $r_{n'} \in \frac{1}{2}\mathbb{N}_0 \setminus \mathbb{N}_0$). Generalizing the notation that s denotes the number of boxes and r the number of columns of Young diagrams, we put $s := l + \sum_{i=1}^{r} s_i$ and $r = \sum_{j=1}^{n'} r_j$ now. That is, s, r and $r_{n'}$ are integers (half integers) if they correspond to a tensor (spinor) representation. At the same time we will use the convention that in expressions with the subscript r, namely s_r, $\mathbf{a}_r$ and $\mathbf{A}_r$, we consider implicitly the integer part of r i.e. $s_r := s_{\lfloor r \rfloor}$ etc. so (1.9) is consistent. Beside this we define "subsums" s^x also for nonintegral superscripts as

$$s^{i+1/2} := 1/2 + s^i, \ i \in \{0, \dots, \lfloor r \rfloor\} \quad \text{for spinor representations,} \tag{1.10}$$

cf. (1.1). Let us emphesize we will use both descriptions from (1.9) and often switch between them without a further mention.

Although we need primarily irreducible representation, it is sometimes convenient to work with nonirreducible ones. Firstly, we shall approach irreducible spinor representation $(\mathbb{E}_\lambda)_{(\pm)} \boxtimes \mathbb{V}_{(\pm)}[w]$ where $\mathbb{V} = \mathbb{E}(s_1, \dots, s_r)_0[w]$ using (1.9) and the inclusion and the projection

$$(\mathbb{E}_\lambda)_{(\pm)} \boxtimes \mathbb{V}_{(\pm)} \hookrightarrow \mathbb{E}_{(\pm)}(\frac{1}{2}; s_1, \dots, s_r)_0[w] \tag{1.11}$$

$$\mathbb{E}_{(\pm)}(\frac{1}{2}; s_1, \dots, s_r)_0[w] \twoheadrightarrow (\mathbb{E}_\lambda)_{(\pm)} \boxtimes \mathbb{V}_{(\pm)}[w], \tag{1.12}$$

respectively. Secondly, our calculus will be mostly independent of the sign and we will work preferably with the whole representation $\mathbb{E}\{r_1, \dots, r_{n'}\}_0[w]$. For $r_{n'} > 0$ and n even, we have the inclusion and projection

$$\mathbb{E}_\pm\{r_1, \dots, r_{n'}\}_0[w] \hookrightarrow \mathbb{E}\{r_1, \dots, r_{n'}\}_0[w]$$

$$\mathbb{E}\{r_1, \dots, r_{n'}\}_0[w] \twoheadrightarrow \mathbb{E}_\pm\{r_1, \dots, r_{n'}\}_0[w],$$

respectively. We shall use them without a further mention.

We shall often suppress the spinor indices in the notation for elements. However, if necessary, we follow (1.6) and attach the spinor index λ (or exceptionally λ' and λ''). That is, we will use

$$v = v_{\mathbf{a}_1 \cdots \mathbf{a}_r} = v_{\lambda \mathbf{a}_1 \cdots \mathbf{a}_r} \in \mathbb{E}(\frac{1}{2}; s_1, \dots, s_r)_0 \tag{1.13}$$

where $\mathbf{a}_i = \mathbf{a}_i^{s_i} = [a_i^1 \cdots a_i^{s_i}]$ etc. as in (1.6).

Finally, we describe duals of representations (1.9). Clearly $\mathbb{E}[w]^* = \mathbb{E}[-w]$, $(\mathbb{E}_a)^* = \mathbb{E}_a[2]$ using $\boldsymbol{g}^{ab} \in \mathbb{E}^{(ab)}[-2]$ and $(\mathbb{E}_\lambda)^* = \mathbb{E}_\lambda[1]$ using $\boldsymbol{\varepsilon}^{\lambda_1\lambda_2} \in \mathbb{E}^{\lambda_1\lambda_2}[-1]$ (the spinor inner product, see 1.2.1 for details). In general, we have the isomorphism

$$(\mathbb{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w])^* \cong \mathbb{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[-w + 2s]. \qquad (1.14)$$

The sign applies for $n = 2n'$ and $r_{n'} > 0$. One can show that the signs are the same on both sides for n' even and different for n' odd.

The real case: representations of $\mathfrak{g}_0' = \mathfrak{so}_{p,q} \oplus \mathbb{R}$

Most of the construction above is independent on the choice of scalars and we shall use all the developed notation also for $\mathfrak{so}_{p,q} \oplus \mathbb{R}$. In particular, we have the real representations $\mathbb{E}_{\mathbf{a}^k}$, $\mathbb{E}_\lambda$ and $\mathbb{E}[w]$ for $w \in \mathbb{R}$, and we can use Weyl's construction. That is, the Theorem above holds if we replace $\mathfrak{so}_n(\mathbb{C})$ by $\mathfrak{so}_{p,q}$. However, the condition "if the sign applies" in the Theorem has now different meaning because we can normalize the volume form $\boldsymbol{\epsilon} \in \mathbb{E}_{\mathbf{a}^n}[n]$ by a real scalar only. Summarizing the real case, we have the $\mathfrak{so}_{p,q} \oplus \mathbb{R}$–representations

$$\mathbb{V}_{(\pm)} := \mathbb{E}_{(\pm)}(l; s_1, \ldots, s_r)_0[w] = \mathbb{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w], \quad w \in \mathbb{R} \qquad (1.15)$$

and we need to know when endomorphisms $\tilde{\boldsymbol{\epsilon}} \in \mathrm{End}\ \mathbb{E}_{\mathbf{a}^{n'}}$ and $\bar{\boldsymbol{\epsilon}} \in \mathrm{End}\ \mathbb{E}_\lambda$ have real eigenvalues. Assume n is even. For $\boldsymbol{g}$ of signature (p, q), we can suppose $\boldsymbol{\epsilon}_{\mathbf{a}}\boldsymbol{\epsilon}^{\mathbf{a}} = (-1)^q n!$ where $\mathbf{a} = \mathbf{a}^n$. Using this, one can compute

$$\boldsymbol{\epsilon}_{\mathbf{bc}}\boldsymbol{\epsilon}^{\mathbf{dc}} = k!(n-k)!(-1)^q\,\mathrm{id}_{\mathbb{E}^{\mathbf{d}}} \quad \text{hence} \quad \boldsymbol{\epsilon}_{\mathbf{bc}}\boldsymbol{\epsilon}^{\mathbf{cd}} = k!(n-k)!(-1)^{r(n-r)+q}\,\mathrm{id}_{\mathbb{E}^{\mathbf{d}}}$$

where $|\mathbf{bc}| = n$ and $\mathbf{b} = \mathbf{b}^k$ and $\mathbf{d} = \mathbf{d}^k$. For $n = 2n'$ and $k = n'$, the sign $(-1)^{r(n-r)+q}$ for $\boldsymbol{\epsilon}_{\mathbf{bc}}\boldsymbol{\epsilon}^{\mathbf{cd}}$ is equal to $(-1)^{n'-q} = (-1)^{n'-p}$ as revealed by a short computation. This is exactly the sign of $\tilde{\boldsymbol{\epsilon}}^2$ i.e. contrary to the complex case, we can assume only $\tilde{\boldsymbol{\epsilon}}^2 = (-1)^{n'-p}\mathrm{id}$. The computation for $\bar{\boldsymbol{\epsilon}}$ is similar, the result is $\bar{\boldsymbol{\epsilon}}^2 = (-1)^{\frac{1}{2}(n-2q)(n-2q+1)}\mathrm{id}$ after an appropriate normalization by a real scalar [44, Appendix]. For $n = 2n'$, this sign is $(-1)^{n'-q} = (-1)^{n'-p}$ hence we can assume $\bar{\boldsymbol{\epsilon}}^2 = (-1)^{n'-p}\mathrm{id}$.

We have shown the eigenvalues of $\tilde{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\epsilon}}$ are ± 1 for $n' - p$ even and there are no real eigenvalues for $n' - p$ odd. Thus the sign in (1.15) applies i.e. we have the decomposition $\mathbb{V} = \mathbb{V}_+ \oplus \mathbb{V}_-$ if and only if n is even, $r_{n'} > 0$ and $n' - p$ is even. We shall use this convention for the remainder of the thesis.

We can use the symbolism of Dynkin diagrams corresponding to $\mathfrak{so}_n(\mathbb{C}) \oplus \mathbb{C}$ also for an irreducible representation $\mathbb{W}$ of $\mathfrak{so}_{p,q} \oplus \mathbb{R}$. It turns out [42, 48] that $\mathbb{W}$ corresponds either to one labelled Dynkin diagram or to a couple of labelled Dynkin diagrams. In the latter case, both highest weights are either equal or mutually symmetric with respect to a symmetry of the Dynkin diagram.

1.2 Geometric structure and tractor calculus

1.2.1. Riemannian, conformal and spin structure. Let us consider an n–dimensional smooth manifold M, $n \geq 3$. We shall use the notation for representations developed in 1.1.3 also for bundles and their spaces of (local) sections by replacing $\mathbb{E}$ in (1.9) by E and $\mathcal{E}$, respectively. The indices here are abstract in the sense of [43]. Recall we raise and lower tensor indices using the metric g or the conformal metric $\boldsymbol{g}$. There are certain topological obstructions to existence of some of structures discussed below. Nevertheless, they exist at least locally.

Pseudo–Riemannian structure

A *pseudo–Riemannian structure* on M is a pair (M, g) where $g = g_{ab} \in \mathcal{E}_{(ab)}$ is a pseudometric. The Levi–Civita connection corresponding to g will be denoted by ∇. That is, ∇ is torsion–free and $\nabla g = 0$. An *oriented pseudo-Riemannian structure* on M is a triple (M, g, ϵ) where (M, g) is a pseudo–Riemannian structure on M and $\epsilon \in \mathcal{E}_{\mathbf{a}^n}$ is a volume form satisfying $\frac{1}{n!}\epsilon_{\mathbf{a}}\epsilon^{\mathbf{a}} = 1$ and $\nabla\epsilon = 0$. The prefix pseudo- will be usually omitted. *Complex (oriented) Riemannian structures* are defined analogously.

In the language of bundles, an (oriented) pseudo–Riemannian structure on M is a reduction of the linear frame bundle P^1M over M to the subgroup $O_{p,q}$ (or $SO_{p,q}$), $n = p + q$ in the real setting and $O_n(\mathbb{C})$ (or $SO_n(\mathbb{C})$) in the complex one. Here (p, q) is the signature of the pseudometric g. Recall any tensor $\mathfrak{so}_{p,q}$–representation lifts to a $SO_{p,q}$–representation. In particular, $TM \cong T^*M =: E_a = \mathcal{G} \times_{SO_{p,q}} \mathbb{E}_a$.

Curvature R of ∇ is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)f^c = R_{ab}{}^c{}_d f^d$$

for $f^c \in \mathcal{E}^c$. It satisfies $R = R_{abcd} \in \mathcal{E}(2,2)$ because ∇ is torsion–free. The structure is called *flat* if $R_{abcd} = 0$. We can decompose $\mathcal{E}(2,2)$ into $\mathcal{E}(2,2)_0$ and the trace part. This yields

$$R_{abcd} = C_{abcd} + 2g_{c[a}P_{b]d} + 2g_{d[b}P_{a]c} \tag{1.16}$$

where $C_{abcd} \in \mathcal{E}(2,2)_0[2]$ is the *Weyl curvature* and $P_{ab} \in \mathcal{E}_{(ab)}$ is the *Rho–tensor*. The Rho tensor is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^c{}_b$ and vice versa: $\mathrm{Ric}_{ab} = (n-2)P_{ab} + P\boldsymbol{g}_{ab}$ where $P = P_a{}^a \in \mathcal{E}$. The *Cotton tensor* is defined by

$$A_{abc} := 2\nabla_{[b}P_{c]a}.$$

Via the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ this is related to the divergence of the

Weyl tensor as follows:

$$\nabla^p C_{pabc} = (n-3)A_{abc} \quad \text{and} \quad \nabla^b P_{ab} = \nabla_a P. \tag{1.17}$$

Spin pseudo-Riemannian structure

We can define the *spin pseudo-Riemannian structure* on a manifold M as a principal $Spin_{p,q}$–bundle $\mathcal{G}$ over M. Equivalently, this a 4–tuple (M, g, β, ϵ) where (M, g, ϵ) is an oriented pseudo–Riemannian structure on M and the Clifford section $\beta_a \in \mathcal{E}_a \otimes \text{End } \mathcal{E}_\lambda$ satisfies the Clifford relation $2\beta_{(a}\beta_{b)} = -g_{ab}\text{id}$. Note all $\mathfrak{so}_{p,q}$–representations lift to $Spin_{p,q}$–representations. In particular, we have the spin bundle $E^\lambda \cong E_\lambda = \mathcal{G} \times_{Spin_{p,q}} \mathbb{E}_\lambda$. Non–oriented and complex versions are defined analogously.

The Levi–Civita connection ∇ on E_a determines a connection on E_λ, determined uniquely by the property $\nabla\beta = 0$. We will term this also the Levi–Civita connection. Furthermore β and ϵ yield a canonical form $\bar{\epsilon} \in \text{End } \mathcal{E}_\lambda$ and an inner product $\varepsilon \in \mathcal{E}^{\lambda\omega}$ on E_λ called *spin metric*. Both are preserved by the connection i.e. $\nabla\bar{\epsilon} = \nabla\varepsilon = 0$. We shall describe them in detail in the conformal setting below.

Conformal structure

A *conformal structure* of signature (p, q) on M is a pair $(M, [g])$ where $[g]$ is a class of conformally equivalent pseudometrics of signature (p, q). Recall that metrics g and $\hat{g}$ are *conformally equivalent* if $\hat{g} = \Omega^2 g$ for a smooth positive function Ω on M. An *oriented conformal structure* is defined similarly as the class $(M, [(g, \epsilon)])$ where (g, ϵ) and $(\Omega^2 g, \Omega^n \epsilon)$ are conformally equivalent. The complex versions are defined analogously.

We may equivalently view the conformal structure as a smooth ray subbundle $\mathcal{Q} \subset E_{(ab)}$ whose fibre over $x \in M$ consists of conformally related signature-(p, q) pseudometrics at the point x. Sections of $\mathcal{Q}$ are pseudometrics from $[g]$ on M. The principal bundle $\pi : \mathcal{Q} \longrightarrow M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \text{End}\,\mathbb{R}$ induces a natural line bundle on $(M, [g])$ that we term the conformal density bundle $E[w]$.

Using principal bundles, oriented conformal structure on M is a principal $CO_{p,q}$–bundle $\mathcal{G}$ over M where $CO_{p,q} = SO_{p,q} \times \mathbb{R}$. (Non–oriented and complex versions are defined analogously.) In this setting, the density bundle is $E[w] = \mathcal{G} \times_{CO_{p,q}} \mathbb{E}[w]$. In general, any tensor representation of $\mathfrak{so}_{p,q} \oplus \mathbb{R}$ yields a conformal bundle in this way. In particular $T^*M =: E_a = \mathcal{G} \times_{CO_{p,q}} \mathbb{E}_a$ and $TM = T^*M[2] = E_a[2] = E^a$.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $E_{(ab)} \otimes E[2]$ determined by the conformal structure. This will be used to identify E^a with $E_a[2]$. Given a choice of metric g from the conformal class, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Note $E[w]$ is trivialised by a choice of metric g from the conformal class, and we write ∇ for the connection corresponding to this trivialisation. It follows immediately that (the coupled) ∇_a preserves the conformal metric.

Beside the conformal metric $\boldsymbol{g} \in \mathcal{E}_{ab}[2]$, we have also the tautological section of $E_{\mathbf{a}^n} \otimes E[n]$ i.e. the *conformal volume form* $\boldsymbol{\epsilon} \in \mathcal{E}_{\mathbf{a}^n}[n]$. This satisfies $\nabla\boldsymbol{\epsilon} = 0$ and $\boldsymbol{\epsilon}_{\mathbf{a}}\boldsymbol{\epsilon}^{\mathbf{a}} = n!$. Hence we can define conformal structures as triples $(M, \boldsymbol{g}, \boldsymbol{\epsilon})$ where $\boldsymbol{g}$ and $\boldsymbol{\epsilon}$ are conformal metric and conformal volume form, respectively.

We will often need to compare the Levi–Civita connection ∇ and $\hat{\nabla}$ corresponding to metrics g and $\hat{g}$, respectively, from the conformal class. We will always suppose the relation $\hat{g} = e^{2\Upsilon}g$ for a smooth function positive Υ on M and use the notation $\Upsilon_a = \nabla_a\Upsilon$. Quantities corresponding to $\hat{g}$ will be denoted by hats. In particular we have the curvature $\hat{R}_{abcd}$ of $\hat{g}$ and also the components $\hat{C}_{abcd}$, $\hat{P}_{ab}$ and $\hat{P}$ defined by (1.16). One computes that g and $\hat{g}$ have the same Weyl curvature i.e. $C = \hat{C}$. In the dimension 3, moreover $\nabla_{[a}P_{b]c} = \hat{\nabla}_{[a}\hat{P}_{b]c}$.

We will need the difference between ∇f and $\hat{\nabla} f$ for various sections f. This is computed in Appendix B in detail, here we only summarize the results. First,

$$\hat{\nabla}_a f = \nabla_a f + w\Upsilon_a f$$
$$\hat{\nabla}_a f_b = \nabla_a f_b - \Upsilon_a f_b - \Upsilon_b f_a + \Upsilon_p f^p \boldsymbol{g}_{ab}$$

for $f \in \mathcal{E}[w]$ and $f_a \in \mathcal{E}_a$, respectively. Since both connections satisfy the Leibnitz rule, we easily compute the general case

$$\begin{aligned}\hat{\nabla}_a f_{b_1\cdots b_s} =& \nabla_a f_{b_1\cdots b_s} + (w-s)\Upsilon_a f_{b_1\cdots b_s} - \Upsilon_{b_1} f_{ab_2\cdots b_s} \cdots - \Upsilon_{b_s} f_{b_1\cdots b_{s-1}a} \\ &+ \Upsilon^p f_{pb_2\cdots b_k}\boldsymbol{g}_{ab_1} \cdots + \Upsilon^p f_{b_1\cdots b_{s-1}p}\boldsymbol{g}_{ab_s}\end{aligned} \tag{1.18}$$

for $f_{b_1\cdots b_s} \in \mathcal{E}_{b_1\cdots b_s}[w]$. This simplifies on forms. It follows immediately from the last display that

$$\begin{aligned}\hat{\nabla}_{[b^0} f_{\mathbf{b}^k]} &= \nabla_{[b^0} f_{\mathbf{b}^k]} + w\Upsilon_{[b^0} f_{\mathbf{b}^k]} \\ \hat{\nabla}^{b^1} f_{\mathbf{b}^k} &= \nabla^{b^1} f_{\mathbf{b}^k} + (n+w-2k)\Upsilon^{b^1} f_{\mathbf{b}^k}\end{aligned} \tag{1.19}$$

for $f_{\mathbf{b}^k} \in \mathcal{E}_{\mathbf{b}^k}[w]$. Recall we use the form index notation developed in 1.1.3. Let us also note the transformation of the Rho–tensor

$$\hat{P}_{ab} = P_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon^c\Upsilon_c\boldsymbol{g}_{ab}. \tag{1.20}$$

A conformal structure is called *conformally flat* or just *flat* if there is a pseudo–metric g in the conformal class such that the corresponding pseudo–Riemannian structure is flat. For dimension $n \geq 4$, this happens if and only if $C = 0$. In dimension 3, a manifold M is conformally flat if and only if $\nabla_{[a}P_{b]c} = 0$. The flat model is the pseudosphere $S^{(p,q)}$, see 1.1.2.

Spin conformal structure

We say that two oriented pseudo–Riemannian structures (g, β, ϵ) and $(\hat{g}, \hat{\beta}, \hat{\epsilon})$ on M are conformally equivalent if $\hat{g} = \Omega^2 g$, $\hat{\beta} = \Omega\beta$ and $\hat{\epsilon} = \Omega^n \epsilon$ for a smooth positive function Ω on M. An *oriented spin conformal structure* of signature (p, q) on M is a pair $(M, [(g, \beta, \epsilon)])$ where $[(g, \beta, \epsilon)]$ is a class of conformally equivalent oriented pseudo–Riemannian structures of signature (p, q) on M. Equivalently, this structure is a 4–tuple $(M, \boldsymbol{g}, \boldsymbol{\beta}, \boldsymbol{\epsilon})$ where $(M, \boldsymbol{g}, \boldsymbol{\epsilon})$ is defined above and and $\boldsymbol{\beta}$ is given by (1.24) and satisfies $\nabla\boldsymbol{\beta} = 0$. We have also the spinor metric $\boldsymbol{\varepsilon}_{\lambda\rho} \in \mathcal{E}_{\lambda\rho}[1]$ which we use to raise and lower spinor indices. Non–oriented and complex versions are defined analogously.

Using principal bundles, an oriented spin conformal structure of signature (p, q) on M is a principal $Spin_{p,q} \times \mathbb{R}$–bundle $\mathcal{G}$ over M. Every representation $\mathbb{V}$ of $\mathfrak{so}_{p,q} \oplus \mathbb{R}$ integrates to a $Spin_{p,q} \times \mathbb{R}$–representation and yields the bundle $V = \mathcal{G} \times_{Spin_{p,q} \times \mathbb{R}} \mathbb{V}$. Hence we have a $1-1$ correspondence between $\mathfrak{so}_{p,q} \times \mathbb{R}$–representations and oriented spin conformal bundles on M. In particular, we have the spin bundle E^λ and its dual $E_\lambda[1] \cong E^\lambda$ with respect to $\boldsymbol{\varepsilon}$.

As in the tensor case, we can compute the difference between ∇f and $\hat{\nabla} f$ for a spinor section f. (Here we consider the (coupled) spinor Levi–Civita connections.) It is computed in Appendix B that the result for $f \in \mathcal{E}_\lambda[w]$ is

$$\hat{\nabla} f = \nabla f + (w-1)\Upsilon_a f - \boldsymbol{\beta}_a \Upsilon_p \boldsymbol{\beta}^p f. \tag{1.21}$$

Notation for spinors

We will consider both the real case of signature (p, q) and the complex case. (We formally put $p := n$ and $q := 0$ for the latter.) Here we review the basic tools for the study of spinors. We will mostly follow [44, Appendix] but we will adapt this to the setting of conformal structure and pass to the bundle level.

As mentioned above, the spinor bundle is denoted by E^λ but shall usually work with its dual E_λ. (See Table 1.2 for the corresponding representations.) That is, the spinor indices will be denoted by greek letters. In even dimensions, E_λ decomposes in the complex case and if $n' - p$ is even in the real one. Then two irreducible components of E_λ will be denoted by primed and two primed indices i.e.

$$E_\lambda = E_{\lambda'} \oplus E_{\lambda''}$$

where $E_{\lambda'} = E_+(\frac{1}{2})$ etc. But we will mostly consider the whole bundle E_λ. The Clifford relation is

$$\boldsymbol{\beta}_{a\lambda}{}^{\omega}\boldsymbol{\beta}_{b\omega}{}^{\gamma} + \boldsymbol{\beta}_{b\lambda}{}^{\omega}\boldsymbol{\beta}_{a\omega}{}^{\gamma} = -\boldsymbol{g}_{ab}\delta_{\lambda}{}^{\gamma}, \quad \boldsymbol{\beta}_{a\lambda}{}^{\omega} \in \mathcal{E}_{a\lambda}{}^{\omega}[1], \; \boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2] \qquad (1.22)$$

where $\boldsymbol{\beta}$ and $\boldsymbol{g}$ are the conformal Clifford symbol and the conformal metric, respectively. The normalization on the right hand side differs from [44] and follows [8].

We have a non-degenerate density valued spinor metric on E_λ, denoted be $\boldsymbol{\varepsilon}$. This can be shown using a direct construction (see [44] how to construct $\boldsymbol{\varepsilon}$ from $\boldsymbol{\beta}$ in the complex case) or proved by theoretical means (see [12, 54] for a general treatment). The spin metric $\boldsymbol{\varepsilon}$ and its inverse are sections

$$\boldsymbol{\varepsilon}^{\lambda\omega} \in \mathcal{E}^{\lambda\omega}[-1] \quad \text{and} \quad \boldsymbol{\varepsilon}_{\lambda\omega} \in \mathcal{E}_{\lambda\omega}[1].$$

This yields the identification $E_\lambda \cong E^\lambda[-1]$. In the other words, $\boldsymbol{\varepsilon}$ allows us to raise and lower spinor indices but since $\boldsymbol{\varepsilon}$ is not, in general, symmetric, we have to state conventions. Following [44], we write

$$f^\lambda := \boldsymbol{\varepsilon}^{\lambda\omega} f_\omega \quad \text{and} \quad \bar{f}_\lambda := \bar{f}^\omega \boldsymbol{\varepsilon}_{\omega\lambda} \qquad (1.23)$$

for $f_\lambda \in \mathcal{E}_\lambda$ and $\bar{f}^\lambda \in \mathcal{E}^\lambda$. Let us note the composition of "lowering" and "raising" applied to a given upper index (and similarly for a lower index) is the identity. This is due to the relation

$$\boldsymbol{\varepsilon}_{\lambda}{}^{\omega} = \boldsymbol{\varepsilon}_{\lambda\rho}\boldsymbol{\varepsilon}^{\omega\rho} = \boldsymbol{\varepsilon}_{\rho\lambda}\boldsymbol{\varepsilon}^{\rho\omega} = \delta^{\omega}_{\lambda}.$$

Uniqueness and symmetry/skew–symmetry of $\boldsymbol{\varepsilon}$ depends, in general, on the residues $[n]_8$ and $[p-q]_8$ where (p, q) is the signature, see [12, 54]. We will not need these details. But let us note that in the complex case such that $[n]_8 \notin \{[2]_8, [6]_8\}$, $\boldsymbol{\varepsilon}$ is unique (up to a complex multiple) if considered only for irreducible bundles. (That is, E_λ for n odd or $E_{\lambda'}$ and $E_{\lambda''}$ for n even.) In the complex case $[n]_8 \in \{[2]_8, [6]_8\}$, $\boldsymbol{\varepsilon}$ exists only on the whole (reducible) bundle E_λ and can be both symmetric or skew and interchanges $E_{\lambda'}$ and $E_{\lambda''}$. The real case depends on the signature and we have generally many possibilities for $\boldsymbol{\varepsilon}$. But for both scalars, we can choose $\boldsymbol{\varepsilon}$ on E_λ which is symmetric for $[n']_4 \in \{[0]_4, [3]_4\}$ and skew symmetric for $[n']_4 \in \{[1]_4, [2]_4\}$. This will be henceforth our assumption. Then in the complex case, $\boldsymbol{\beta}_a{}^{\lambda\gamma} \in \mathcal{E}_a{}^{\lambda\gamma}$ is symmetric for $[n']_4 \in \{[0]_4, [1]_4\}$ and skew symmetric for $[n']_4 \in \{[2]_4, [3]_4\}$ on the spinor indices [44]. (See [12, 54] for information about real cases.)

We shall use notation with all the spinor indices when necessary but actually we can often suppress them completely. (Recall the description of irreducible spin conformal representations/bundles from 1.1.3 does not need more than one spinor index.) In the spinor index–free notation, the conformal Clifford relation becomes

$$\boldsymbol{\beta}_a\boldsymbol{\beta}_b + \boldsymbol{\beta}_b\boldsymbol{\beta}_a = -\boldsymbol{g}_{ab}, \quad \boldsymbol{\beta}_a \in \mathcal{E}_a[1] \otimes \operatorname{End} \mathcal{E}_\lambda,\ \boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2], \tag{1.24}$$

cf. (1.22). Note the symbols possessing "hidden" spinor indices are noncommutative and to avoid confusion, we have to state certain conventions. First, the omitted spinor index is always located downstairs e.g. $f_{\mathfrak{a}} \in \mathcal{E}_{\lambda\mathfrak{a}}$. (Here $\mathfrak{a}$ denotes any system of indices.) Further, omitted spinor indices in the Clifford section are distributed as above i.e. $\boldsymbol{\beta}_a = \boldsymbol{\beta}_{a\lambda}{}^{\omega}$ (and not e.g. $\boldsymbol{\beta}_a{}^{\omega}{}_{\lambda}$ or $\boldsymbol{\beta}_a{}^{\lambda\omega}$).

Then the application of $\boldsymbol{\beta}_a$ is defined uniquely e.g. $\boldsymbol{\beta}^p \nabla_p f_{\mathfrak{a}} = \boldsymbol{\beta}^p{}_\lambda{}^\omega \nabla_p f_{\omega\mathfrak{a}}$ or $\boldsymbol{\beta}_a \boldsymbol{\beta}^p \nabla_p f_{\mathfrak{a}} = \boldsymbol{\beta}_{a\lambda}{}^\omega \boldsymbol{\beta}^p{}_\omega{}^\gamma \nabla_p f_{\gamma\mathfrak{a}}$ etc. If we need to violate any of these conventions or work with more complicated expressions, we shall write all the spinor indices explicitly.

1.2.2. Background: parabolic geometries. The aim of this section is to indicate that many tools and methods we use and develop in the thesis have analogues for a broader class of so called *parabolic geometries*. Their theory can be found in [18] so the reader should look there for exact definitions. Also we develop a notation for $\mathfrak{g}_0$–components (see below) that we will need later.

Cartan geometries of a type (G, P) are "curved analogs" of homogeneous spaces G/P where P is a closed subgroup of a Lie group G. This means that the bundle $G \longrightarrow G/P$ and a Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ are replaced by a principal P-bundle $\mathcal{G} \longrightarrow M$ equipped with a one–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which satisfies some (but not all) of the properties of the Maurer–Cartan form. Here M is a smooth manifold, $\mathfrak{g}$ denotes the Lie algebra of G, $\mathcal{G}$ is called *Cartan bundle* and ω is called the *Cartan connection*. Note that Cartan connection is not the connection in the classical sense.

Having a principal P–bundle $\mathcal{G}$ over M, we can construct an associated bundle $V = \mathcal{G} \times_P \mathbb{V}$ for each P–module $\mathbb{V}$. For example, we can identify the tangent bundle as $TM \simeq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ where $\mathfrak{p}$ is the Lie algebra of P and $\mathfrak{g}/\mathfrak{p}$ is a P–module via the adjoint action of G on $\mathfrak{g}$. If $\mathbb{V}$ is a G–module then

$$V = \mathcal{G} \times_P \mathbb{V}$$

is called a *tractor bundle*. (Here we consider the action of P as a restriction of the action of G.) In the flat case, this is the canonical trivial bundle $V \simeq (G/P) \times \mathbb{V}$ and therefore V admits a canonical (flat) linear connection.

In the curved case, a Cartan connection ω on $\mathcal{G}$ induces a linear connection on V, called *tractor connection*.

The construction of principal bundles equipped with Cartan connections from underlying structures is non–trivial. It is solved for *parabolic geometries* i.e. if P is a parabolic subgroup of semisimple Lie group G. This will be henceforth our assumption. Moreover, we obtain *normal Cartan connection* by an appropriate normalization of the curvature. The same is true for tractor connections and it turns out [15], tractor bundles with the normal tractor connection can be characterized directly from the underlying structure and these determine the normal Cartan bundle and connection. We will use especially the *standard tractor bundle* corresponding to the standard representation of G.

Since tractor bundles are vector bundles, their geometrical interpretation is usually easier than in the case of Cartan connections on $\mathcal{G}$. Moreover, we have a construction of tractor bundles and connections for so called *irreducible* parabolic geometries directly from the underlying structure (i.e. without use of Cartan bundles and connections), see [15, 16]. Recall that irreducible parabolic geometries (also known as almost Hermitian structures) are characterized by the fact that $\mathfrak{g}_+$ is irreducible as a representation of $\mathfrak{g}_0$. [15, 16] use the following data which are easily available from underlying structures: a G_0–principle bundle over M, a certain class of preferred affine connections on M, an appropriate interpretation of the their curvatures and a G–module V. The result is the normal tractor connection on $V = \mathcal{G}_0 \times_P \mathbb{V}$.

The conformal geometry is an irreducible parabolic geometry where the groups $G_0 \subseteq P \subseteq G = SO_{p,q}$ are mentioned in 1.1.2, the G_0–principal bundle over M is the set of all conformal isometries $\mathfrak{g}_- \longrightarrow T_xM$ (where $\mathfrak{g}_-$ plays the role of coordinates), the class of preferred connections consists of Levi–Civita connections etc. In the case of the spin conformal geometry, P is an appropriate parabolic subgroup of $G = Spin_{p,q}$. See 1.2.3 and 1.2.4 for details about the tractor connection on the standard tractor bundle and the tractor bundle corresponding to the spin representation, respectively.

The advantage of parabolic geometries is the relatively well–known algebraic structure of parabolic subalgebras (or subgroups) and their representations. Since the subgroup G_0 is reductive, we can consider the decomposition of P–modules $\mathbb{V}$ into irreducible G_0–modules. To do this on the bundle level, we need a G_0–structure i.e. a reduction $\mathcal{G}' \longrightarrow M$ of $\mathcal{G} \longrightarrow M$ to the subgroup G_0. Then, for a G_0–submodule $\mathbb{W} \subseteq \mathbb{V}$, we can consider the subbundle $W = \mathcal{G}' \times_{G_0} \mathbb{W} \subseteq V$ and the corresponding inclusion $pr : W \hookrightarrow V$. We will call pr a $\mathfrak{g}_0$*–component* of V. The corresponding projection will be denoted by $pr^* : V \twoheadrightarrow W$. *Irreducible* $\mathfrak{g}_0$*–component* is a $\mathfrak{g}_0$–component corresponding to an irreducible G_0–module $\mathbb{W}$.

A $\mathfrak{g}_0$–component pr of V yields a $\mathfrak{g}_0$–component of any subbundle $V' \subseteq V$ denoted also by pr in the obvious way. However two different $\mathfrak{g}_0$–components pr_1 and pr_2 can yield the same $\mathfrak{g}_0$–component of V'. (For example, both can be trivial for V'.) Further pr yields a $\mathfrak{g}_0$–component of the bundle $V \otimes U$ for any bundle U which will be also referred as pr. Strictly speaking, the latter is actually $pr \otimes \mathrm{id}$ but we shall use this in the situations when there is no danger of confusion.

The corresponding terminology will be used for a section $f \in \mathcal{V}$ of V i.e. $pr^* f$ denotes a section of $W \subseteq V$. A *projecting part* of f will mean $\mathfrak{g}_0$–component pr of V such that $pr^* f$ is invariant i.e. $pr^* f$ does not depend on the reduction of $\mathcal{G} \longrightarrow M$ to G_0. Let us note if $f \neq 0$ then there is always a nonvanishing irreducible projecting part of f. We have always a filtration on $\mathbb{V}$ as a P–module. If every $\mathfrak{g}_0$–component pr' of V of degree (with respect to the filtration) higher than pr satisfies $(pr')^* f = 0$ then pr is a projecting part of f. More details can be found in 1.2.6, in particular in Lemma therein.

If U is a bundle and $\Phi : \mathcal{U} \longrightarrow \mathcal{V}$ an invariant differential operator, we will use the analogous terminology i.e. $pr^*\Phi := pr^* \circ \Phi$ is a differential operator $\mathcal{U} \longrightarrow \mathcal{W}$. A *projecting part* of Φ will be any $\mathfrak{g}_0$–component pr of V such that $pr^*\Phi$ is invariant. From the same reasons as above, every nontrivial invariant differential operator has a nontrivial irreducible projecting part.

1.2.3. Standard conformal tractor bundle. Using the notation from Section 1.2.2, the *standard tractor bundle* E^A is defined as $E^A := \mathcal{G} \times_P \mathbb{E}^A$ where the $\mathfrak{g}$–module $\mathbb{E}^A$ is given by the standard representation of $\mathfrak{so}_{n+2}(\mathbb{C})$ i.e.

$$\overset{1}{\circ}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{0}{\circ} \quad \text{or} \quad \overset{1}{\circ}\!-\!\overset{0}{\circ}\cdots-\!\overset{0}{\circ}\!<^{\circ\, 0}_{\circ\, 0}$$

for n odd or even, respectively. Hence we immediately get the algebraic structure of E^A. The $\mathfrak{g}$–module $\mathbb{E}^A$ is

$$\mathbb{E}^A = \overset{1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{0}{\circ} \mathbin{\subset\!\!\!\!+} \overset{-1}{\times}\!-\!\overset{1}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{0}{\circ} \mathbin{\subset\!\!\!\!+} \overset{-1}{\times}\!-\!\overset{0}{\circ}\cdots\overset{0}{\circ}\!\Rightarrow\!\overset{0}{\circ} \quad \text{or}$$

$$\mathbb{E}^A = \overset{1}{\times}\!-\!\overset{0}{\circ}\cdots-\!\overset{0}{\circ}\!<^{\circ\, 0}_{\circ\, 0} \mathbin{\subset\!\!\!\!+} \overset{-1}{\times}\!-\!\overset{1}{\circ}\cdots-\!\overset{0}{\circ}\!<^{\circ\, 0}_{\circ\, 0} \mathbin{\subset\!\!\!\!+} \overset{-1}{\times}\!-\!\overset{0}{\circ}\cdots-\!\overset{0}{\circ}\!<^{\circ\, 0}_{\circ\, 0}$$

as a $\mathfrak{p}$–module for n odd or even, respectively. Using the index notation (cf. Table 1.2) and passing to the bundle level this means

$$E^A = E[1] \mathbin{\subset\!\!\!\!+} E_a[1] \mathbin{\subset\!\!\!\!+} E[-1]. \tag{1.25}$$

The real case is analogous, in particular we also get the semidirect sum in the previous display. Let us note the semidirect sum notation means that the subspaces $E[-1]$ and $E_a[1] \mathbin{\subset\!\!\!\!+} E[-1]$ and quotient spaces $E^A/(E_a[1] \mathbin{\subset\!\!\!\!+} E[-1])$ and $E^A/E[-1]$ of E^A are conformally invariant.

There are other constructions of E^A more suitable for our purpose than the associated bundle $\mathcal{G} \times_P \mathbb{E}^A$. We will prefer the approach whose starting point is the conformal structure as the class of metrics rather then the Cartan bundle $\mathcal{G}$. A choice of a metric provides an isomorphism of E^A with the direct sum $E[1] \oplus E_a[1] \oplus E[-1]$. Our aim is to describe sections of E^A and the tractor connection on E^A for a chosen metric together with their behaviour after rescaling.

One way of the constructing E^A is to define this bundle as the quotient of the jet bundle $J^2(E[1])$ by its smooth subbandle $E_{(ab)_0}[1]$ i.e. by the exact sequence

$$0 \longrightarrow E_{(ab)_0}[1] \longrightarrow J^2(E[1]) \longrightarrow E^A \longrightarrow 0,$$

see [16]. A Levi–Civita connection ∇ from the conformal class provides the isomorphism

$$\begin{aligned} \iota_\nabla : \mathcal{E}^A &\longrightarrow \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1], \text{ by} \\ j^2\sigma &\mapsto (\sigma, \nabla_a\sigma, -(\triangle + P)\sigma) \end{aligned}$$

for $\sigma \in \mathcal{E}[1]$. This is not conformally invariant; denoting the image of $j^2\sigma$ corresponding to ι_∇ and $\iota_{\hat{\nabla}}$ by (σ, μ_a, ρ) and $\widehat{(\sigma, \mu_a, \rho)} = (\hat{\sigma}, \hat{\mu}_a, \hat{\rho})$, respectively, and using the matrix notation, one can easily check the transformation rule

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_p\Upsilon^p & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_a + \Upsilon_a\sigma \\ \rho - \Upsilon_a\mu^a - \frac{1}{2}\Upsilon^2\sigma \end{pmatrix}. \quad (1.26)$$

Another possibility is to define $[E^A]_g = E[1] \oplus E_a[1] \oplus E[-1]$ for each $g \in [g]$ and identify $(\sigma, \mu, \rho) \in [\mathcal{E}^A]_g$ with $(\hat{\sigma}, \hat{\mu}, \hat{\rho}) \in [\mathcal{E}^A]_{\hat{g}}$ by the transformation (1.26). It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the standard tractor bundle E^A over the conformal manifold M.

Let us note all these constructions yield the same bundle E^A (up to isomorphism) due to uniqueness of the normal conformal tractor bundle [15]. Henceforth we shall follow the last one as this suits best for the calculus we will develop in Section 2.1.

The bundle E^A admits an invariant metric h_{AB} and an invariant connection (unique after a proper normalization [15]), which we shall also denote by ∇_a, preserving h_{AB}. This connection induces a connection on $\bigotimes E^A$ (also denoted by ∇_a) and is called *normal tractor connection*. In a conformal scale g, the metric h_{AB} and ∇_a on E^A are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\rho + P_{ab}\sigma \\ \nabla_a \rho - P_{ab}\mu^b \end{pmatrix}. \qquad (1.27)$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. (See [21] for the detailed computation.) Note that h_{AB} defines a section of $E_{AB} = E_A \otimes E_B$, where E_A is the dual bundle of E^A. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of E_A, E^A and their tensor products. Clearly if the conformal structure has signature (p, q) then h_{AB} will have signature $(p+1, q+1)$.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by

$$X_A \in \mathcal{E}_A[1], \quad Z_{Aa} \in \mathcal{E}_{Aa}[1], \quad Y_A \in \mathcal{E}_A[-1],$$

where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ we can raise indices and lower indices and define X^A, Z^{Aa}, Y^A. Then we immediately see that $Y_A X^A = 1$, $Z_{Ab} Z^A{}_c = \boldsymbol{g}_{bc}$ and that all other quadratic combinations that contract the tractor index vanish. In the other words, the metric h_{AB}

defined by (1.27) is the section

$$h_{AB} = Y_A X_A + Z_{Aa} Z_B{}^a + X_A Y_B \in \mathcal{E}_{(AB)}.$$

The sections X, Y and Z give rise to the "XYZ"–calculus for tractor bundles [16, 29, 35] which is less evocative than the matrix notation but more suitable for many computations. We shall describe this for tractor forms. Using X, Y and Z, sections $f_A \in \mathcal{E}_A$ of the form $[f_A]_g = (\sigma, \mu_a, \rho)$ can be expressed as

$$f_A = Y_A \sigma + Z_A^a \mu_a + X_A \rho \tag{1.28}$$

where $\sigma \in \mathcal{E}[1]$, $\mu_a \in \mathcal{E}_a[1]$ and $\rho \in \mathcal{E}[-1]$. Here $[f_A]_g$ denotes $f_A \in \mathcal{E}_A$ for a given choice $g \in [g]$. Since σ, μ_a, ρ will transform according to (1.26) if we change a metric g to $\hat{g} = e^{2\Upsilon} g$ and we require $\hat{f} = f$, the corresponding transformation of X, Y and Z is

$$\hat{Y}_A = Y_A - \Upsilon_b Z_A^b - \frac{1}{2}\Upsilon_b \Upsilon^b X_A, \quad \hat{Z}_A^a = Z_A^a + \Upsilon^a X_A, \quad \hat{X}_A = X_A. \tag{1.29}$$

Comparing (1.28) with the form of $\nabla_a f_A$ given by (1.27), we immediately get

$$\nabla_a Y_A = P_{ab} Z_A^b, \quad \nabla_a Z_A^b = -P_a{}^b X_A - \delta_a^b Y_A, \quad \nabla_a X_A = Z_{Aa}. \tag{1.30}$$

Since the tractor connection ∇_a is invariant on $\mathcal{E}^A$, its curvature called *tractor curvature* $\Omega_{abCD} \in \mathcal{E}_{[ab][CD]}$ given by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) f^C = \Omega_{ab}{}^C{}_D f^D$$

for $f^C \in \mathcal{E}^C$, is invariant as well. Using (1.27) or (1.30) and the formulae for the Riemannian curvature, a direct computation yields the curvature tensor

$$\Omega_{abCD} = \mathbb{Z}_{CD}^{c\,d} C_{abcd} - 2\mathbb{X}_{CD}^{\ d} A_{dab}. \tag{1.31}$$

The tractor connection ∇ cannot be iterated in conformally invariant way i.e. $\nabla_a \nabla_b f_{\mathfrak{B}}$ for $f_{\mathfrak{B}} \in \mathcal{E}_{\mathfrak{B}}[w]$ is not, in general, invariant. (∇_a is not invariant on E_b therefore also not invariant on $E_{b\mathfrak{B}}$ etc.) However, we have a 2nd order operator which partly addresses this problem, called *tractor D–operator*

$$D_A : \mathcal{E}_{\mathfrak{B}}[w] \longrightarrow \mathcal{E}_{A\mathfrak{B}}[w-1] \tag{1.32}$$

$$D_A f_{\mathfrak{B}} = w(n+2w-2) Y_A f_{\mathfrak{B}} + (n+2w-2) Z_A^a \nabla_a f_{\mathfrak{B}} - X_A(\triangle + wP) f_{\mathfrak{B}}$$

The definition (1.32) is for a given choice of the conformal scale but actually D_A is conformally invariant as can be checked directly using the formulae for rescaling (1.29) above. It is sufficient to do this for densities (i.e. when $\mathfrak{B} = \emptyset$), see [21] for a detailed computation. The general case follows from the observation that the relation between $\hat{\nabla}_a f_{\mathfrak{B}}$ and $\nabla_a f_{\mathfrak{B}}$ is formally the same as for densities and that one does not need to commute ∇'s in the calculation of invariance. (Let us note there are also conformally invariant constructions of D_A, see e.g. [30].) Furthermore, since $D_A h_{BC} = 0$, the tractor D–operator commutes with the raising and lowering of indices. We can consider also the commutator $(D_A D_B - D_B D_A) f$ for $f \in \mathcal{E}_{\mathfrak{B}}[w]$. The result is of the first order and depends on w, see [35] for details.

We can demonstrate the notion of projecting parts and its importance for invariant operators. It is clear from the definition of D_A which is given in the matrix notation by

$$D_A f_{\mathfrak{F}} = \begin{pmatrix} w(n+2w-2) f_{\mathfrak{F}} \\ (n+2w-2) \nabla_a f_{\mathfrak{F}} \\ -(\triangle + wP) f_{\mathfrak{F}} \end{pmatrix}$$

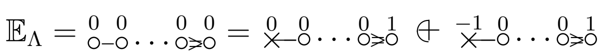
that $Y_A : \mathcal{E}_{\mathfrak{F}}[w] \longrightarrow \mathcal{E}_{A\mathfrak{F}}[w-1]$ is a projecting part of $D_A f_{\mathfrak{F}}$ and the projection yields a nonzero multiple of $f_{\mathfrak{F}}$ if $w \neq 0$ and $w \neq \frac{2-n}{2}$. For $w = 0$, Z_A^a is a projecting part and the projection yields $\nabla_a f_{\mathfrak{F}}$. If $w = \frac{2-n}{2}$ then X_A will be a projecting part. The corresponding projection shows invariance of the operator

$$\Box f_{\mathfrak{F}} := \triangle f_{\mathfrak{F}} + wPf_{\mathfrak{F}}$$

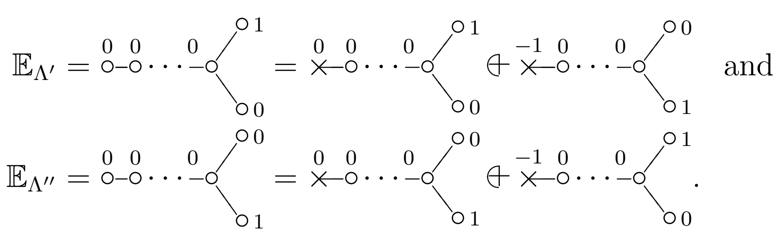
for $f_{\mathfrak{F}} \in \mathcal{E}_{\mathfrak{F}}[1-n/2]$ i.e. invariance of tractor twisted conformal Laplacian.

1.2.4. Spinor tractor bundle. Now we describe tractor spinors and develop an equivalent of the "XYZ"–calculus from 1.2.3. We shall mostly follow [8] but adapt the notation therein to be consistent with the conventions used in the thesis.

As in the case of the standard tractor bundle, there are several ways to define the spinor tractor bundle E^{Λ} for the spin conformal structure $(M, \boldsymbol{g}, \boldsymbol{\beta}, \boldsymbol{\epsilon})$ of signature (p,q). We denote tractor spinor indices by greek capitals. From the Cartan bundle point of view we have $E^{\Lambda} := \mathcal{G} \times_P \mathbb{E}^{\Lambda}$ where $P \subseteq G = Spin_{p+1,q+1}$ is the corresponding parabolic subgroup, $\mathcal{G}$ is the Cartan bundle and $\mathbb{E}^{\Lambda}$ the spinor representation of G. We shall consider directly the dual $E_{\Lambda} = (E^{\Lambda})^*$ because we prefer the form notation and actually $\mathbb{E}_{\Lambda} \cong \mathbb{E}^{\Lambda}$. To describe $\mathbb{E}_{\Lambda}$, let us pass to the algebra level and to the complex setting first. Then $\mathbb{E}_{\Lambda}$ is a $\mathfrak{g} = \mathfrak{so}_{n+2}(\mathbb{C})$–module and we have the structure $\mathfrak{g}_0 \subseteq \mathfrak{p} \subseteq \mathfrak{g}$, see (1.2) for details. Considering $\mathbb{E}_{\Lambda}$ as a $\mathfrak{g}_0$–module, we obtain the decomposition to $\mathfrak{g}_0$–irreducibles and the semidirect ($\mathfrak{p}$–) structure on $\mathbb{E}_{\Lambda}$:



for n odd and $\mathbb{E}_{\Lambda} = \mathbb{E}_{\Lambda'} \oplus \mathbb{E}_{\Lambda''}$ for n even where



We will use the primed and two–primed tractor spinor indices in a similar way as for spinor ones but we will mostly work with the whole bundle E_Λ. One can derive the decomposition in the real case from the complex one [42, 48]. The details depend on the signature, $\mathfrak{g}$–irreducible modules are $\mathbb{E}_\Lambda$ (for n even such that $n'-p$ is odd and for n odd) and $\mathbb{E}_{\Lambda'}$ and $\mathbb{E}_{\Lambda''}$ (for n even such that $n'-p$ is even). These $\mathfrak{g}$–irreducible modules are always semidirect sum of two $\mathfrak{g}_0$–component. Summarizing, we have

$$E_\Lambda = E_\lambda[1] \oplus\!\!\!\!+ E_\lambda \tag{1.33}$$

(in all cases) in the index notation. The normal Cartan connection on $\mathcal{G}$ yields an invariant connection on E_Λ.

Another way to construct E_Λ is to define this bundle as the quotient of the jet bundle $J^1(E_\lambda[1])$ by its smooth subbandle $E_{|a\lambda|_0}[1] := E(\frac{1}{2};1)_0[1]$. That is, $|..|_0$ denotes the "Clifford free" part of $E_{a\lambda}[1]$. (This means, β^a vanishes for all sections of $E_{|a\lambda|_0}[1]$.) Let us note the operator $\sigma_\lambda \mapsto \nabla_{|a}\sigma_{\lambda|_0}$ is conformally invariant for $\sigma_\lambda \in \mathcal{E}_\lambda[1]$, cf. (1.21). E_Λ is defined by the exact sequence

$$0 \longrightarrow E_{|a\lambda|_0}[1] \longrightarrow J^1(E_\lambda[1]) \longrightarrow E_\Lambda \longrightarrow 0.$$

A Levi–Civita connection ∇ from the conformal class provides the isomorphism

$$\begin{array}{rcl} \iota_\nabla : E_\Lambda & \longrightarrow & E_\lambda[1] \oplus E_\lambda, \text{ by} \\ j^1\sigma & \mapsto & (\sigma, \frac{2}{n}\boldsymbol{\beta}^p\nabla_p\sigma) \end{array}$$

for $\sigma \in \mathcal{E}_\lambda[1]$. This is not conformally invariant; denoting the image of $j^1\sigma$ corresponding to ι_∇ and $\iota_{\hat{\nabla}}$ by (σ,τ) and $\widehat{(\sigma,\tau)} = (\hat{\sigma},\hat{\tau})$, respectively, and using the vertical notation, one can easily check (using (1.21)) the transformation rule

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\tau} \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ \Upsilon^p \boldsymbol{\beta}_p & \mathrm{id} \end{pmatrix} \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} \sigma \\ \tau + \Upsilon^p \boldsymbol{\beta}_p \sigma \end{pmatrix}. \tag{1.34}$$

Following this, we put $[E_\Lambda]_g := E_\lambda[1] \oplus E_\lambda$ for every metric $g \in [g]$ and define E_Λ as the equivalence class of $[E_\Lambda]_g$ in the following way. We identify $(\sigma, \tau) \in [\mathcal{E}_\Lambda]_g$ with $(\hat{\sigma}, \hat{\tau}) \in [\mathcal{E}_\Lambda]_{\hat{g}}$ by the transformation (1.34). It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the spinor tractor bundle E_Λ over the spin conformal manifold M.

The bundle E_Λ admits an invariant connection (unique after a proper normalization [15]), which we shall denote by ∇_a. In a conformal scale g, the connection ∇_a on E_Λ is given by

$$\nabla_a \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma + \boldsymbol{\beta}_a \tau \\ \nabla_a \tau + P_a{}^p \boldsymbol{\beta}_p \sigma \end{pmatrix} \tag{1.35}$$

[8]. It is readily verified that this is conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$.

Now we want to introduce spinor–tractor analogues of the tractors X_A, Z_A^a and Y_A we have for the standard tractor bundle. We define the projections

$$Y = Y_\Lambda^\lambda \in \mathcal{E}_\Lambda^\lambda[-1] = \mathrm{Hom}\,(\mathcal{E}_\lambda[1], \mathcal{E}_\Lambda) \quad \text{and} \quad X = X_\Lambda^\lambda \in \mathcal{E}_\Lambda^\lambda = \mathrm{Hom}\,(\mathcal{E}_\lambda, \mathcal{E}_\Lambda)$$

by the relation

$$f = Y\sigma + X\tau = Y_\Lambda^\lambda \sigma_\lambda + X_\Lambda^\lambda \tau_\lambda \quad \text{for every } f \in \mathcal{E}_\Lambda \tag{1.36}$$

where $[f]_g = (\sigma, \tau) \in \mathcal{E}_\lambda[1] \oplus \mathcal{E}_\lambda$ denotes f in a metric $g \in [g]$. That is, X and Y are defined for a fixed metric g. Since σ and τ transform according to (1.34) if we change the metric g to $\hat{g}$ and $\hat{f} = f$, the corresponding transformation of X and Y is

$$\hat{Y} = Y - \Upsilon^p X \boldsymbol{\beta}_p, \quad \hat{X}_A = X_A. \tag{1.37}$$

Comparing (1.36) with the form of $\nabla_a f$ given by (1.35), we see that

$$\nabla_a Y = P_a{}^p X \boldsymbol{\beta}_p, \quad \nabla_a X = Y \boldsymbol{\beta}_a. \tag{1.38}$$

The conformally invariant tractor D–operator for spinors $D = D^\lambda_\Lambda$: $\mathcal{E}_\lambda[w] \longrightarrow \mathcal{E}_\Lambda[w-1]$ is defined by

$$Df = \begin{pmatrix} (n+2w-2)f \\ 2\boldsymbol{\beta}^p \nabla_p f \end{pmatrix} = (n+2w-2)Yf + 2X\boldsymbol{\beta}^p \nabla_p f \tag{1.39}$$

for $f \in \mathcal{E}_\lambda[w]$ [8]. More generally, this is a conformally invariant operator $D : \mathcal{E}_{\lambda\mathfrak{B}}[w] \longrightarrow \mathcal{E}_{\Lambda\mathfrak{B}}[w-1]$ for a system of tractor indices $\mathfrak{B}$.

We have also tractor analogues of $\boldsymbol{\beta}_a$ and $\boldsymbol{\varepsilon}^{\lambda\gamma}$. These are the tractor Clifford section and the tractor spinor inner product

$$\boldsymbol{\beta}_A = \boldsymbol{\beta}_{A\Lambda}{}^\Gamma \in \mathcal{E}_A \otimes \mathrm{End}\,(\mathcal{E}_\Lambda) \quad \text{and} \quad \boldsymbol{\varepsilon}^{\Lambda\Gamma} \in \mathcal{E}^{\Lambda\Gamma},$$

respectively, such that $\boldsymbol{\beta}_A \boldsymbol{\beta}_B + \boldsymbol{\beta}_B \boldsymbol{\beta}_A = -h_{AB}\mathrm{id}$ and $\nabla_a \boldsymbol{\beta}_A = \nabla_a \boldsymbol{\varepsilon}^{\Lambda\Gamma} = 0$. We will use them only rarely (so there will no danger of confusion with $\boldsymbol{\beta}_a$ and $\boldsymbol{\varepsilon}^{\lambda\gamma}$) as we actually need mainly their existence. This is established in [8] for $\boldsymbol{\beta}_A$ and then, existence of $\boldsymbol{\varepsilon}^{\Lambda\Gamma}$ follows by theoretical means [12, 54]. We will use $\boldsymbol{\varepsilon}^{\Lambda\Gamma}$ and its dual $\boldsymbol{\varepsilon}_{\Lambda\Gamma}$ to raise and lower tractor spinor indices. It will be convenient later to use the conventions *opposite* to (1.23) i.e.

$$f^\Lambda := \boldsymbol{\varepsilon}^{\Omega\Lambda} f_\Omega \quad \text{and} \quad \bar{f}_\Lambda := \bar{f}^\Omega \boldsymbol{\varepsilon}_{\Lambda\Omega}. \tag{1.40}$$

In the complex setting, we know exactly whether $\boldsymbol{\varepsilon}^{\lambda\gamma}$ and $\boldsymbol{\beta}_a{}^{\lambda\gamma}$ are symmetric or skew (depending on $[n']_4$), see 1.2.1. Using this we can describe $\boldsymbol{\varepsilon}^{\Lambda\Omega}$ explicitly. First, denoting by $\langle \sigma, \sigma' \rangle := \boldsymbol{\varepsilon}^{\lambda\gamma} \sigma_\lambda \sigma'_\gamma$ the spinor inner product for $\sigma, \sigma' \in \mathcal{E}_\lambda$ we define the *formal adjoint* $\boldsymbol{\beta}_a^* \in \mathcal{E}_a \otimes \mathrm{End}\ (\mathcal{E}_\lambda)[1]$ of $\boldsymbol{\beta}_a$ by the relation

$$\forall \sigma, \sigma' \in \mathcal{E}_\lambda : \ \langle \boldsymbol{\beta}_a \sigma, \sigma' \rangle = \langle \sigma, \boldsymbol{\beta}_a^* \sigma' \rangle. \tag{1.41}$$

From this and (skew)symmetry of ${\boldsymbol{\beta}_a}^{\lambda\gamma}$ and $\boldsymbol{\varepsilon}^{\lambda\gamma}$, it is easily checked that $\boldsymbol{\beta}_a^* = (-1)^{n'} \boldsymbol{\beta}_a$.

Now we define $\boldsymbol{\varepsilon}^{\Lambda\Gamma}$, denoted also by $\langle f, f' \rangle := \boldsymbol{\varepsilon}^{\Lambda\Gamma} f_\Lambda f'_\Gamma$ for $f, f' \in \mathcal{E}_\Lambda$ in the index free notation. We put

$$\boldsymbol{\varepsilon}^{\Lambda\Gamma} = \begin{pmatrix} 0 & \boldsymbol{\varepsilon}^{\lambda\gamma} \\ (-1)^{n'+1}\boldsymbol{\varepsilon}^{\lambda\gamma} & 0 \end{pmatrix} \quad \text{i.e. } \langle \begin{pmatrix} \sigma \\ \tau \end{pmatrix}, \begin{pmatrix} \sigma' \\ \tau' \end{pmatrix} \rangle = (-1)^{n'+1} \langle \tau, \sigma' \rangle + \langle \sigma, \tau' \rangle. \tag{1.42}$$

One easily computes that the conformal transformation for metrics $g, \hat{g} \in [g]$ of the second expression yields the term $\Upsilon^p \big((-1)^{n'+1} \langle \boldsymbol{\beta}_p \sigma, \sigma' \rangle + \langle \sigma, \boldsymbol{\beta}_p \sigma' \rangle \big)$ using (1.37). But the latter vanishes using (1.41) because $\beta_a^* = (-1)^{n'} \beta_a$. Let us note that $\boldsymbol{\varepsilon}^{\Lambda\Gamma}$ symmetric for $[n'+1]_4 \in \{[0]_4, [3]_4\}$ and skew for $[n'+1]_4 \in \{[1]_4, [2]_4\}$, cf. with $\boldsymbol{\varepsilon}^{\lambda\gamma}$ in 1.2.1. Further (1.40) yields the projectors $Y^{\Lambda\lambda} \in \mathcal{E}^{\Lambda\lambda}[-1]$ and $X^{\Lambda\lambda} \in \mathcal{E}^{\Lambda\lambda}$. They satisfy $X^{\Lambda\lambda} Y_\Lambda^\gamma = (-1)^{n'+1} \boldsymbol{\varepsilon}^{\lambda\gamma}$ and $Y^{\Lambda\lambda} X_\Lambda^\gamma = \boldsymbol{\varepsilon}^{\lambda\gamma}$. Hence

$$\boldsymbol{\varepsilon}^{\Lambda\Gamma} = Y_\lambda^\Lambda X^{\Gamma\lambda} + (-1)^{n'+1} X_\lambda^\Lambda Y^{\Gamma\lambda} \in \mathcal{E}^{\Lambda\Gamma}. \tag{1.43}$$

1.2.5. Tractor forms. This was pioneered in [11] but we will use a modification of the notation therein. Following the tractor calculus for E_A from 1.2.3, we will prepare similar tools for the bundle

$$T^k = E_{\mathbf{A}^{k+1}} := \mathcal{G} \times_P \mathbb{E}_{\mathbf{A}^{k+1}} \quad \text{where} \quad \mathbb{T}^k = \mathbb{E}_{\mathbf{A}^{k+1}} := \bigwedge^{k+1} \mathbb{E}_A, \tag{1.44}$$

$0 \leq k \leq n'$ now. This notation is designed so that the superscript in T^k corresponds to the tensor valence of the top slot and this will be convenient later. It follows from the semidirect composition series of $\mathcal{E}_A$ that the corresponding decomposition of $\mathcal{E}_{\mathbf{A}^k}$ is

$$\mathcal{E}_{\mathbf{A}^k} = \mathcal{E}_{[A^1 \cdots A^k]} \simeq \mathcal{E}^{k-1}[k] \oplus \Big(\mathcal{E}^k[k] \oplus \mathcal{E}^{k-2}[k-2]\Big) \oplus \mathcal{E}^{k-1}[k-2], \qquad (1.45)$$

see [11]. Given a choice of metric g from the conformal class this determines a splitting of this space into four components (a replacement of the $\oplus$s with $\oplus$s is effected) and the projectors (or splitting operators) X, Y, Z for $\mathcal{E}_A$ determine corresponding projectors $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}$ for $\mathcal{E}_{\mathbf{A}^{k+1}}$, $k \geq 1$ as follows.

$$\begin{aligned}
\mathbb{Y}^k &= \mathbb{Y}_{A^0 A^1 \cdots A^k}{}^{a^1 \cdots a^k} &&= \mathbb{Y}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} &&= Y_{A^0} Z_{A^1}{}^{a^1} \cdots Z_{A^k}{}^{a^k} &&\in \mathcal{E}_{\mathbf{A}^{k+1}}^{\mathbf{a}^k}[-k-1] \\
\mathbb{Z}^k &= \mathbb{Z}_{A^1 \cdots A^k}{}^{a^1 \cdots a^k} &&= \mathbb{Z}_{\mathbf{A}^k}{}^{\mathbf{a}^k} &&= Z_{A^1}{}^{a^1} \cdots Z_{A^k}{}^{a^k} &&\in \mathcal{E}_{\mathbf{A}^k}^{\mathbf{a}^k}[-k] \\
\mathbb{W}^k &= \mathbb{W}_{A' A^0 A^1 \cdots A^k}{}^{a^1 \cdots a^k} &&= \mathbb{W}_{A' A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} &&= X_{[A'} Y_{A^0} Z_{A^1}{}^{a^1} \cdots Z_{A^k]}{}^{a^k} &&\in \mathcal{E}_{\mathbf{A}^{k+2}}^{\mathbf{a}^k}[-k] \\
\mathbb{X}^k &= \mathbb{X}_{A^0 A^1 \cdots A^k}{}^{a^1 \cdots a^k} &&= \mathbb{X}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} &&= X_{A^0} Z_{A^1}{}^{a^1} \cdots Z_{A^k}{}^{a^k} &&\in \mathcal{E}_{\mathbf{A}^{k+1}}^{\mathbf{a}^k}[-k+1]
\end{aligned} \qquad (1.46)$$

where $k \geq 0$. The superscript k in $\mathbb{Y}^k$, $\mathbb{Z}^k$, $\mathbb{W}^k$ and $\mathbb{X}^k$ shows always the corresponding tensor valence. (This is slightly different than in [11], where k concerns the tractor valence.) Note that $Y = \mathbb{Y}^0$, $Z = \mathbb{Z}^1$ and $X = \mathbb{X}^0$ and $\mathbb{W}^0 = X_{[A'} Y_{A^0]}$. Using these projectors, a section $f_{\mathbf{A}^{k+1}} \in \mathcal{E}_{\mathbf{A}^{k+1}}$ written as a 4-tuple $[f_{\mathbf{A}^{k+1}}]_g = \left(\begin{smallmatrix} & \sigma & \\ \mu & & \varphi \\ & \rho & \end{smallmatrix} \right)$ for a metric $g \in [g]$ is of the form

$$f_{\mathbf{A}^{k+1}} = \mathbb{Y}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} \sigma_{\mathbf{a}^k} + \mathbb{Z}_{A^0 \mathbf{A}^k}{}^{a_0 \mathbf{a}^k} \mu_{a_0 \mathbf{a}^k} + \mathbb{W}_{A^0 \mathbf{A}^k}{}^{\dot{\mathbf{a}}^k} \varphi_{\dot{\mathbf{a}}^k} + \mathbb{X}_{A^0 \mathbf{A}^k}{}^{\mathbf{a}^k} \rho_{\mathbf{a}^k}$$

for forms $\sigma, \mu, \varphi, \rho$ of weight and valence according to the relationship given in (1.45).

The conformal transformation (1.29) yields the transformation formulae for the projectors:

$$
\begin{aligned}
\widehat{\mathbb{Y}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k}} =& \mathbb{Y}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} - \Upsilon_{a^0}\mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} - k\Upsilon^{a^1}\mathbb{W}_{A^0\mathbf{A}^k}^{\ \ \dot{\mathbf{a}}^k} \\
& - \frac{1}{2}\Upsilon^k\Upsilon_k\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} + k\Upsilon_p\Upsilon^{a^1}\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ p\dot{\mathbf{a}}^k} \\
\widehat{\mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k}} =& \mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} + (k+1)\Upsilon^{a^0}\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} \\
\widehat{\mathbb{W}_{A^0\mathbf{A}^k}^{\ \ \dot{\mathbf{a}}^k}} =& \mathbb{W}_{A^0\mathbf{A}^k}^{\ \ \dot{\mathbf{a}}^k} - \Upsilon_{a^1}\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} \\
\widehat{\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k}} =& \mathbb{X}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k}
\end{aligned}
\tag{1.47}
$$

for metrics $\hat{g}$ and g from the conformal class. The normal tractor connection on $(k+1)$-form-tractors is

$$
\nabla_p \begin{pmatrix} \sigma_{\mathbf{a}^k} \\ \mu_{a^0\mathbf{a}^k} \ \ \varphi_{\dot{\mathbf{a}}^k} \\ \rho_{\mathbf{a}^k} \end{pmatrix} = \begin{pmatrix} \nabla_p\sigma_{\mathbf{a}^k} - (k+1)\mu_{p\mathbf{a}^k} - \boldsymbol{g}_{pa^1}\varphi_{\dot{\mathbf{a}}^k} \\ \left\{ \begin{smallmatrix} \nabla_p\mu_{a^0\mathbf{a}^k} \\ +P_{pa^0}\sigma_{\mathbf{a}^k} + g_{pa^0}\rho_{\mathbf{a}^k} \end{smallmatrix} \right\} \quad \left\{ \begin{smallmatrix} \nabla_p\varphi_{\dot{\mathbf{a}}^k} \\ +kP_p^{\ a^1}\sigma_{\mathbf{a}^k} - k\delta_p^{a^1}\rho_{\mathbf{a}^k} \end{smallmatrix} \right\} \\ \nabla_p\rho_{\mathbf{a}^k} - (k+1)P_p^{\ a^0}\mu_{a^0\mathbf{a}^k} + P_{pa^1}\varphi_{\dot{\mathbf{a}}^k} \end{pmatrix} \tag{1.48}
$$

or equivalently

$$
\begin{aligned}
\nabla_p\mathbb{Y}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} =& P_{pa_0}\mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} + kP_p^{\ a^1}\mathbb{W}_{A^0\mathbf{A}^k}^{\ \ \dot{\mathbf{a}}^k} \\
\nabla_p\mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} =& -(k+1)\delta_p^{a^0}\mathbb{Y}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} - (k+1)P_p^{\ a^0}\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} \\
\nabla_p\mathbb{W}_{A^0\mathbf{A}^k}^{\ \ \dot{\mathbf{a}}^k} =& -\boldsymbol{g}_{pa^1}\mathbb{Y}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} + P_{pa^1}\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ a^1\dot{\mathbf{a}}^k} \\
\nabla_p\mathbb{X}_{A^0\mathbf{A}^k}^{\ \ \mathbf{a}^k} =& \boldsymbol{g}_{pa^0}\mathbb{Z}_{A^0\mathbf{A}^k}^{a^0\,\mathbf{a}^k} - k\delta_p^{a^1}\mathbb{W}_{A^0\mathbf{A}^k}^{\ \ \dot{\mathbf{a}}^k}
\end{aligned}
\tag{1.49}
$$

where the sequentially labelled indices at the same level are skew over i.e. $[a^1\dot{\mathbf{a}}^k]$ or $[a^0\mathbf{a}^k]$ in the last display.

For later use we need also the formulae

$$\begin{aligned}
\Delta \mathbb{Y}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\sigma_{\mathbf{a}^k} =& \mathbb{Y}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}(\Delta - P)\sigma_{\mathbf{a}^k} + \mathbb{Z}_{A_0\mathbf{A}^k}^{a^0\,\mathbf{a}^k}\big[(\nabla_{a_0}P) + 2P_{a_0}{}^p\nabla_p\big]\sigma_{\mathbf{a}^k} \\
&+ k\mathbb{W}_{A_0\mathbf{A}^k}{}^{\dot{\mathbf{a}}^k}\big[(\nabla^{a_1}P) + 2P^{a_1p}\nabla_p\big]\sigma_{\mathbf{a}^k} \\
&+ \mathbb{X}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\big[2kP_{a_1}{}^pP_p{}^q\sigma_{q\dot{\mathbf{a}}^k} - P^{pq}P_{pq}\sigma_{\mathbf{a}^k}\big] \\
\Delta \mathbb{Z}_{A_0\mathbf{A}^k}^{a^0\,\mathbf{a}^k}\mu_{a^0\mathbf{a}^k} =& -2(k+1)\mathbb{Y}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\nabla^{a^0}\mu_{a^0\mathbf{a}^k} \\
&+ \mathbb{Z}_{A_0\mathbf{A}^k}^{a^0\,\mathbf{a}^k}\big[\Delta\mu_{a^0\mathbf{a}^k} - 2(k+1)P_{a^0}{}^p\mu_{p\mathbf{a}^k}\big] \\
&- (k+1)\mathbb{X}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\big[(\nabla^pP) + 2P^{pq}\nabla_q\big]\mu_{p\mathbf{a}^k} \qquad (1.50) \\
\Delta \mathbb{W}_{A_0A^1\dot{\mathbf{A}}^k}{}^{\dot{\mathbf{a}}^k}\nu_{\dot{\mathbf{a}}^k} =& -2\mathbb{Y}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\nabla_{a^1}\nu_{\dot{\mathbf{a}}^k} \\
&+ \mathbb{W}_{A_0A^1\dot{\mathbf{A}}^k}{}^{\dot{\mathbf{a}}^k}\big[(\Delta - 2P)\nu_{\dot{\mathbf{a}}^k} + 2(k-1)P_{a^2}{}^p\nu_{p\ddot{\mathbf{a}}^k}\big] \\
&+ \mathbb{X}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\big[(\nabla_{a^1}P) - 2P_{a^1}{}^p\nabla_p\big]\nu_{\dot{\mathbf{a}}^k} \\
\Delta \mathbb{X}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\rho_{\mathbf{a}^k} =& (-n+2k)\mathbb{Y}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\rho_{\mathbf{a}^k} + 2\mathbb{Z}_{A_0\mathbf{A}^k}^{a^0\,\mathbf{a}^k}\nabla_{a_0}\rho_{\mathbf{a}^k} \\
&- 2k\mathbb{W}_{A_0\mathbf{A}^k}{}^{\dot{\mathbf{a}}^k}\nabla^{a^1}\rho_{\mathbf{a}^k} + \mathbb{X}_{A_0\mathbf{A}^k}{}^{\mathbf{a}^k}\big[\Delta - P\big]\rho_{\mathbf{a}^k}
\end{aligned}$$

The volume form $\boldsymbol{\epsilon}_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^n}[n]$ determines the conformally invariant *tractor volume form*

$$\boldsymbol{\epsilon}_{\mathbf{A}^{n+2}} := \mathbb{W}_{A^1A^2\ddot{\mathbf{A}}^{n+2}}{}^{\mathbf{c}}\boldsymbol{\epsilon}_{\mathbf{c}} \in \mathcal{E}_{\mathbf{A}^{n+2}}.$$

This satisfies $\boldsymbol{\epsilon}_{\mathbf{A}}\boldsymbol{\epsilon}^{\mathbf{A}} = (n+2)!$. Using the tractor volume form, we can decompose the bundle $\mathbb{E}_{\mathbf{B}^{n'+1}}$ into two eigenspaces of the appropriate action of $\boldsymbol{\epsilon}_{\mathbf{A}}$ if $n = 2n'$ and $n' - p$ is even. Here (p,q) is signature of the conformal structure.

Beside the form tractor bundle $\mathcal{E}_{\mathbf{A}^{k+1}}$, it will be convenient to introduce also invariant quotient spaces and invariant subspaces of this bundle. Their existence is visible form the composition series (1.45). For example, the subbundle

$$(\mathcal{E}_{\mathbb{X}})_{\mathbf{A}^{k+1}} := \Big\{ f_{\mathbf{A}^{k+1}} \in \mathcal{E}_{\mathbf{A}^{k+1}} \mid X_{[A^0}f_{\mathbf{A}^{k+1}]} = X^{A^1}f_{\mathbf{A}^{k+1}} = 0 \Big\} \subseteq \mathcal{E}_{\mathbf{A}^{k+1}}$$

Invariant subspaces of $\mathcal{E}_{\mathbf{A}^{k+1}}$ and $\mathcal{E}_\Lambda$			
Notation	**Invariant condition for** $f_{\mathbf{A}^{k+1}} \in \mathcal{E}_{\mathbf{A}^{k+1}}$ **or** $f_\Lambda \in \mathcal{E}_\Lambda$	**Composition series**	**Figure**
$(\mathcal{E}_{\mathbb{X}})_{\mathbf{A}^{k+1}}$	$X_{[A^0} f_{\mathbf{A}^{k+1}]} = X^{A^1} f_{\mathbf{A}^{k+1}} = 0$	$\mathcal{E}^k[k-1]$	$\left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & * & \end{smallmatrix}\right)$
$(\mathcal{E}_{\mathbb{XZ}})_{\mathbf{A}^{k+1}}$	$X^{A^1} f_{\mathbf{A}^{k+1}} = 0$	$\begin{array}{c} \mathcal{E}^{k+1}[k+1] \\ \overset{\frown}{+} \\ \mathcal{E}^k[k-1] \end{array}$	$\left(\begin{smallmatrix} & 0 & \\ * & & 0 \\ & * & \end{smallmatrix}\right)$
$(\mathcal{E}_{\mathbb{XW}})_{\mathbf{A}^{k+1}}$	$X_{[A^0} f_{\mathbf{A}^{k+1}]} = 0$	$\begin{array}{c} \mathcal{E}^{k-1}[k-1] \\ \overset{\frown}{+} \\ \mathcal{E}^k[k-1] \end{array}$	$\left(\begin{smallmatrix} & 0 & \\ 0 & & * \\ & * & \end{smallmatrix}\right)$
$(\mathcal{E}_X)_\Lambda$	$X^\gamma_{[\Gamma} f_{\Lambda]} = 0$	$\mathcal{E}_\lambda$	$\left(\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right)$

Invariant quotient spaces of $\mathcal{E}_{\mathbf{A}^{k+1}}$ and $\mathcal{E}_\Lambda$			
Notation	**Invariant definition**	**Composition series**	**Figure**
$(\mathcal{E}_{\mathbb{Y}})_{\mathbf{A}^{k+1}}$	$\mathcal{E}_{\mathbf{A}^{k+1}}/((\mathcal{E}_{\mathbb{XZ}})_{\mathbf{A}^{k+1}} \oplus (\mathcal{E}_{\mathbb{XW}})_{\mathbf{A}^{k+1}})$	$\mathcal{E}^k[k+1]$	$\left(\begin{smallmatrix} & * & \\ - & & - \\ & - & \end{smallmatrix}\right)$
$(\mathcal{E}_{\mathbb{YZ}})_{\mathbf{A}^{k+1}}$	$\mathcal{E}_{\mathbf{A}^{k+1}}/(\mathcal{E}_{\mathbb{XW}})_{\mathbf{A}^{k+1}}$	$\begin{array}{c} \mathcal{E}^k[k+1] \\ \overset{\frown}{+} \\ \mathcal{E}^{k+1}[k+1] \end{array}$	$\left(\begin{smallmatrix} & * & \\ * & & - \\ & - & \end{smallmatrix}\right)$
$(\mathcal{E}_{\mathbb{YW}})_{\mathbf{A}^{k+1}}$	$\mathcal{E}_{\mathbf{A}^{k+1}}/(\mathcal{E}_{\mathbb{XZ}})_{\mathbf{A}^{k+1}}$	$\begin{array}{c} \mathcal{E}^k[k+1] \\ \overset{\frown}{+} \\ \mathcal{E}^{k-1}[k-1] \end{array}$	$\left(\begin{smallmatrix} & * & \\ - & & * \\ & - & \end{smallmatrix}\right)$
$(\mathcal{E}_Y)_\Lambda$	$\mathcal{E}_\Lambda/(\mathcal{E}_X)_\Lambda$	$\mathcal{E}_\lambda[1]$	$\left(\begin{smallmatrix} * \\ - \end{smallmatrix}\right)$

Table 1.4: Invariant substructures of $\mathcal{E}_{\mathbf{A}^{k+1}}$ and $\mathcal{E}_\Lambda$.

is conformally invariant. Other possibilities are shown in Table 1.4 on p. 45 including spinor tractors. The last column shows a schematic description in the matrix notation. Here $*$ means "arbitrary form of admissible valence and weight" and - indicates an invariant subspace in the definition of invariant quotient spaces. Obviously, sections of these bundles can be described via an appropriate modification of transformation rules for $\mathbb{X}, \mathbb{Y}, \mathbb{W}$ and $\mathbb{Z}$. Let us note that we do not have an invariant connection on these bundles.

1.2.6. Notation for $\mathfrak{g}_0$–components. Given a choice $g \in [g]$ of the metric, we have defined a $\mathfrak{g}_0$–component pr of a tensor/tractor bundle V as the homomorphism $pr : W \hookrightarrow V$ for a $\mathfrak{g}_0$–subbundle $W \subseteq V$, see in 1.2.2. Here we introduce a notation for $\mathfrak{g}_0$–components based on the $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$–notation for form tractor bundles and a similar X, Y–notation for the tractor–spinor bundle.

The $\mathfrak{g}_0$–components of sections of tensor/tractor bundles have certain features which sometimes give rise to properties of the full tractor section or operator. For example, the tractor D–operator on $\mathcal{E}[w]$ is

$$D_A = c_1 Y_A + c_2 Z_A^a \nabla_a - X_A(\Delta + wP) \tag{1.51}$$

for appropriate scalars c_1 and c_2. The three summands of D_A are Y_A, $Z_A^a \nabla_a$ and $X_A(\Delta + wP)$ up to scalar multiples. We can observe the sum of the homogenity of Y_A, Z_A^a and X_A, which is $+1$, 0 and -1, respectively, plus the order of the corresponding operator which is id, ∇_a and $(\Delta + wP)$, respectively, is equal to 1 for all three slots. That is, the sum is an invariant quantity for D_A. We can also consider the tensor valence of these three slots which is 0, 1 and 0, respectively. The purpose of this section is to define these quantities for more complicated tensor/tractor bundles and operators.

We will usually consider tractor operators in the form

$$\Phi = \sum_{i \in I} c_i pr_i \Phi_i, \quad \text{where} \quad \Phi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2 \tag{1.52}$$

where Φ_i is a tensor operator, c_i a scalar, pr_i is a string of $\mathbb{X}$, $\mathbb{Y}$ and I a finite set, cf. (1.51). We will need this notation especially for bundles V_1 and V_2 which are tensor products of form tractor bundles.

We apply tools developed below in crucial constructions in 2.1.5, in particular in the Theorem therein. But let us note these tools are merely technical (this section is indeed rather technical) and necessary for proofs but not for the formulation of main results.

We will call a bundle U *natural* if U lives in a tensor product of $E[w]$, E_a, E_λ, E_A, E_Λ and their duals. In particular, a bundle $E_{\mathfrak{T}}[w]$ is natural for any system of indices $\mathfrak{T}$. Let us consider systems $\mathfrak{A}$ of tractor/spinor tractor indices and $\mathfrak{a}$ of tensor/spinor indices. The number of indices in $\mathfrak{a}$ and $\mathfrak{A}$, where every form index is considered as a system of several tensor or tractor indices, will be denoted $|\mathfrak{a}|$ and $|\mathfrak{A}|$, respectively. For example, $|\mathbf{A}^k\mathbf{B}^l\Lambda| = k + l + 1$.

Tractor form product bundles and components

Let us start with a simple example of a $\mathfrak{g}_0$–component $pr : W \hookrightarrow V$, see 1.2.2 on page 29. The tractor $\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}$ (see 1.2.5) yields the $\mathfrak{g}_0$–component

$$\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} : \mathcal{E}_{\mathbf{a}}[w] \hookrightarrow \mathcal{E}_{[A^0\mathbf{A}]}[w - k + 1]$$

of the bundle $E_{[A^0\mathbf{A}]}[w-k+1]$ where $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{a} = \mathbf{a}^k$. The corresponding projection pr^* is given by the section $\mathbb{Y}^{A^0\mathbf{A}}{}_{\mathbf{a}}$ as

$$(\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}})^* = \mathbb{Y}^{A^0\mathbf{A}}{}_{\mathbf{a}} : \mathcal{E}_{[A^0\mathbf{A}]}[w - k + 1] \twoheadrightarrow \mathcal{E}_{\mathbf{a}}[w].$$

Similarly, we can consider $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{W}$ as $\mathfrak{g}_0$–components of $E_{[A^0\mathbf{A}]}$ and also $X = X^\lambda_\Lambda$ and $Y = Y^\lambda_\Lambda$ as $\mathfrak{g}_0$–components of E_Λ. The $\mathfrak{g}_0$–components $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$,

$\mathbb{W}$ and X, Y will be called *TFP–components* of $E_{[A^0\mathbf{A}]}$ and E_Λ, respectively. Note they are, in general, not irreducible.

We generalize this notion in the following way. Bundles of the form $E_{\mathbf{A}_1\cdots\mathbf{A}_p}$ and $E_{\mathbf{A}_1\cdots\mathbf{A}_p\Lambda}$ will be called *tractor form product bundles* or TFP–bundles. (That is, they are tensor products of form tractor bundles, possibly with the spinor tractor bundle.) For a TFP–bundle V we define the set of *tractor form product components* or *TFP–components* denoted by $TFPC(V)$ inductively as follows. Firstly,

$$TFPC(E_{[A^0\mathbf{A}]}) = \{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\} \quad \text{and} \quad TFPC(E_\Lambda) = \{X, Y\}.$$

Secondly, if bundles V_1 and V_2 are TFP–bundles and also the bundle $V_1 \otimes V_2$ is a TFP–bundle, we put

$$TFPC(V_1 \otimes V_2) := \big\{ pr_1 pr_2 := pr_1 \otimes pr_2 \mid pr_i \in TFPC(V_i),\ i \in \{1,2\} \big\}. \tag{1.53}$$

Further we put $TFPC(V[w]) := TFPC(V)$ in an obvious way and we will consider all the notation for $pr \in TFPC(V)$ developed in this section also for $V[w]$. Recall $pr \in TFPC(V)$ defines also a $\mathfrak{g}_0$–component of any subbundle $W \subseteq V$ and the bundle $V \otimes U$ for a natural bundle U. (The former is just pr followed by the projection on W and the latter $pr \otimes \mathrm{id}_U$, see 1.2.2 on p. 29.)

As we denote the tensor product of two TFP–components simply by juxtaposition (see $pr_1 pr_2$ in (1.53)), TFP–components of V are denoted by juxtapositions of p symbols $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$, followed by X or Y in the spinor case. (We can use $\mathbb{W}$ or $\mathbb{Z}$ only if the tractor valence of the corresponding form tractor bundle is at least 2 or at most n, respectively.) For example, $\mathbb{X}_\mathbf{A}\mathbb{Y}_\mathbf{B} \in TFPC(E_{\mathbf{AB}})$. To simplify the notation, we will write also $\mathbb{X}\mathbb{Y} \in TFPC(E_{\mathbf{AB}})$ if the order of the form tractor indices $\mathbf{AB}$ is fixed. Therefore $\mathbb{X}\mathbb{Y} \neq \mathbb{Y}\mathbb{X}$ in this case but $\mathbb{X}_\mathbf{A}\mathbb{Y}_\mathbf{B} = \mathbb{Y}_\mathbf{B}\mathbb{X}_\mathbf{A}$.

In general, let us consider the system of indices $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_p$ or $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_p \Lambda$ with the fixed order of the indices in $\mathfrak{A}$. We will often write TFP–components of $E_{\mathfrak{A}}$ as $pr = pr_{\mathfrak{A}} = pr_{\mathfrak{A}}^{\mathfrak{a}} \in TFPC(E_{\mathfrak{A}})$ for an appropriate system $\mathfrak{a}$ of tensor/spinor indices. For example, possible notations for the bundle $E_{\mathfrak{A}}$, $\mathfrak{A} = \mathbf{A}_1\mathbf{A}_2\Lambda$ with the fixed order $\mathbf{A}_1\mathbf{A}_2\Lambda$ of indices are

$$\begin{aligned} pr : &= \mathbb{X}\mathbb{W}Y = pr_{\mathfrak{A}} = (\mathbb{X}\mathbb{W}Y)_{\mathfrak{A}} = \mathbb{X}_{\mathbf{A}_1}\mathbb{W}_{\mathbf{A}_2}Y_{\Lambda} = Y_{\Lambda}\mathbb{W}_{\mathbf{A}_2}\mathbb{X}_{\mathbf{A}_1} \\ &= pr_{\mathfrak{A}}^{\mathfrak{a}} = (\mathbb{X}\mathbb{W}Y)_{\mathfrak{A}}^{\mathfrak{a}} = \mathbb{X}_{\mathbf{A}_1}^{\mathbf{a}_1}\mathbb{W}_{\mathbf{A}_2}^{\mathbf{a}_2}Y_{\Lambda}^{\lambda} = \mathbb{W}_{\mathbf{A}_2}^{\mathbf{a}_2}Y_{\Lambda}^{\lambda}\mathbb{X}_{\mathbf{A}_1}^{\mathbf{a}_1} \\ &\in TFPC(E_{\mathfrak{A}}) \end{aligned}$$

where $\mathfrak{a} := \mathbf{a}_1\mathbf{a}_2\lambda$ and the valences of $\mathbf{a}_1$ and $\mathbf{a}_2$ are $|\mathbf{A}_1| - 1$ and $|\mathbf{A}_2| - 2$, respectively.

Another possible way to indicate valences and avoid the indices is to replace $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{W}$ by $\mathbb{X}^i$, $\mathbb{Y}^i$, $\mathbb{Z}^i$ and $\mathbb{W}^i$, see (1.46) for the definition. As mentioned above, $\mathbb{X}^i$, $\mathbb{Y}^i$, $\mathbb{Z}^i$ and $\mathbb{W}^i$ and also X and Y are not necessarily irreducible for $i = n'$, n even. If they decompose, we shall use also the notation

$$\mathbb{X}_{\pm}^{n'} : \mathcal{E}_{\pm}^{n'}[w] \longrightarrow \mathcal{T}^{n'}[w - n' + 1] \tag{1.54}$$

and similarly $\mathbb{Y}_{\pm}^{n'}$, $\mathbb{Z}_{\pm}^{n'}$, $\mathbb{W}_{\pm}^{n'}$, $X_{\pm}$ and $Y_{\pm}$ to distinguish the (two) irreducible components. Recall here the superscript n' always denotes the tensor valence, cf. (1.44).

Consider a TFP–bundle $V = E_{\mathfrak{A}}$. Using the developed notation, it is easy to describe explicitly the projection pr^* for $pr \in TFPC(V)$. Recall $pr : W \hookrightarrow V$ and $pr^* : V \twoheadrightarrow W$ is the corresponding projection, see 1.2.2. We define the *dual TFP–component* $pr^{\perp} \in TFPC(V)$ of $pr \in (V)$ in the following way. Firstly, we put

$$\mathbb{X}^{\perp} = \mathbb{Y}, \mathbb{Y}^{\perp} = \mathbb{X},\ \mathbb{Z}^{\perp} = \mathbb{Z},\ \mathbb{W}^{\perp} = \mathbb{W},\ X^{\perp} = Y,\ Y^{\perp} = X \tag{1.55}$$

for TFP–components of $V = E_{\mathbf{A}}$ and $V = E_{\Lambda}$. Secondly, if V_1 and V_2 are TFP–bundles and also $V_1 \otimes V_2$ is a TFP–bundle, we put

$$(pr)^{\perp} = pr_1^{\perp} pr_2^{\perp} \quad \text{for } pr = pr_1 pr_2 \in TFPC(V_1 \otimes V_2),\ pr_i \in TFPC(V_i)$$

where $i \in \{1, 2\}$. Now consider $pr = pr^{\mathfrak{a}}_{\mathfrak{A}} \in TFPC(E_{\mathfrak{A}})$. That is, $pr : E_{\mathfrak{a}}[w] \hookrightarrow E_{\mathfrak{A}}$ and raising/lowering the indices, we observe

$$pr^{*} = (pr^{*})^{\mathfrak{A}}_{\mathfrak{a}} = (pr^{\perp})^{\mathfrak{A}}_{\mathfrak{a}} : E_{\mathfrak{A}} \twoheadrightarrow E_{\mathfrak{a}}[w]. \tag{1.56}$$

This follows from the definition of raising and lowering indices if $\mathfrak{A} = \mathbf{A}$ and $\mathfrak{A} = \Lambda$. From this, the general case follows.

We have defined TFP–component as a string of $\mathbb{X}$'s, $\mathbb{Y}$'s etc. Now we introduce several quantities for TFP–component related to the notions of valence and homogenity. Also, we introduce parameters related to the order of differential operators and formulae for them. Below we use the term *order* of a differential operator in the sense usual in differential geometry.

Valence and homogenity

Let us consider a TFP–bundle $V = E_{\mathfrak{A}}$ and $pr = pr^{\mathfrak{a}}_{\mathfrak{A}} \in TFPC(E_{\mathfrak{A}})$. We define the *(tensor) valence* as the mapping $v : TFPC(V) \longrightarrow \mathbb{N}_0$ where $v(pr)$ counts the number of tensor indices in $\mathfrak{a}$. (That is, $v(pr) = |\mathfrak{a}|$ if there is no spinor index in $\mathfrak{a}$ and $v(pr) = |\mathfrak{a}| - 1$ otherwise.) Further, consider a $\mathfrak{g}_0$–component pr_2 of a natural bundle U. Then $pr' := pr \otimes pr_2$ is a $\mathfrak{g}_0$–component of $V \otimes U$ and we define the *(tensor) valence of pr' restricted to V* as $v_V(pr') = v_{\mathfrak{A}}(pr') := v(pr)$. We shall use $v_{\mathfrak{A}}(pr)$ especially for $pr \in TFPC(V)$ considered as a $\mathfrak{g}_0$–component of $V \otimes U$.

We define the *homogenity* as the mapping $h : TFPC(V) \longrightarrow \frac{1}{2}\mathbb{Z}$ defined inductively using the relations

1. $h(\mathbb{Y}) = 1$, $h(\mathbb{W}) = h(\mathbb{Z}) = 0$, $h(\mathbb{X}) = -1$, $h(Y) = \frac{1}{2}$, $h(X) = -\frac{1}{2}$

2. $h(pr_1 pr_2) = h(pr_1) + h(pr_2)$ for $pr_i \in TFPC(V_i)$, $i \in \{1, 2\}$ and $pr_1 pr_2 \in TFPC(V_1 \otimes V_2)$.

Note $h(pr^\perp) = -h(pr)$ which follows immediately from the definition of $pr^\perp$ above. As mentioned above, we use the definition of the homogenity also for weighted bundles $V = E_{\mathfrak{A}}[w]$. If this is an unweighted tractor bundle, it is easy to verify the following fact. Let us consider $pr : W \hookrightarrow V$ where W is irreducible. That is, $\mathbb{W}$ is an irreducible $\mathfrak{g}_0$–submodule of $\mathbb{V}$ where $V = \mathcal{G} \times_P \mathbb{V}$, (see 1.2.2). Then the grading element $E \in \mathfrak{g}_0$ defined in 1.1.1 acts on $\mathbb{W} \subseteq \mathbb{V}$ as multiplication by $h(pr)$. That is, the homogenity of TFP–components of V corresponds to the grading on $\mathbb{V}$.

Further we define the *highest homogenity* as the mapping hh on TFP–bundles with values in $\frac{1}{2}\mathbb{Z}$ inductively using the relations $hh(E_{\mathbf{A}}) = 1$, $hh(E_\Lambda) = \frac{1}{2}$ and $hh(V_1 \otimes V_2) = hh(V_1) + hh(V_2)$. Clearly every $pr \in TFPC(V)$ satisfies $h(pr) \leq hh(V)$ and the equality is possible only for the (unique) TFP–component which can be expressed as juxtaposition of only $\mathbb{Y}$'s and Y's.

If $pr \in TFPC(V)$, $V = E_{\mathfrak{A}}[w]$ and pr_2 is a $\mathfrak{g}_0$–component of a natural bundle U then $pr' := pr \otimes pr_2$ is a $\mathfrak{g}_0$–component of $V \otimes U$ and we define the *homogenity of* pr' *restricted to* V as $h_V(pr') = h_{\mathfrak{A}}(pr') := h(pr)$. Here we suppose that if U is associated with a system of indices then these indices are distinct from the indices in $\mathfrak{A}$. We shall use this especially for $pr \in TFPC(V)$ considered as a $\mathfrak{g}_0$–component of $V \otimes U$, see 1.2.2. Similarly, we shall define the *highest homogenity of* $V \otimes U$ *restricted to* V as $hh_V(V \otimes U) = hh_{\mathfrak{A}}(V \otimes U) := hh(V)$.

Proposition. *Consider a TFP–bundle* $V = E_{\mathfrak{A}}[w]$. *That is,* $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_p$ *or* $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_p \Lambda$. *Then every* $pr \in TFPC(V)$ *satisfies*

$$v(pr) \leq |\mathfrak{A}| - \lceil |h(pr)| \rceil. \tag{1.57}$$

More generally, consider a natural bundle U. Then an analogous statement holds for the bundle $V \otimes U$ if we replace v and h by v_V and h_V, respectively.

Proof. Assume $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_p$ first. Clearly if $\widetilde{pr} \in TFPC(V)$ satisfies $v(\widetilde{pr}) = |\mathfrak{A}|$ then $\widetilde{pr} := \mathbb{Z} \cdots \mathbb{Z}$. Let us try to replace some of the $\mathbb{Z}$'s in $\widetilde{pr}$ by $\mathbb{X}$, $\mathbb{Y}$ or $\mathbb{W}$ to obtain a given $pr \in TFPC(V)$. Since any single replacement changes the homogenity by at most one and $h(\widetilde{pr}) = 0$, we have to replace at least $|h(pr)|$ of them. But any single replacement lowers the valence by at least one hence we will reduce the valence by at least $|h(pr)|$. This means, $v(pr) \leq |\mathfrak{A}| - |h(pr)|$.

Assume $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_p \Lambda$. Then every $\widetilde{pr} \in TFPC(V)$ satisfies $v(\widetilde{pr}) \leq |\mathfrak{A}| - 1$. Clearly if $v(\widetilde{pr}) = |\mathfrak{A}| - 1$ then either $\widetilde{pr} = \mathbb{Z} \cdots \mathbb{Z}X$ or $\widetilde{pr} = \mathbb{Z} \cdots \mathbb{Z}Y$. These two possibilities for $\widetilde{pr}$ satisfy $h(\widetilde{pr}) = \pm\frac{1}{2}$ by definition hence $\lceil |h(\widetilde{pr})| \rceil = 1$. Now using the same arguments as for $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_p$, the inequality (1.57) follows. The statement for $V \otimes U$ is obvious. □

Recall that if pr is a $\mathfrak{g}_0$–component of a bundle V and $f \in \mathcal{V}$ is section of V, the projection pr^* yields the section $pr^* f$, see 1.2.2. Let us demonstrate this on $f_A = D_A \sigma$, $\sigma \in \mathcal{E}[w]$. (Note $E_A = E_{\mathbf{A}^1}$ is a TFP–bundle.) Then e.g. $X = X_A$ is a $\mathfrak{g}_0$–component of E_A and $X^* f = Y^A f_A = -(\Delta + wP)\sigma$ because $(X_A)^\perp = Y_A$ and $(X^*)^A = (X^\perp)^A = Y^A$. Recall we defined $\mathfrak{g}_0$–components also for differential operators, see p. 31.

We need especially to know when is $pr^* f$ invariant i.e. when is pr a projecting part of f, see p. 31. In the example above, Y_A is a projecting part because $(Y^*)f = X^A f_A = c_1 \sigma$, see (1.51). If $w = 0$ then also $Z = Z_A^a$ is a projecting part because $(Z^*)_a^A f_A = Z_a^A f_A = c_2 \nabla_a \sigma$ etc. A general sufficient condition for pr to be a projecting part of $f \in \mathcal{V}$ was given in 1.2.2 using the gradation on $\mathbb{V}$ where $V = \mathcal{G} \times_P \mathbb{V}$. The following Lemma formulates this observation using TFP–components.

Lemma. *Let $V = E_{\mathfrak{A}}[w]$ be a TFP–bundle and W, U natural bundles.*

(i) Consider a section $f \in \mathcal{V}$ and a TFP–component $pr \in TFPC(V)$. If the condition

$$\forall \widetilde{pr} \in TFPC(V) : h(\widetilde{pr}) > h(pr) \Longrightarrow \widetilde{pr}^* f = 0 \tag{1.58}$$

is satisfied then pr is a projecting part of f. More generally, if $f \in \mathcal{V} \otimes \mathcal{W}$ and (1.58) is satisfied then pr (considered as a $\mathfrak{g}_0$–component of $V \otimes W$) is a projecting part of f.

(ii) Similarly, if $E : \mathcal{U} \longrightarrow \mathcal{V}$ is an invariant differential operator such that

$$\forall \widetilde{pr} \in TFPC(V) : h(\widetilde{pr}) > h(pr) \Longrightarrow \widetilde{pr}^* E = 0 \tag{1.59}$$

then pr is a projecting part of of E. More generally, if $E : \mathcal{U} \otimes \mathcal{W} \longrightarrow \mathcal{V} \otimes \mathcal{W}$ is an invariant differential operator and (1.59) is satisfied then pr is a projecting part of E.

Proof. Let us consider a TFP–component $pr \in TFPC(E_{\mathfrak{A}}[w])$. This is a bundle homomorphism $pr^{\mathfrak{a}}_{\mathfrak{A}} : E_{\mathfrak{a}}[w'] \hookrightarrow E_{\mathfrak{A}}[w]$ for an appropriate system of indices $\mathfrak{a}$. Then we have $(pr^{\perp})^{\mathfrak{a}}_{\mathfrak{A}} \in TFPC(E_{\mathfrak{A}}[w])$, see (1.55), and the projection $(pr^*)^{\mathfrak{A}}_{\mathfrak{a}} = (pr^{\perp})^{\mathfrak{A}}_{\mathfrak{a}} : E_{\mathfrak{A}}[w] \twoheadrightarrow E_{\mathfrak{a}}[w']$. All these homomorphisms depend on the choice of a metric $g \in [g]$ from the conformal class. Recall pr is given by a string of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$, X and Y. Each of them is a TFP–component of a tractor form bundle or a tractor spinor bundle.

Consider another metric $\hat{g} = e^{2\Upsilon} g$. This yields TFP–components $\hat{\mathbb{X}}$, $\hat{\mathbb{Y}}$, ... of a tractor form bundle or a tractor spinor bundle, see (1.47) and (1.37). Consider the string which defines pr where we replaced $\mathbb{X}$, $\mathbb{Y}$, ... by $\hat{\mathbb{X}}$, $\hat{\mathbb{Y}}$, ..., respectively. This defines a TFP–component of $E_{\mathfrak{A}}[w]$ which we denote by $\widehat{pr}$. That is, $pr^{\mathfrak{a}}_{\mathfrak{A}}, \widehat{pr}^{\mathfrak{a}}_{\mathfrak{A}} : E_{\mathfrak{a}}[w'] \hookrightarrow E_{\mathfrak{A}}[w]$ but, in general, $pr \neq \widehat{pr}$.

We need to know the difference $(pr^*)^{\mathfrak{A}}_{\mathfrak{a}} - (\widehat{pr}^*)^{\mathfrak{A}}_{\mathfrak{a}} = (pr^\perp)^{\mathfrak{A}}_{\mathfrak{a}} - (\widehat{pr}^\perp)^{\mathfrak{A}}_{\mathfrak{a}} : E_{\mathfrak{A}}[w] \twoheadrightarrow E_{\mathfrak{a}}[w']$. It follows from (1.47) and (1.37) that

$$pr^\perp - \widehat{pr}^\perp = \sum_{\substack{pr_i^\perp \in TFPC(V) \\ h(pr_i^\perp) < h(pr^\perp)}} \psi_i(\Upsilon) pr_i^\perp$$

where $\psi_i(\Upsilon)$ is a homomorphism depending on Υ. But if $h(pr_i^\perp) < h(pr^\perp)$ then $h(pr_i) > h(pr)$ which means $pr_i^* f = 0 = (pr_i^\perp)^{\mathfrak{A}}_{\mathfrak{a}} f_{\mathfrak{A}}$ using (1.58). If we apply the last display to $f_{\mathfrak{A}}$, the right hand side will vanish and the left hand side will be $(pr^* - \widehat{pr}^*)f = 0$. Therefore $pr^* f = \widehat{pr}^* f$ i.e. $pr^* f$ does not depend on the choice of g. That is, pr is a projecting part of f.

The same proof applies for $f \in \mathcal{V} \otimes \mathcal{W}$ and clearly the statements for invariant operators follow from the statements for sections. $\square$

Formulae for differential operators

We shall work with differential operators given by formulae throughout the thesis. First we define explicitly what a "formula" means for us.

Definition. (i) *Tractor formula* or *formula* (for a differential operator) is a finite sum $\Phi = \sum_i c_i \mathrm{Proj}_i \Phi_i$ where c_i denote scalars, Proj_i projections and Φ_i is a juxtaposition of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$, X, Y, ∇, R, P, $\boldsymbol{g}$, $\boldsymbol{\beta}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\epsilon}$, h with an arbitrary position of indices which makes a sense. That is, $\mathbb{X}$ possess one form tensor and one form tractor index of appropriate valences, ∇ one tensor index etc. Every index is used either once (a free index) or twice. The latter indicates a partial contraction. We assume Proj_i can be expressed by Young symmetrizations of free indices and projections to kernels of $\boldsymbol{g}$, h, $\boldsymbol{\varepsilon}$ or $\boldsymbol{\beta}$ on free indices of an appropriate type. Proj_i will be omitted if $\mathrm{Proj}_i = \mathrm{id}$.

(ii) A *tensor formula* is a formula without tractor or spinor tractor indices.

(iii) Let Φ_1 and Φ_2 be juxtapositions, c_i a scalar and Proj_i a projection as defined in (i). We put

- $c_i\mathrm{Proj}_i\Phi_1\nabla_a X\Phi_2 \sim c_i\mathrm{Proj}_i\Phi_1 Y\boldsymbol{\beta}_a\Phi_2 + c_i\mathrm{Proj}_i\Phi_1 X\nabla_a\Phi_2$ and similarly for all relations in (1.38) and (1.49)
- $c_i\mathrm{Proj}_i\Phi_1\Psi\Psi'\Phi_2 \sim c_i\mathrm{Proj}_i\Phi_1\Psi'\Psi\Phi_2$ for $\Psi', \Psi \in \{\mathbb{X}, \mathbb{Y}, \cdots, \boldsymbol{\epsilon}, h\}$ from (i) such that $\Psi', \Psi \neq \nabla$
- $c_i\mathrm{Proj}_i\Phi_1\mathbb{X}^{\mathbf{A}}\mathbb{Y}_{\mathbf{A}}\Phi_2 \sim \frac{c_i}{k}\mathrm{Proj}_i\Phi_1\Phi_2$, $\mathbf{A} = \mathbf{A}^k$ and analogously for all remaining (partial) contraction and raising/lowering of indices

Then $\sim$ generates an equivalence relation (on the set of the formulae), denoted also by $\sim$. We say that the formulae Φ and Φ' are *equivalent*, if $\Phi \sim \Phi'$.

Remarks. 1. Consider a formula Φ and $f \in \mathcal{V}$ for a natural bundle V. Let us suppose Φf, equipped with indices, makes sense as in the Definition (i). Then the formula Φ yields a differential operator on V. We will interpret Φ in the usual way: ∇ is the tensor product of a Levi–Civita and spin connection (from the conformal class) and the normal tractor connection, R is curvature of the Levi–Civita connection (for a given scale), P the corresponding Rho–tensor or its trace, $\boldsymbol{g}$ the conformal metric, h the tractor metric, $\boldsymbol{\beta}$ the Clifford section, $\boldsymbol{\epsilon}$ the volume form, $\boldsymbol{\varepsilon}$ the spinor metric and $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$ and X, Y sections defined in 1.2.5 and 1.2.4, respectively. We will consider ∇ as acting to the end of the juxtaposition. That is, $\Phi_1\nabla\Phi_2 f$ means $\Phi_1\nabla(\Phi_2 f)$ for juxtapositions Φ_1 and Φ_2.

We will not always distinguish between formulae and corresponding operators, if the source space is specified. Also note, if $\Phi \sim \Phi'$ then Φ and Φ' yield the same differential operator.

2. We will often use $\mathrm{Proj}_i = \mathrm{id}$ as any projection from Definition (i) can be expressed as an appropriate sum, cf. $\nabla_{(a}\nabla_{b)} = \frac{1}{2}\nabla_a\nabla_b + \frac{1}{2}\nabla_b\nabla_a$. If we write a formula in the form $\Phi = \sum_i c_i\Phi_i$ we will always assume that the Φ_i's are juxtapositions as in the Definition. We shall also use the notation

$\Phi = \Phi_{\mathfrak{A}}$ where $\mathfrak{A}$ denotes the system of free (spinor) tractor indices of Φ if they are all covariant ("downstairs").

3. If $\Phi = \sum_i c_i \Phi_i$ and $\Psi = \sum_j d_j \Phi_j$ are two formulae, we put

$$\Phi \circ \Psi = \Phi\Psi := \sum_{i,j} c_i d_j \Phi_i \Psi_j.$$

If $\Phi\Psi$ is a formula (i.e. if this expressions makes sense as in the Definition), it corresponds to a composition of differential operators.

4. Using the relation $\sim$, see the Definition (iii), all the symbols $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$, X, Y in a juxtaposition can be "moved to the left". Also we can avoid contractions of tractor form and spinor indices. That is, every tractor formula $\Phi' = \Phi'_{\mathfrak{A}} = \sum_i c_i \Phi'_i$ is equivalent to a formula of the form

$$\Phi = \sum_{i \in I} c_i pr_i \Phi_i$$

(cf. (1.52)) where Φ_i are tensor formulae and $I \subseteq \mathbb{N}_0$ a finite set. Suppose $\Phi_{\mathfrak{A}}$ is as in the last display and $E_{\mathfrak{A}}$ is a TFP–bundle. Then the pr_i in (1.52) are formally TFP–components $pr_i \in TFPC(E_{\mathfrak{A}})$. Consider an arbitrary $pr \in TFPC(E_{\mathfrak{A}})$ and the projection pr^*. Then we get the tensor formula

$$pr^* \Phi = \sum_{i \in I} c_i pr^* pr_i \Phi_i \sim k \sum_{\substack{i \in I \text{ s.t.} \\ pr = pr_i}} c_i \Phi_i$$

where k is a (nonzero) scalar multiple. In particular, any tractor formula without free tractor (form or spin) indices is equivalent to a tensor formula. Recall $pr^*\Phi$ is a $\mathfrak{g}_0$–component of the formula (or the corresponding operator) Φ.

Our next aim is to define a quantity for formulae reflecting the order of differential operators. (Cf. the discussion of the tractor D–operator at the beginning of this section.) We define $oh(\Psi) \in \frac{1}{2}\mathbb{Z}$ (which stands for "order homogenity") for a formula Ψ inductively as follows.

1. If $\Psi \in \{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{Y}, X, Y\}$ then $oh(\Psi) := h(\Psi)$.
2. $oh(\nabla) := 1$, $oh(R) = oh(P) := 2$ and if $\Psi \in \{\boldsymbol{g}, \boldsymbol{\beta}, \boldsymbol{\epsilon}, h, \boldsymbol{\varepsilon}\}$ then $oh(\Psi) := 0$
3. If Ψ is such that $oh(\Psi)$ is well–defined and $c\mathrm{Proj}\Psi$ is a formula, then $oh(c\mathrm{Proj}\Psi) := oh(\Psi)$
4. If Ψ_1, Ψ_2 are such that $oh(\Psi_1)$, $oh(\Psi_2)$ is well–defined and $\Psi_1\Psi_2$ is a formula then $oh(\Psi_1\Psi_2) := oh(\Psi_1) + oh(\Psi_2)$. If moreover $oh(\Psi_1) = oh(\Psi_2)$ then $oh(\Psi_1 + \Psi_2) := oh(\Psi_1)$.

For example, $oh(D_A) = 1$, see p. 46. In general, $oh(\Phi)$ is defined for a formula $\Phi = \sum_i c_i \mathrm{Proj}_i \Phi_i$ such that $oh(\Phi_i)$ is defined for every i and moreover $oh(\Phi_i) = oh(\Phi_j)$, $i \neq j$. **Throughout this thesis, we shall always work with formulae $\sum_i c_i \mathrm{Proj}_i \Phi_i$ satisfying this property.** This is actually natural and non–restrictive. The conformal invariant calculus can be build on the tractor D–operator and this property is satisfied for formulae of the form of a projection to the composition $D \cdots D$.

It is an important property of oh that if $oh(\Phi)$ and $oh(\Psi)$ is defined for two formulae $\Phi \sim \Psi$ then $oh(\Phi) = oh(\Psi)$. It is straightforward to verify this from the Definition (iii) above.

Assume the formula Φ has no free tractor or spinor tractor indices. (It follows from the Definition (iii) that if $\Phi \sim \Phi'$ then also Φ' has no free (spinor) tractor indices.) We define the *formal order* $fo(\Phi)$ of Φ as $fo(\Phi) := oh(\Phi)$ if the right hand side is defined. Clearly if a differential operator of the order o is given by a tensor formula Φ such that $fo(\Phi)$ is defined then $o \leq fo(\Phi)$. Moreover

$$fo(\Phi) = 1 \Longrightarrow \Phi \equiv 0 \text{ or } o = fo(\Phi) = 1. \tag{1.60}$$

where $\Phi \equiv 0$ means that the operator Φ vanishes. This follows from the definition of oh. If $oh(\Phi) = 1$ then every summand in $\Phi = \sum_i c_i \mathrm{Proj}_i \Phi_i$

involves ∇ once hence there is no summand of the zero order.

Let us consider a TFP–bundle $E_{\mathfrak{A}} = V_1 \otimes V_2$ and a formula $\Phi = \Phi_{\mathfrak{A}}$ such that $oh(\Phi)$ is defined. If $pr \in TFPC(V_1)$ then $oh(pr^*\Phi) = oh(\Phi) - h(pr)$ using $h(pr^\perp) = -h(pr)$ and (1.56). Thus, if $pr \in TFPC(E_{\mathfrak{A}})$ then $fo(pr^*\Phi) = oh(\Phi) - h(pr)$.

Example. The following simple examples follow directly from the corresponding definitions. Clearly $oh(\mathbb{X}_{\mathbf{A}}) = -1$ and $oh(\mathbb{X}_A \nabla^p \nabla_p + w\mathbb{X}_A P) = 1$ for a scalar w. Note, e.g. $\mathbb{X}_{\mathbf{A}}$ defines the operator $f \mapsto \mathbb{X}_{\mathbf{A}} f$. Also $fo(\nabla^p \nabla_p) = fo(P) = 2$ but the latter operator (which acts by muplitplication by P) has the order 0. Since also $oh(c_1 \mathbb{Z}_A^a \nabla_a + c_2 \mathbb{Y}_A) = 1$, we verified that $oh(D_A) = 1$. Further, it follows from (1.39) that $oh(D^\lambda_\Lambda) = \frac{1}{2}$. Thus, for example, $oh(D_A D^\lambda_\Lambda) = \frac{3}{2}$.

Summary. We briefly summarize the notation developed above.

- We will use $\mathfrak{g}_0$–components $pr^{\mathfrak{a}}_{\mathfrak{A}}$. These take the form of a string or 'word' of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$, X and Y
- We will use formulae Φ of differential operators of the form a of (formal) sum of strings of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$, X, Y, ∇, R, P, $\boldsymbol{g}$, h, $\boldsymbol{\beta}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\varepsilon}$
- $v(pr^{\mathfrak{a}}_{\mathfrak{A}})$ counts the number of tensor indices in $\mathfrak{a}$
- $h(pr^{\mathfrak{a}}_{\mathfrak{A}})$ counts 1 for every $\mathbb{Y}$, -1 for every $\mathbb{X}$, $\frac{1}{2}$ for every Y and $-\frac{1}{2}$ for every X
- $oh(\Phi)$, for a chosen string in the sum Φ, counts 1 for every ∇, 2 for every R and P, and $+1$, -1, $\frac{1}{2}$ and $-\frac{1}{2}$ for every $\mathbb{Y}$, $\mathbb{X}$, Y and X, respectively, if the result is independent on the choice of the string
- $fo(\Phi) = oh(\Phi)$ for a formula Φ with no free tractor (form or spinor) tractor index.

1.3 Invariant differential operators

In this section we mainly review basic facts known about conformally invariant differential operators. This is especially the classification in the flat case based on parabolic representation theory. Also we review methods available for curved cases. The (real or complex) algebras $\mathfrak{p} \subseteq \mathfrak{g}$ have been defined in 1.1.1 (see also 1.2.2). We will use the notation $\mathbb{E}, \mathbb{V}, \mathbb{W}, \ldots$ for representation spaces, $E, V, W, \ldots$ for bundles and $\mathcal{E}, \mathcal{V}, \mathcal{W}, \ldots$ for sections.

1.3.1. Notation for representation spaces.

Let us consider a highest weight Λ of a complex irreducible $\mathfrak{g}_0$–, $\mathfrak{p}$– or $\mathfrak{g}$–representation. We denote by $\mathbb{V}^\Lambda$ the representation dual to the representation with the highest weight Λ. In the other words, $\mathbb{V}^\Lambda$ denotes the representation with the lowest weight $-\Lambda$. We will use $\mathbb{V}^\Lambda$ also for corresponding representation of Lie groups.

There are two reasons for this notation. First, we will need certain cohomology of Lie algebras $H(\mathfrak{g}_-; \mathbb{V})$ where $\mathbb{V}$ is a $\mathfrak{g}$–representation. (See Appendix A for the definition.) The $\mathfrak{p}$–representation on $H(\mathfrak{p}_+; \mathbb{V})$ decomposes into irreducibles and we can easily compute highest weights Γ of these $\mathfrak{p}$–components [40]. (See also Theorem A.1.1.) Moreover, we have the duality $H(\mathfrak{g}_-; \mathbb{V}^*) \cong H(\mathfrak{p}_+; \mathbb{V})^*$ of $\mathfrak{p}$–representations. Now the notation $\mathbb{V}^\Gamma \subseteq H(\mathfrak{g}_-; \mathbb{V}^\Lambda)$ means that Γ is a highest weight of a $\mathfrak{p}$–irreducible component of $H(\mathfrak{p}_+; (\mathbb{V}^\Lambda)^*)$.

Expressing a highest weight a in the basis of fundamental weights (see 1.1.1 for details), we obtain *coefficients* of a highest weight. Later we will write a highest weight using its coefficients as labels of the nodes the of a Dynkin diagram from Table 1.1. This notation requires a choice Δ_+ of positive roots, see 1.1.1. We will always consider such labelled Dynkin diagrams with respect to Δ_+ corresponding to upper block triangular matrices in the matrix presentation of $\mathfrak{so}_n(\mathbb{C})$. (This is the usual setting.) However, $\mathfrak{p}$ is given by lower block triangular matrices, see (1.2). From this it follows that an irreducible $\mathfrak{p}$–representation $\mathbb{V}^\Gamma$ is given by a Dynkin diagram with coefficients of Γ over the nodes.

1.3.2. Flat parabolic geometries. Assume the complex setting. Recall that a flat parabolic geometry is the homogeneous space $M = G/P$ together with the P–principal bundle $G \longrightarrow G/P$ equipped with the Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$. We shall consider linear differential operators i.e. operators between homogeneous vector bundles V^{Γ_1} and V^{Γ_2}. These are associated to the P–bundle $G \longrightarrow G/P$ with respect to the P–representations $\mathbb{V}^{\Gamma_1}$ and $\mathbb{V}^{\Gamma_2}$, respectively. Here P is a parabolic subgroup of a semisimple group G, see 1.2.2. An *operator* is a mapping $\mathcal{V}^{\Gamma_1} \longrightarrow \mathcal{V}^{\Gamma_2}$. A kth order *differential* operator can be described as a bundle map $J^kV^{\Gamma_1} \longrightarrow V^{\Gamma_2}$ on the k–jet prolongation $J^kV^{\Gamma_1}$. This operator is *G–invariant* if it commutes with the induced action of G on sections $\mathcal{V}^{\Gamma_1}$ and $\mathcal{V}^{\Gamma_2}$. The real case is analogous.

To classify these operators we can use the Lie representation theory. We have the identification $J^kV^{\Gamma_1} = G \times_P J^k\mathbb{V}^{\Gamma_1}$ for an appropriate P–module $J^k\mathbb{V}^{\Gamma_1}$. Then the invariant differential operators $\mathcal{V}^{\Gamma_1} \longrightarrow \mathcal{V}^{\Gamma_2}$ correspond bijectively to P-homomorphisms $J^k\mathbb{V}^{\Gamma_1} \longrightarrow \mathbb{V}^{\Gamma_2}$ (a version of Frobenius reciprocity) or dually to $(\mathbb{V}^{\Gamma_2})^* \longrightarrow (J^k\mathbb{V}^{\Gamma_1})^*$. Now we need the identification of the latter with generalized Verma modules. Recall that these are $(\mathfrak{g}, P)$–modules

$$\mathbb{V}^\Gamma_{\mathfrak{p}} := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V}^\Gamma$$

for algebras $\mathfrak{p} \subseteq \mathfrak{g}$ and a P–module $\mathbb{V}^\Gamma$ where $\mathfrak{U}(\_)$ denotes the universal enveloping algebra. We have $V^\Gamma_{\mathfrak{p}} = \mathfrak{U}(\mathfrak{g}_-) \otimes \mathbb{V}^\Gamma$ as vector spaces (by virtue of the Poincare–Birkhoff–Witt theorem) and $(J^k\mathbb{V}^\Gamma)^* = \mathfrak{U}_k(\mathfrak{g}_-) \otimes (\mathbb{V}^\Gamma)^*$ where $\mathfrak{U}_k(\mathfrak{g}_-) \subseteq \mathfrak{U}(\mathfrak{g}_-)$ is given by the filtration of $\mathfrak{U}(\mathfrak{g}_-)$ by degree. Here we have used the identification $\mathfrak{g}_- \cong T_xM \cong (T_x^*M)^*$. Moreover, the identifications

$$(J^k\mathbb{V}^\Gamma)^* \cong \mathfrak{U}_k(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} (\mathbb{V}^\Gamma)^* \hookrightarrow \mathbb{V}_{\mathfrak{p}}^{\Gamma^*} \cong (J^\infty\mathbb{V}^\Gamma)^*$$

can be realized as P–homomorphisms and the last isomorphism is actually a $(\mathfrak{g}, P)$–homomorphism. Therefore, instead of P–homomorphisms $J^k\mathbb{V}^{\Gamma_1} \longrightarrow \mathbb{V}^{\Gamma_2}$ we can consider dually P–homomorphisms $\mathbb{V}^{\Gamma_2^*} \longrightarrow \mathbb{V}_{\mathfrak{p}}^{\Gamma_1^*}$ where Γ_1^* and Γ_2^* denote the duals of Γ_1 and Γ_2, respectively. The last step is Frobenius reciprocity

$$\mathrm{Hom}_{\,P}\left(\mathbb{V}^{\Gamma_2^*}, \mathbb{V}_{\mathfrak{p}}^{\Gamma_1^*}\right) = \mathrm{Hom}_{\,(\mathfrak{g},P)}\left(\mathbb{V}_{\mathfrak{p}}^{\Gamma_2^*}, \mathbb{V}_{\mathfrak{p}}^{\Gamma_1^*}\right).$$

So we have passed from k–order G–invariant differential operators $\mathcal{V}^{\Gamma_1} \longrightarrow \mathcal{V}^{\Gamma_2}$ to homomorphisms of generalized Verma modules $\mathbb{V}_{\mathfrak{p}}^{\Gamma_2^*} \longrightarrow V_{\mathfrak{p}}^{\Gamma_1^*}$.

To simplify the situation, we omit the discussion of possible Lie groups P with the Lie algebra $\mathfrak{p}$ i.e. we will consider generalized Verma modules as $\mathfrak{g}$–modules. In the complex setting, the Harish-Chandra theorem (see e.g. [37]) provides a necessary condition for existence of a homomorphism $\mathbb{V}_{\mathfrak{p}}^{\Gamma} \longrightarrow \mathbb{V}_{\mathfrak{p}}^{\Gamma'}$: the weights $\Gamma + R$ and $\Gamma' + R$ (where R denotes the lowest form, see 1.1.1) have to be conjugated by an element $w \in W$ i.e. $w(\Gamma + R) = \Gamma' + R$. That is, Γ and Γ' are on the same orbit of the affine action of the Weyl group W.

All the homomorphisms are injections and determined uniquely up to a scalar multiple by the source and target spaces. Their complete classification is known in the conformal case (see below) and is also known in the case of true Verma modules i.e. when $\mathfrak{p} = \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ [55, 3] (see also [4]). The complete classification for the general parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is probably not solved yet.

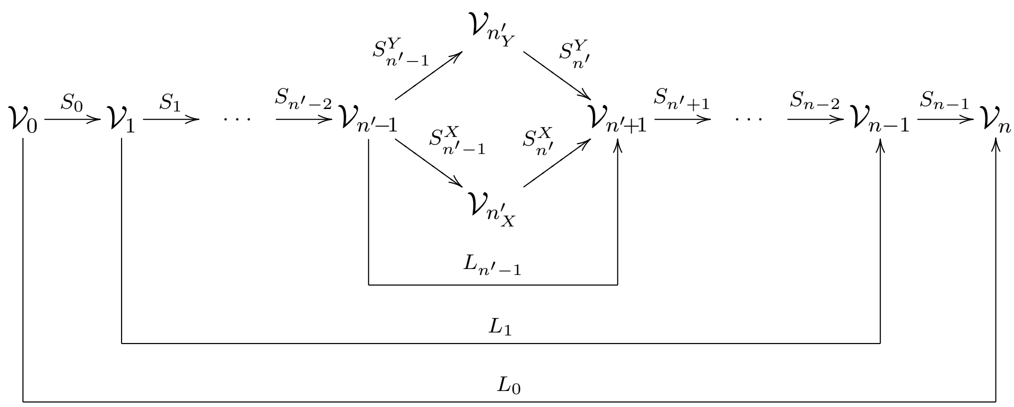
1.3.3. Flat conformal geometries. The classification of conformally invariant differential operators described below is essential for the thesis. All of them appear in the pattern corresponding to the Hasse graph structure on $W^{\mathfrak{p}}$ displayed in 1.1.1 in the complex setting. We shall describe this in detail. Then we briefly comment upon the real case.

The classification of homomorphism of generalised complex Verma modules corresponding to the parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g} = \mathfrak{so}_{n+2}(\mathbb{C})$ is known due [5, 6]. Some of them are provided by homomorphisms of true Verma modules corresponding to a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{p}$ since every generalised Verma module $\mathbb{V}_{\mathfrak{p}}^{\Gamma}$ is a quotient of the Verma module $\mathbb{V}_{\mathfrak{b}}^{\Gamma}$. Of course, many homomorphisms of true Verma modules will vanish when we pass to the quotients. Those who survive are called *standard* and the remaining ones *non–standard*. The same terminology will be used for operators 'dual' to these and their curved analogues. In the case of regular pattern, standard homomorphisms correspond exactly to the arrows in the pattern and to composition of two arrows in the middle diamond for n even, see 1.1.1 for the terminology. Some homomorphisms appear on the pattern with a $\mathfrak{p}$–dominant non–integral weight or on singular patterns, see p. 7. In the latter case, the homomorphisms and corresponding operators will be said to be *singular*.

This terminology and the summary of results below is taken from [24] which demonstrates the classification from [6] together with the Hasse graph structure on $W^{\mathfrak{p}}$. We formulate these results for operators which go in the opposite direction than the dual homomorphisms of generalized Verma modules. They are given uniquely up to a scalar multiple (and hence determined by the source and target bundle) and operate between local sections of these bundles. For a given weight Λ, the pattern consists of vector bundles $V_k := V^{w_k \cdot \Lambda}$, $w_k \in W^{\mathfrak{p}}$ and of $V_{n'_2} := V^{w_{n'_2} \cdot \Lambda}$ and $V_{n'_1} := V^{w_{n'_1} \cdot \Lambda}$ for n even, see the patterns in Table 1.1. The subscripts $k = |w_k| \in \{0, \dots, n\}$ and $n' = |w_{n'_1}| = |w_{n'_2}|$ will be referred as the *degree*. We shall display the patterns on the level of sections. This notation makes sense only for those of weights $w_k.\Lambda$, $w_{n'_1}.\Lambda$ and $w_{n'_2}.\Lambda$ which are $\mathfrak{p}$–dominant. We shall describe the results separately for n even and odd, respectively. In both cases, $V_0 = V^\Lambda$ where Λ is the weight of the pattern, see p. 7.

Even dimensional complex cases

The pattern obtained from 1.1.1 by replacing elements of $W^\mathfrak{p}$ by the space of sections of corresponding associate bundle together with all possible G–invariant operators is the following:



where $\{n'_X, n'_Y\} = \{n'_1, n'_2\}$. (We shall comment upon the latter notation below.) Let us suppose first Λ is $\mathfrak{g}$–dominant i.e. we have a *regular* pattern. The operators denoted by S together with the operator

$$L_{n'-1} = S^Y_{n'} \circ S^Y_{n'-1} = -S^X_{n'} \circ S^X_{n'-1}$$

are standard, the remaining ones $L_0, \dots, L_{n'-2}$ are nonstandard. (Note we shall construct $L_{n'-1}$ in a different way, without use of $S^Y_{n'-1}$ and $S^Y_{n'}$.) We shall also use the notation $\mathcal{V}_{n'} = \mathcal{V}_{n'_X} \oplus \mathcal{V}_{n'_Y}$ and

$$S_{n'-1} : \mathcal{V}_{n'-1} \longrightarrow \mathcal{V}_{n'}, \quad S_{n'} : \mathcal{V}_{n'} \longrightarrow \mathcal{V}_{n'+1}$$

for the direct sums $S^Y_{n'-1} \oplus S^X_{n'-1}$ and $S^Y_{n'} \oplus S^X_{n'}$, respectively. Positions in the pattern will be denoted by the subscripts $0, 1, \dots, n'-1, n'_X, n'_Y, n'+1, \dots, n$ and called *regular* positions. We distinguish the two positions in the middle in such a way that order of the operator $S^Y_{n'-1}$ is lower (or equal) than order of $S^X_{n'-1}$. Using the symbolism of Dynkin diagrams, it is easy to see whether $V_{n'_X} = V_{n'_1}$, $V_{n'_Y} = V_{n'_2}$ or $V_{n'_X} = V_{n'_2}$, $V_{n'_Y} = V_{n'_1}$, see 2.2.1 and 3.1.2 for details. The regular pattern without nonstandard operators is known as *generalized Bernstein–Gelfand–Gelfand (gBGG) resolution* or *sequence*.

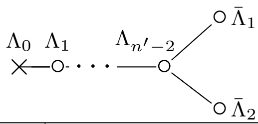
In the case of *singular* pattern i.e. when Λ is a singular weight, not all weights in the pattern will be $\mathfrak{p}$–dominant. Actually [24], a $\mathfrak{p}$–dominant weight appears only if there are two coefficients -1 in Λ and they are over "the legs" (nodes labelled by n'_1 and n'_2 in Table 1.1) or there is only one coefficient -1, cf. the pattern in Table 2.1. (Here we consider Λ as a vector of coefficients over the Dynkin diagram.) The former case with two coefficients -1 is completely degenerated with identical bundles in the middle diamond and there are no nontrivial operators. The latter case yields operators $L_1, \ldots, L_{n'}$ as shown in Table 1.5. (The case $\bar{\Lambda}_2 = -1$ is not displayed since this is completely analogous to the case $\bar{\Lambda}_1 = -1$.) These homomorphisms are non–standard with the exception of $L_{n'}$ and all of them are singular. Positions in singular patterns will be denoted by couples of subscripts of identified bundles "$0, 1$", ..., "$n-1, n$" in cases with one coefficient -1 and by the word "middle" in the case with two $-1'$s. (We will not need to distinguish between couples "$n'-1, n'_X$" and "$n'-1, n'_Y$". Both will be referred as "$n'-1, n'$" and the similar convention will be used for the position "$n', n'+1$".) All these position will be called *singular* positions.

Odd dimensional complex cases

Contrary to the previous case, there are only standard operators denoted by $S_0, \ldots, S_{n-1}$ in the regular pattern

$$\mathcal{V}_0 \xrightarrow{S_0} \mathcal{V}_1 \xrightarrow{S_1} \cdots \xrightarrow{S_{n'-1}} \mathcal{V}_{n'} \xrightarrow{S_{n'}} \mathcal{V}_{n'+1} \xrightarrow{S_{n'+1}} \cdots \xrightarrow{S_{n-2}} \mathcal{V}_{n-1} \xrightarrow{S_{n-1}} \mathcal{V}_n$$

which therefore coincides with the gBGG sequence. These operators will be

The pattern for singular weights	
	
$\bar{\Lambda}_1 = \bar{\Lambda}_2 = -1$	$\mathcal{V}_{n'-1} = \mathcal{V}_{n'_Y} = \mathcal{V}_{n'+1}$, $\mathcal{V}_{n'-1} = \mathcal{V}_{n'_X} = \mathcal{V}_{n'+1}$
$\Lambda_i = -1$ $0 \leq i \leq n'-2$	$\mathcal{V}_i = \mathcal{V}_{i+1} \xrightarrow{L_{i+1}} \mathcal{V}_{n-i-1} = \mathcal{V}_{n-i}$
$\bar{\Lambda}_1 = -1$	$\mathcal{V}_{n'_X} = \mathcal{V}_{n'-1} \xrightarrow{L_{n'}} \mathcal{V}_{n'_Y} = \mathcal{V}_{n'+1}$

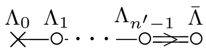
The pattern for nonintegral weights	
	
$\Lambda_0 \in \left(\frac{1}{2}\mathbb{N} \setminus \mathbb{N}\right) \cup \{-\frac{1}{2}\}$	$\mathcal{V}_0 \xrightarrow{L_0} \mathcal{V}_n$
$\Lambda_{i-1}, \Lambda_i \in \left(\frac{1}{2}\mathbb{N} \setminus \mathbb{N}\right) \cup \{-\frac{1}{2}\}$ $1 \leq i \leq n'-1$	$\mathcal{V}_i \xrightarrow{L_i} \mathcal{V}_{n-i}$
$\Lambda_{n'-1} \in \left(\frac{1}{2}\mathbb{N} \setminus \mathbb{N}\right) \cup \{-\frac{1}{2}\}$	$\mathcal{V}_{n'} \xrightarrow{L_{n'}} \mathcal{V}_{n'+1}$

Table 1.5: Operators on non–regular patterns.

referred as *short* operators. Positions in this pattern are called *regular* and denoted by subscripts of the bundles $0, 1, \ldots, n$.

If Λ is a singular weight, some of weights $w.\Lambda$, $w \in W^{\mathfrak{p}}$ can be $\mathfrak{p}$–dominant. It follows from the pattern in Table 2.1 that the coefficient -1 can be only once in Λ. Corresponding *singular* positions will be denoted by couples "$0, 1$", ..., "$n-1, n$" as in the even dimensional case. However, there are no nontrivial operators in this case i.e. there are no singular homomorphisms for n odd.

We can obtain more operators – non–standard ones – on patterns with a $\mathfrak{p}$–dominant non–integral weight Λ. This patterns will be called *non–standard*. There can be at most two non–integral coefficients and they have to be half integral greater or equal $-\frac{1}{2}$, cf. Table 2.1. Possible choices for half integral coefficients are shown in Table 1.5. Positions on these patterns will be called *non–standard positions* and will be denoted by subscripts of bundles from Table 1.5 i.e. by $0, 1, \ldots, n$.

In all dimensions, we denote the operators using the developed notation i.e. by S_i, L_j, $S^X_{n'-1}$ etc. possibly with specification of positions i.e. $S_i : i \longrightarrow i+1$, $L_j : j-1, j \longrightarrow n-j, n-j+1$, $L_j : j \longrightarrow n-j$ etc. This determines an operator uniquely. The operators denoted here by L and S will be called *long* and *short* operators, respectively.

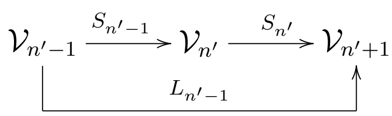
We have seen that if an irreducible conformal bundle admits an operator from the pattern then the coefficient over the crossed node (in the notation of Dynkin diagrams) is an integer for n even and an integer or a half integer for n odd. Translating this to the notation developed in 1.1.3 (based on Young diagrams) the same is true for the conformal weight. Such a weight shall be called *admissible*. That is, the set of admissible weights is of the form

$$\mathbb{AW} = \begin{cases} \mathbb{Z} & n \text{ even} \\ \frac{1}{2}\mathbb{Z} & n \text{ odd.} \end{cases} \tag{1.61}$$

Real cases

A real irreducible bundle V appears in a (real) pattern if any (equivalently every) irreducible component of $V(\mathbb{C})$ appears in a complex pattern. This determines uniquely the positions of V, the type of the real pattern (regular, singular, non-standard) and the operators.

The structure of real patterns can be classified in exactly the same way as in the complex case with one exception, see e.g. [48]. If $n = 2n'$ and the signature (p, q) satisfies $n' - p$ is odd then the bundle $V_{n'}$ can be irreducible. This happens if and only if $\mathbb{V}^\Lambda = \mathbb{E}\{r_1, \cdots, r_{n'-1}, 0\}_0[w]$ where Λ is the weight of the pattern. That is, $\mathbb{V}^\Lambda$ is a tensor representation and there is no column of the length $n' = \frac{n}{2}$ in the Young diagram corresponding to $\mathbb{V}^\Lambda$. Then the latter property is satisfied for all positions in the pattern with the exception of n'. (Recall $n' - p$ odd means the action $\tilde{\boldsymbol{\epsilon}}$ on $V_{n'}$ has no real eigenvalues.) If $V_{n'}$ is irreducible, the middle diamond degenerates to



(We do not need analogues of complex operators $S^Y_{n'-1}$ and $S^Y_{n'}$ for $L_{n'-1}$.) With this exception, we can use the same notation for real and complex flat operators.

1.3.4. Curved conformal geometries. Let us consider conformal structure as P–principal bundle $\mathcal{G} \longrightarrow M$ equipped with the Cartan connection ω. Vector bundles natural for the conformal structure are of the form

$V^\Gamma = \mathcal{G} \times_P \mathbb{V}^\Gamma$ for a linear P–representation $\mathbb{V}^\Gamma$. *Differential* operators between $\mathcal{V}^{\Gamma_1}$ and $\mathcal{V}^{\Gamma_2}$ are, as in the flat case, bundle mappings $J^k V^{\Gamma_1} \longrightarrow V^{\Gamma_2}$ on the k–the jet prolongation. *Invariant* differential operators are, roughly speaking, differential operators $\mathcal{V}^{\Gamma_1} \longrightarrow \mathcal{V}^{\Gamma_2}$ defined for fixed representations $\mathbb{V}^{\Gamma_1}$ and $\mathbb{V}^{\Gamma_2}$ for *all* conformal structures in a universal way independent on any other choices. (See [39] for a general theory). Contrary to the flat case, we shall not provide a precise definition because we do not need it. The operators discussed in this thesis will be *natural* in the following sense. An operator between tensor/spinor bundles is natural if, for a given metric from the conformal class, it can be expressed by a tensor formula defined in Definition 1.2.6 (i) on p. 54. More generally, an operator between natural bundles $\Phi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ is natural if $pr_2^* \Phi pr_1 : \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ is a natural tensor operator for any irreducible $\mathfrak{g}_0$–components $pr_1 : W_1 \hookrightarrow V_1$ and $pr_2 : W_2 \hookrightarrow V_2$ of V_1 and V_2, respectively. In particular, if an operator is given by a tractor formula in the sense of Definition 1.2.6 (i) on p. 54 then it is natural. A natural operator is *invariant* if the formula does not depend on the choice of the metric. Henceforth, if we use the term 'operator' without any specification, we will mean a differential operator which is natural and invariant.

We are primarily interested in operators on general (curved) structures which are non–trivial upon restriction to the flat case. In the other words, we would like to know which operators on conformally flat manifolds extend to all conformal geometries i.e. which of them have *curved analogues*. Most of them do (see below) but some so not and the complete answer is not known yet. Moreover, there can be more curved analogues for a given operator on flat manifolds.

In the discussion on operators in flat parabolic geometries 1.3.2 we considered operators $\mathcal{V}^{\Gamma_1} \longrightarrow \mathcal{V}^{\Gamma_2}$ via the composition $J^k V^{\Gamma_1} \cong \mathcal{G} \times_P J^k \mathbb{V}^{\Gamma_1} \longrightarrow$

V^{Γ_1}. It turns out this approach fails in curved cases for $k \geq 2$, see e.g. [18]. However, the situation is simple for $k = 1$ and we have the following proposition. See [25] for the proof in the (spin) Riemannian case and [49] for the (spin) pseudoriemannian one. (See also [18] for the same result treating all parabolic geometries.)

Proposition. *Every first order operator on flat conformal manifolds has a unique curved analogue.*

There are algebraic techniques [24] which provide existence results for curved analogues of the operators from the pattern in 1.3.3. [24] shows existence of curved analogues for all operators from 1.3.3 with the exception of L_0 in the even dimensional case. Actually, the operator L_0 has a curved analogue if it acts on functions but this result is more subtle [36]. All these results are summarized, together with a nonexistence result from [33], in the following theorem.

Theorem. *Let us consider the classification of the operators on conformally flat manifolds from 1.3.3 and the notation therein. Then:*

(i) All operators from the pattern with the exception of L_0 for n even have curved analogues and also L_0 for n even has a curved analogue if acting on functions. The latter is the operator $L_0 : \mathcal{E} \longrightarrow \mathcal{E}[-n]$.

(ii) The operator $L_0 : \mathcal{E}[w] \longrightarrow \mathcal{E}[-w-n]$, $w \in \mathbb{R}$ for n even has no curved analogue for $w > 0$.

The construction developed in this thesis shall not treat all operators mentioned in Theorem (i). We say that an operator $\mathcal{V}_1 \longrightarrow \mathcal{V}_2$ is *strongly invariant* if it can be written by a formula (in the sense of Definition 1.2.6 on p. 54) which, interpreting the Levi–Civita connection as the coupled Levi–Civita–tractor connection, yields also an operator $\mathcal{V}_1 \otimes \mathcal{E}_{\mathfrak{T}} \longrightarrow \mathcal{V}_2 \otimes \mathcal{E}_{\mathfrak{T}}$ where

$\mathcal{E}_{\mathfrak{T}}$ denotes any tractor bundle. We will also say that the formula is strongly invariant. It is shown in [30] that there is no strongly invariant operator $\mathcal{E} \longrightarrow \mathcal{E}[-n]$ for n even with a leading term $\Delta^{n'}$ i.e., a curved analogue of the flat operator $L_0 : \mathcal{E} \longrightarrow \mathcal{E}[-n]$. (See [21] for a detailed treatment of the operator $L_0 : \mathcal{E} \longrightarrow \mathcal{E}[-4]$ for $n = 4$. This operator is given by a formula which does not depend on the choice $g \in [g]$. However, to verify the latter fact we need to use $\nabla_{[a}\nabla_{b]}f = 0$ which is satisfied for $f \in \mathcal{E}$ but not for $f \in \mathcal{E} \otimes \mathcal{E}_{\mathfrak{T}}$.)

1.3.5. Main aim of the thesis. *We will construct formulae of strongly invariant curved analogues of all invariant operators in the flat case for which such curved analogues are known to exist. These are all operators from the pattern with the exception of L_0 in the even dimensional case. The formulae will be strongly invariant and provided in a compact form in the tractor calculus.*

Formulae for many operators in conformal geometry are known. In particular, there is an algorithm for all short operators [28], see also [19, 14] where other parabolic geometries are treated. The latter exploits representation theory (namely Casimir computation). Among these, [14] provides the simplest algorithm for short operators. This result does not concern the operator $S_{n'}$ for n odd.

A route for getting formulae for long operators is shown in [24] but we have no algorithm for these operators comparable to the results mentioned in the previous paragraph. (Of course, this thesis provides one.) Formulae for the critical operator L_0 on densities for n even are available in [35]: following [17], the result from [36] is translated to the tractor calculus and an algorithm for formulae is developed. (They are computed in terms of the Levi–Civita connection for small orders.) Recall our construction does not concern this operator. A special form of long operators on density–valued forms is constructed in [11]. See also references therein for other related Branson's results.

Our formulae in the tractor calculus will not require any additional information from representation theory. Also note that it is straightforward to express tractor expressions in terms of the Levi–Civita connection and its curvature using the definitions of the tractor connection and tractor D–operator from 1.2.3 and 1.2.4 (eventually 1.2.5). This procedure is tedious for operators of higher orders and can be done by computers using, for example, the software developed for [35].

1.3.6. Curved translation principle. This is a general procedure which can build complicated operators from simpler ones. In the flat case, this is provided by the Jantzen–Zuckermann translation functor [56]. This result yields the translation for (dual) homomorphisms of generalized Verma modules. A key point here is to use the action of the centre of $\mathfrak{U}(\mathfrak{g})$. See also [24] for more details.

The "simple" operators we start with are the exterior derivative d, the conformal Laplacian $\Box$ and the Dirac operator $D\!\!\!/$. Considering the pattern from 1.3.3, $d : \mathcal{E}^k \longrightarrow \mathcal{E}^{k+1}$ is the short operator S_k for $\Lambda = 0$ (the weight of the pattern). This pattern is the deRham complex. Operators $\Box : \mathcal{E}[1 - \frac{n}{2}] \longrightarrow \mathcal{E}[-1-\frac{n}{2}]$ and $D\!\!\!/ : \mathcal{E}_\lambda[1-\frac{n}{2}] \longrightarrow \mathcal{E}_\lambda[-\frac{n}{2}]$ appear on appropriate singular and non–standard patterns as $L_{n'-1}$. In fact, these are strongly conformally invariant operators on all curved manifolds.

The curved version of the translation functor is the Eastwood's *curved translation principle* introduced in [23]. We obtain the results promised in 1.3.5 by an implementation of this technique in the tractor calculus. Before we discuss the general case, we demonstrate the translation on curved analogues of the flat operator $\Delta^k : \mathcal{E}[k - \frac{n}{2}] \longrightarrow \mathcal{E}[-k - \frac{n}{2}]$, $k \geq 1$. (See [22] for the invariance of Δ^k in the flat case.) $\Box$ is a curved analogue of Δ^2. For $k > 1$, we follow Eastwood and Gover in [21] (see also [35] for more details). We define $\Box_2 := \Box$ and we "translate" this operator to a curved analogue of Δ^k, denoted by $\Box_{2k}$, as follows. We define

$$\begin{aligned} \Box_{2k} : \mathcal{E}[k-n/2] &\longrightarrow \mathcal{E}[-k-n/2] \\ D^{A_1} \cdots D^{A_{k-1}} \Box D_{A_{k-1}} \cdots D_{A_1} f &= \left(\prod_{i=2}^{k} (n-2i)(i-1) \right) \Box_{2k} f \end{aligned} \tag{1.62}$$

for $k \geq 2$. The scalar on the right hand side is nonzero for n even and $k = n' - i < n'$ i.e. $i > 0$ or for n odd. Note the condition $k < \frac{n}{2}$ for n even means that $\Box_{2k}$ does not provide a formula for L_0 on functions. This was inevitable because the operator $\Box_{2k}$, given here as a composition of strongly invariant operators, is necessarily strongly invariant whereas L_0 is not.

The first step in the translation is to apply a *splitting operator* (or just a *splitting*) which is a differential operator between natural bundles $Split : \mathcal{V} \longrightarrow \mathcal{V}'$ such that there is a $\mathfrak{g}_0$–component $pr : \mathcal{V} \hookrightarrow \mathcal{V}'$ of V' satisfying $pr^* Split = \mathrm{id}_{\mathcal{V}}$. Typically, V will be a tensor bundle and V' a tractor bundle and *Split* "puts" or "extends" the section $f \in \mathcal{V}$ into tractors. A trivial example is $f_{\mathfrak{a}} \mapsto X_A f_{\mathfrak{a}}$. (See [21] for several nontrivial (but simple) examples.) In the translation (1.62) above, $f \mapsto D_{A_{k-1}} \cdots D_{A_1} f$ is a splitting on $\mathcal{E}[k - n/2]$.

To give some idea of the translation in the general case we give an informal account of this construction. Suppose we need a curved analogue of an operator Q from the pattern, acting on a bundle V_1. First we need an appropriate (invariant) splitting

$$\mathcal{V}_1 \ni f \mapsto Split(f) = \begin{pmatrix} \vdots \\ 0 \\ f \\ * \\ \vdots \end{pmatrix} \in \mathcal{V}_1'.$$

Then we apply a (strongly invariant) operator $Op \in \{d, \Box, \not{D}\}$ which yields

$$\mathcal{V}_1 \ni f \mapsto Op \circ Split(f) = \begin{pmatrix} \vdots \\ * \\ Q'f \\ * \\ \vdots \end{pmatrix} \in \mathcal{V}_2'$$

where $Q = Q'$ in the flat case. That is, we will be always able to guarantee that such $Q'f$ appears among the slots on the right hand side and the slots affecting invariance of Q' (i.e. the slots displayed "above" Q') are zero in the flat case. In the curved case, there can be curvature terms in these slots which means the projection to Q' would not be invariant. To solve this problem, we will replace the projection to Q' by formal adjoint of (generally another) splitting operator $Split_2$. This will be denoted by $Split_2^* : \mathcal{V}_2' \longrightarrow \mathcal{V}_2$. Note formal adjoints of splitting operators are sort of dual operations to the splitting: $Split_2^*$ goes from tractors back to tensors in an invariant way, see details in 2.1.8. This use of dual splittings was pioneered by Branson and Gover [10]. So the resulting form of the operator Q obtained by the described realization of the curved translation principle is

$$Q = Split_2^* \circ Op \circ Split_1. \tag{1.63}$$

1.3.7. gBGG splitting operator. The existence of appropriate splitting operators is the crucial question for the translation. It has been established for short operators in [20] (see also [13]). Let us start with the complex case. Consider a regular pattern with the ($\mathfrak{g}$–dominant) weight Λ and a bundle $V^{w.\Lambda}$ with $w \in W^{\mathfrak{p}}$, see 1.3.3. Denoting $k = |w|$ the length of w, it follows from Kostant's theorem A.1.1 that the corresponding representation space is a $\mathfrak{g}_0$–submodule $\mathbb{V}^{w.\Lambda} \subseteq \mathbb{E}_{\mathbf{a}^k} \otimes \mathbb{V}^{\Lambda}$. The inclusion $\mathbb{V}^{w.\Lambda} \hookrightarrow \mathbb{E}_{\mathbf{a}^k} \otimes \mathbb{V}^{\Lambda}$ is unique, see Theorem A.1.1.

Definition. In the complex case, let us consider a $\mathfrak{g}$–dominant weight Λ, the tractor bundle V^{Λ} and the irreducible bundle $V^{w.\Lambda}$ for $w \in W^{\mathfrak{p}}$, $|w| = k$.

gBGG splitting operator is a splitting operator $\mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{E}_{\mathbf{a}^k} \otimes \mathcal{V}^{\Lambda}$. In the real case and for V irreducible, $S : \mathcal{V} \longrightarrow \mathcal{V}'$ is the *gBGG splitting operator* if the restriction of $S(\mathbb{C})$ to any irreducible component of $\mathcal{V}(\mathbb{C})$ is the gBGG splitting operator.

Proposition. *Consider the bundle $\mathcal{V}$ in the regular pattern on the position k or k_X or k_Y. Then the gBGG slitting exists on all curved manifolds and has the form $\mathcal{V} \longrightarrow \mathcal{E}_{\mathbf{a}^k} \otimes \mathcal{T}$ where T is a tractor bundle. It is determined uniquely in the flat case.*

Proof. See [20] (or [13] or Chapter 2) for the existence and Appendix A for the flat case. □

There are many differences between the construction in Chapter 2 and [20, 13]. Both build the gBGG splitting from simpler steps. Consider the gBGG splitting $\mathcal{E}_{\mathbf{a}^k}[k+1] \longrightarrow \mathcal{E}_{\mathbf{A}^{k+1}} \cong \mathcal{E}_{\mathbf{a}^0} \otimes \mathcal{E}_{\mathbf{A}^{k+1}}$. Using the notation for quotient spaces from 1.4, the [20, 13] construction can be schematically displayed as

$$\mathcal{E}_{\mathbf{a}^k}[k+1] \ni f \mapsto \begin{pmatrix} & f & \\ - & & - \\ & - & \end{pmatrix} \mapsto \begin{pmatrix} & f & \\ * & & * \\ & - & \end{pmatrix} \mapsto \begin{pmatrix} & f & \\ * & & * \\ & * & \end{pmatrix} \in \mathcal{E}_{\mathbf{A}^{k+1}}$$

whereas our construction will be

$$\mathcal{E}_{\mathbf{a}^k}[k+1] \ni f \mapsto \begin{pmatrix} & f & \\ - & & - \\ & - & \end{pmatrix} \mapsto \begin{pmatrix} & f & \\ - & & * \\ & - & \end{pmatrix} \mapsto \begin{pmatrix} & f & \\ * & & * \\ & * & \end{pmatrix} \in \mathcal{E}_{\mathbf{A}^{k+1}}.$$

Actually, we will later replace operators between quotient bundles by operators between (sub)bundles and (contrary to [20]) decompose the construction of *DSplit* into invariant steps. We will provide strongly invariant tractor formulae for all these steps.

We have no preferred splitting operator for non–regular patterns. However, *DSplit* is well-defined for all irreducible bundles. We shall see later that it is actually suitable for the translation of long operators.

1.3.8. Notes on zero and first order operators. The purpose of this section is to state simple properties concerning invariant differential operators of order 0 and 1. In the following Lemma, we shall use the notation from 1.1.3 describing Young symmetries and corresponding bundles by a sequence of numbers $(s_1, \ldots, s_r)$.

Lemma. *Let us suppose that we have the nontrivial operators*

$$E_1 : \ \mathcal{E}(s_1, \ldots, s_r)_0[w_1] \longrightarrow \mathcal{E}(s'_1, \ldots, s'_{r'})_0[w_2]$$
$$E_2 : \ \mathcal{E}(s_1, \ldots, s_r)_0[w_1] \longrightarrow \mathcal{E}^{s'_1} \otimes \cdots \otimes \mathcal{E}^{s'_u}[w_2] \ \ \textit{where} \ \ s'_1 \geq \cdots \geq s'_u$$

given by a formula of formal order 0*, which does not use the volume form* $\boldsymbol{\epsilon}$*. The conformal weights* w_1 *and* w_2 *are real or complex scalars. According to the constructions in 1.1.3, this means that all free tensor indices of sections of these bundles are covariant ("downstairs"). Recall* $s = \sum_{i=1}^{r} s_i$*.*

(i) The operator E_1 *satisfies* $r = r'$ *and* $s_i = s'_i$ *for all* $i \in \{1, \ldots, r\}$ *and the operator is a multiple of identity.*

(ii) The operator E_2 *satisfies* $s' - s \in 2\mathbb{N}_0$ *where* $s' = \sum_{i=1}^{u} s'_i$*.*

(iii) Assume $\mathcal{E}(s_1, \ldots, s_r)_0[w_1] \subseteq \mathcal{E}_{\mathbf{a}_1 \cdots \mathbf{a}_r}[w_1]$*. Let us suppose* $s = s'$ *where* $s' = \sum_{i=1}^{u} s'_i$*. Then* $s_1 \geq s'_1$*. More generally, suppose* $s = s'$ *and* E_2 *is given by a formula* Φ *satisfying that, for a fixed* $k \in \{1, \ldots, r\}$*, the indices from* $\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}$ *do not appear in* Φ*. Then* $s_1 = s'_1, \ldots, s_{k-1} = s'_{k-1}$ *and* $s_k \geq s'_k$*.*

Proof. E_1 and E_2 are nontrivial algebraic operator because their formal order is zero. (The latter means that there are no ∇'s, R's and P's in formulae for E_1 and E_2.

(i) E_1 is a nontrivial algebraic operator. If the source space is irreducible then $E_1 = C\,\mathrm{id}$, $C \neq 0$ by Schur's lemma and (i) follows. If $s_1 = \frac{n}{2}$ then either $E_1 = C\,\mathrm{id}$, $C \neq 0$ or E_1 is a projection to an irreducible component. The latter requires the volume form.

(ii) Assume f is a section of the source space. Then $E_2 f$ is a sum of terms $\boldsymbol{g} \cdots \boldsymbol{g} f$ with all indices covariant and (ii) follows.

(iii) Following (ii), if $s = s'$ we can suppose $E_2 f$ is a sum of f's (with possibly renamed and (skew)symmetrized indices). Clearly $s_1 \geq s'_1$ because $s_1 < s'_1$ requires more than s_1 skew indices of f which vanishes, see 1.1.3. The more general case follows from the (similar) property that a skew symmetrization over more than s_k indices among $\mathbf{a}_k, \ldots, \mathbf{a}_r$ vanishes. $\square$

Proposition. *Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and let U, V be natural tensor–spinor bundles. Let $\Phi(w)$ be an expression, polynomial in $w \in \mathbb{K}$, such that for every fixed w, $\Phi(w)$ is a formula in the sense of Definition 1.2.6 on p. 54 satisfying $fo(\Phi(w)) = 1$. Suppose $\Phi(w)$ defines a 1–parameter family of differential operators $E(w) : \mathcal{U}[w] \longrightarrow \mathcal{V}[w']$ where $w' \in \mathbb{K}$ depends on w. (That is, $\Phi(w)$ possesses no free (spinor) tractor indices.) Suppose $E(w)$ is invariant for every w. Then $E(w)$ is trivial for every w.*

Proof. Assume U, V are irreducible. Note for a given w, $E(w)$ is either of the first order and not algebraic or $E(w)$ vanishes. ($fo(\Phi(w)) = 1$ guarantees ∇ appears once in every summand of $\Phi(w)$ and any term involving the curvature requires $fo(\Phi(w)) \geq 2$.) That is, if $E(w)$ does not vanish, it will be an irreducible gradient $E(w) = Proj_V \nabla$ where $V \hookrightarrow T^*M \otimes U[w]$ is unique [51]. Summarizing, $E(w) = c(w) Proj_V \nabla$ up to an isomorphism. (Recall $E(w)$ is an operator – not a formula – here.) Here $c(w)$ is a scalar depending on w. Since $\Phi(w)$ depends polynomially on w, it has the form of a finite formal sum $\Phi(w) = \sum_i c_i(w) \mathrm{Proj}_i \Phi_i$ where $c_i(w)$ are polynomials, see the Definition, page 54. Therefore $c(w)$ is polynomial in w. But every gradient is invariant for a unique conformal weight. (See [25] for the Riemannian case and [49] for a generalization to nondefinite and complex cases.) This means that $c(w)$ can be nonzero only for a unique $w \in \mathbb{K}$. Therefore the

$c(w)$ vanishes for every w.

If U, V are not irreducible, we can apply the same reasoning to all irreducible components in U and V. □

Chapter 2

Splitting operators

2.1 Construction of splittings

Let us start with k–forms and splitting operators of the form $\mathcal{E}_{\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{\mathbf{A}^l}[w']$. Looking at possible projecting parts of $\mathcal{E}_{\mathbf{A}^l}$ in (1.45), we have especially the following possibilities for $f \in \mathcal{E}_{\mathbf{a}^k}[w]$:

$$B : f \mapsto \begin{pmatrix} & 0 & \\ 0 & & 0 \\ & f & \end{pmatrix}, \quad M : f \mapsto \begin{pmatrix} & 0 & \\ f & & 0 \\ & * & \end{pmatrix}, \quad T : f \mapsto \begin{pmatrix} & f & \\ * & & * \\ & * & \end{pmatrix}, \quad f \mapsto \begin{pmatrix} & 0 & \\ 0 & & f \\ & * & \end{pmatrix}. \tag{2.1}$$

(We will not need more details about the last of these. See also 2.1.9 and the operator $\tilde{M}$ therein). All these splittings were constructed in [11] via ambient metric. As we work directly on the tractor bundle, we will obtain explicit formulae for B, M and T.

The operators B, M and T in (2.1) are called the *bottom*, *middle* and *top* operator and are generalised to irreducible tensor bundles in 2.1.1, 2.1.4 and 2.1.5, respectively. That is, they are constructed as operators

$$\mathcal{E}_{\mathfrak{T}}\{r_1, \dots, r_{n'}\}_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}\mathbf{A}^l}\{r_1, \dots, r_{k-1}, r_k - 1, r_{k+1}, \dots, r_{n'}\}_0[w'] \tag{2.2}$$

where $r_k \geq 1$ and the tractor indices in $\mathfrak{T}$ indicate the strong invariance. Obviously $l = k+1$ for the operators B and T and $l = k$ for the operator M.

For reasons that will soon become clear preferred choices are $k := \min\{i \mid r_i \neq 0\}$ for the operator M and $k := \max\{i \mid r_i \neq 0\}$ for the operator B and T. These choices are sufficient for the aim of the thesis. However any value k such that $r_k \neq 0$ is possible, see 2.1.9 for details about M and T (the case of B is obvious). The operators (2.2) concern tensor bundles. The generalisation to spinors is then straightforward, see 2.1.6.

We shall demonstrate the calculus for operators B, M and T mainly on spaces $\mathcal{E}(k)[w]$ and $\mathcal{E}(k,l)_0[w]$ and their spinor versions. The case $\mathcal{E}(k,l)_0[w]$, although a simple one from our point of view has not been studied much previously and formulae for many operators are not known. The Example 3.1.6 shows the result - tractor formulae for curved analogues of *all* (strongly) invariant operators on $\mathcal{E}(k,l)_0[w]$ which exist in the flat case. These formulae are expressed in terms of B, M, T and their formal adjoints.

The results in 2.1.1 below are in a sense obvious but important for the reader as they demonstrate the notation and properties which become more complicated in the case of M and T.

On the other hand, 2.1.2 is not essential for the general construction. The aim here is to demonstrate in special cases the techniques we will use for the construction of M and T. These special cases will be (density valued) 2–forms and the space $\mathcal{E}(2,2)_0[w]$ i.e. the trace–free Young symmetries . Another motivation for 2.1.2 is that it can be applied to the Weyl curvature tensor $C \in \mathcal{E}(2,2)_0[2]$.

Throughout the thesis, we shall use the bundles $E(l; s_1, \cdots, s_r)_0[w]$, $l \in \{0, \frac{1}{2}\}$, or equivalently $E\{r_1, \cdots, r_{n'}\}_0[w]$, which have no attached indices. However, they are defined as subbundles

$$E\{r_1, \cdots, r_{n'}\}_0[w] = E(\frac{1}{2}; s_1, \cdots, s_r)_0[w] \subseteq E_{\lambda \mathbf{a}_1 \cdots \mathbf{a}_r}[w]$$

where $\mathbf{a}_i = \mathbf{a}_i^{s_i}$, $1 \leq i \leq r$ in the spinor case, and similarly in the tensor one. We shall use this index structure implicitly. For example, we will consider $\mathbb{X}_{A_i^0}{}^{\mathbf{a}_i}{}_{\mathbf{A}_i}$ as an operator on $E(\frac{1}{2}; s_1, \cdots, s_r)_0[w]$. Using the previous display, the subscript i determines this operator uniquely.

2.1.1. Bottom operator. The bottom operator on k–forms $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[w]$ is the algebraic operator $B_{A^0}{}^{\mathbf{a}}{}_{\mathbf{A}} f_{\mathbf{a}} := \mathbb{X}_{A^0}{}^{\mathbf{a}}{}_{\mathbf{A}} f_{\mathbf{a}}$ where $\mathbf{A} = \mathbf{A}^k$. B lowers the conformal weight w by $k-1$. Let us consider the the general case $f = f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} \in \mathcal{E}_{\mathfrak{T}}(l; s_1, \ldots, s_r)_0[w]$, $l \in \{0, \frac{1}{2}\}$. Obviously, we can apply $\mathbb{X}_{A^0}{}^{\mathbf{a}}{}_{\mathbf{A}}$ to any form index $\mathbf{a} = \mathbf{a}^{s_j}$ or to use the spinor projection X_Λ^λ. However, it will be convenient for our subsequent constructions to define the *bottom operator* B as

$$\begin{gathered} B_{A_1^0}{}^{\mathbf{a}_1}{}_{\mathbf{A}_1} : \mathcal{E}_{\mathfrak{T}}(0; s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}[A_1^0 \mathbf{A}_1]}(s_2, \ldots, s_r)_0[w - s_1 + 1] \\ B_{A_1^0}{}^{\mathbf{a}_1}{}_{\mathbf{A}_1} f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} = \mathbb{X}_{A_1^0}{}^{\mathbf{a}_1}{}_{\mathbf{A}_1} f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} \end{gathered} \tag{2.3}$$

for tensor representations and

$$\begin{gathered} B_\Lambda^\lambda : \mathcal{E}_{\mathfrak{T}}(\tfrac{1}{2}; s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}\Lambda}(s_1, \ldots, s_r)_0[w] \\ B_\Lambda^\lambda f_{\mathfrak{T}\lambda \mathbf{a}_1 \cdots \mathbf{a}_r} = X_\Lambda^\lambda f_{\mathfrak{T}\lambda \mathbf{a}_1 \cdots \mathbf{a}_r} \end{gathered} \tag{2.4}$$

for spinor ones. That is, we use X_Λ^λ or, if there are only form indices, we use $\mathbb{X}_{A^0}{}^{\mathbf{a}}{}_{\mathbf{A}}$ where $\mathbf{a}$ is a form index of the maximal valence.

Since the bottom operator is of the zero order, it is strongly invariant and can be used repeatedly. Let us suppose we want to apply the bottom operator b times, $1 \leq b \leq r$, $b \in \mathbb{N}$ in the tensor case. The result, also called the *bottom operator*, is the composition

$$\begin{gathered} \big(B^{(b)}\big)_{A_b^0}{}^{\mathbf{a}_b \cdots}{}_{\mathbf{A}_b \cdots A_1^0}{}^{\mathbf{a}_1}{}_{\mathbf{A}_1} =: B_{A_b^0}{}^{\mathbf{a}_b}{}_{\mathbf{A}_b} \cdots B_{A_1^0}{}^{\mathbf{a}_1}{}_{\mathbf{A}_1} : \\ \mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}[A_b^0 \mathbf{A}_b] \cdots [A_1^0 \mathbf{A}_1]}(s_{b+1}, \ldots, s_r)_0[w - s^b + b]. \end{gathered} \tag{2.5}$$

In the spinor case, we compose the bottom operator B_Λ^λ and $\lfloor b \rfloor \in \mathbb{N}$ tensor bottom operators and we put $b := \lfloor b \rfloor + \frac{1}{2} \in \frac{1}{2}\mathbb{N}$ where $\frac{1}{2} \leq b = \lfloor b \rfloor + \frac{1}{2} \leq r$.

This will be denoted by

$$\begin{aligned} \big(B^{(b)}\big)_{A^0_b\mathbf{A}_b\cdots A^0_1\mathbf{A}_1\Lambda}^{\mathbf{a}_b\cdots\ \ \mathbf{a}_1\,\lambda} =: B_{A^0_b\mathbf{A}_b}^{\ \ \mathbf{a}_b}\cdots B_{A^0_1\mathbf{A}_b}^{\ \ \mathbf{a}_1}B_\Lambda^\lambda : \\ \mathcal{E}(\frac{1}{2}; s_1,\ldots,s_r)_0 \longrightarrow \mathcal{E}_{\Lambda\mathfrak{T}[A^0_1\mathbf{A}_1]\cdots[A^0_b\mathbf{A}_b]}(s_{b+1},\ldots,s_r)_0[w-s^b+b]. \end{aligned} \tag{2.6}$$

Recall that in connection with spinors we use the following conventions from 1.1.3. We consider implicitly the integer part $\lfloor b\rfloor$ of b in expressions with non-integer subscript like $\mathbf{a}_b$ or s_b but $s^b = \frac{1}{2} + s^{\lfloor b\rfloor}$. Finally note that since $oh(X^\lambda_\Lambda) = h(X^\lambda_\Lambda) = -\frac{1}{2}$ and $oh(\mathbb{X}_{A^0\mathbf{A}}^{\ \ \mathbf{a}}) = h(\mathbb{X}_{A^0\mathbf{A}}^{\ \ \mathbf{a}}) = -1$ according to the notation in 1.2.6. Therefore (2.5) and (2.6) yield

$$oh(B^{(b)}) = h(B^{(b)}) = -b. \tag{2.7}$$

Definition/Terminology. The operator $B^{(b)}$ will be called *bottom splitting* or *bottom splitting operator*.

Theorem (Properties of the bottom operator).
Let us consider the bottom operator $B^{(b)}$ given by the relation (2.5) or (2.6) and the system of indices $\mathfrak{A} = [A^0_1\mathbf{A}_1]\cdots[A^0_{\lfloor b\rfloor}\mathbf{A}_{\lfloor b\rfloor}]$. The TFP–component

$$pr_b := \begin{cases} \mathbb{X}_{A^0_1\mathbf{A}_1}^{\ \ \mathbf{a}_1}\cdots\mathbb{X}_{A^0_t\mathbf{A}_b}^{\ \ \mathbf{a}_b} \in TFPC(E_\mathfrak{A}) & b,r\in\mathbb{N} \\ \mathbb{X}^\lambda_\Lambda\mathbb{X}_{A^0_1\mathbf{A}_1}^{\ \ \mathbf{a}_1}\cdots\mathbb{X}_{A^0_t\mathbf{A}_b}^{\ \ \mathbf{a}_b} \in TFPC(E_{\Lambda\mathfrak{A}}) & b,r\notin\mathbb{N} \end{cases}$$

is the only non-vanishing projecting part of $B^{(b)}$.

We shall not always strictly distinguish between operators and corresponding formulae. (We study operator which are natural, i.e. given by a formula, see 1.3.4.) But we will write $= 0$ (or $\neq 0$) only for operators. That is, if Φ is and operator/formula then $\Phi = 0$ means that this operator vanishes. (The formula may be nontrivial.) On the other hand, if we write $oh(\Phi)$, this always concern the formula Φ.

2.1.2. Simple examples of M and T. We shall describe the operators B, M and T for $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^2}[w]$ first and then the operator M for $f_{\mathbf{ab}} \in \mathcal{E}(2,2)_0[w]$. The symmetries of the latter will be referred as Weyl tensor symmetries. Recall that $C_{\mathbf{ab}} = C_{a^1a^2b^1b^2} \in \mathcal{E}(2,2)_0[2]$.

According to 1.3.8, the bottom splitting $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[w]$ is just $f_{\mathbf{a}} \mapsto \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{a}}$, $\mathbf{A} = \mathbf{A}^2$, i.e.

$$f_{\mathbf{a}} \xrightarrow{\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}} \begin{pmatrix} & 0 & \\ 0 & & 0 \\ & f_{\mathbf{a}} & \end{pmatrix}$$

in the matrix notation. Now let us try to use this to construct the middle operator. Following [11], we shall use the notation

$$\begin{aligned} (\varepsilon(\Phi)F)_{[A^0\mathbf{A}^k]} &= (k+1)\Phi_{[A_0}F_{\mathbf{A}^k]} \\ (\iota(\Phi)F)_{\dot{\mathbf{A}}^k} &= \Phi^{A_1}F_{\mathbf{A}^k} \end{aligned}$$

for an operator Φ on $\mathcal{E}_{\mathbf{A}^k}[w]$ increasing the valence by one.

Middle operator on $\mathcal{E}_{\mathbf{a}^2}[w]$. Let us start with $\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}}$ according to the matrix notation in (2.1). While this is not invariant as a section of $\mathcal{E}_{\mathbf{A}}[w-2]$, it is invariant as a section of the quotient bundle $(\mathcal{E}_{\mathbb{YZ}})_{\mathbf{A}}[w-2]$. So we need an invariant operator $(\mathcal{E}_{\mathbb{YZ}})_{\mathbf{A}}[w-2] \longrightarrow \mathcal{E}_{\mathbf{A}}[w-2]$. On the other hand, we have the bottom splitting $\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}$ and so we can consider $D^{A^0}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} = \frac{1}{3}\iota(D)\varepsilon(X)$. We are going to show this composition applied to $\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}}$ provides the desired middle operator. (This was used in [27] on the tractor curvature.) In the matrix notation we get

$$f_{\mathbf{a}} \xrightarrow{\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}} \begin{pmatrix} & 0 & \\ f_{\mathbf{a}} & & - \\ & - & \end{pmatrix} \xrightarrow{\iota(D)\varepsilon(X)} \begin{pmatrix} & 0 & \\ c(w)f_{\mathbf{a}} & & 0 \\ & * & \end{pmatrix} =: \tilde{f}_{\mathbf{A}} \qquad (2.8)$$

where $c(w)$ is a scalar depending on w. We have to show the $\mathbb{Y}$ and $\mathbb{W}$–slots of $\tilde{f}_{\mathbf{A}}$ vanish and the $\mathbb{Z}$–slot reveals a multiple of the identity. Since $oh(\mathbb{Z}_{\mathbf{A}}) = 0$, $oh(X_A) = -1$ and $oh(D_A) = 1$ we get $oh\big(\iota(D)\varepsilon(X)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\big) = 0$, see p. 56. That is, every summand in $\iota(D)\varepsilon(X)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}$ (i.e. a juxtaposition of $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$ and a tensor formula in the sense of Definition on p. 54) satisfies this property. Using $oh(\mathbb{Y}\varphi) \geq 1$ for any tensor formula φ, it follows from $oh\big(\iota(D)\varepsilon(X)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\big) = 0$ that $\mathbb{Y}$ does not appear in any summand of $\iota(D)\varepsilon(X)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}$. Therefore the $\mathbb{Y}$–slot of $\tilde{f}_{\mathbf{A}}$ vanishes. Similarly, if $\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\varphi$ is (up to a multiple) a summand in $\iota(D)\varepsilon(X)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}$ then $oh(\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\varphi) = oh\big(\iota(D)\varepsilon(X)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\big) = 0$ hence $fo(\varphi) = 0$. That is, φ is algebraic. Since also $oh(\mathbb{W}) = 0$, we have shown that the $\mathbb{W}$– and $\mathbb{Z}$–slots of $\tilde{f}_{\mathbf{A}}$ can be only multiples of the identity. Since $\mathbb{W}_{A^1A^2}$ possesses no tensor indices, the $\mathbb{W}$–slot of $\tilde{f}_{\mathbf{A}}$ vanishes. Similarly, the $\mathbb{X}$–slot of $\tilde{f}_{\mathbf{A}}$ (i.e. the star in the matrix notation above) is a first order operator on $f_{\mathbf{a}^2}$ and since $\mathbb{X}_{A^1}{}_{A^2}^{a^2}$ possesses only one tensor index, the $\mathbb{X}$–slot is $c'(w)\nabla^p f_{pa^2}$ for an appropriate scalar $c'(w)$.

It remains to identify $c(w)$ in (2.8) and $c'(w)$ above. First, $\varepsilon(X)\mathbb{Z}_{\mathbf{A}}^{\mathbf{a}} = 3\mathbb{X}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}}$. Recall that the formula (1.32) for D^{A^0} has three summands in a scale. We shall refer them as the Y^{A^0}, Z^{A^0} and X^{A^0}–term. Since $3\mathbb{X}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}} \in \mathcal{E}_{[A^0\mathbf{A}]}[w-1]$, the Y^{A^0} term contributes to $c(w)$ with $(w-1)(n+2w-4)$. Looking at the formula for $\nabla^p\mathbb{X}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}}$ in (1.49), we see that the contribution of the Z^{A^0}–term is given by $3(n+2w-4)Z^{A^0}_{a^0}\mathbb{Z}^{a^0}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}}$ which yields the scalar $(n+2w-4)(n-2)$. Finally, looking at the formula for $3\Delta\mathbb{X}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}}$ in (1.50), the contribution of the X^{A^0}–term is determined by $-X^{A^0}(-n+4)\mathbb{Y}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}} f_{\mathbf{a}}$ and this yields the scalar $(n-4)$. Hence the scalar $c(w)$ is equal to

$$(w-1)(n+2w-4)+(n+2w-4)(n-2)+(n-4) = (n+2w-2)(n+w-4).$$

The computation of $c'(w)$ in the $\mathbb{X}$–slot of $\tilde{f}_{\mathbf{A}}$ is similar. The contribution of the Y^{A^0}–term is zero. The contribution of the Z^{A^0}–term is is clearly given by $3(n+2w-4)Z^{A^0p}\mathbb{X}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}}\nabla_p f_{\mathbf{a}}$ which yields the scalar $-2(n+2w-4)$. Finally, the contribution of the X^{A^0}–term is given by $-X^{A^0}(-12\mathbb{W}_{A^0A^1}{}_{A^2}^{a^2})\nabla^{a^1} f_{\mathbf{a}}$ which yields the scalar -4. The result is $c'(w) = -2(n+2w-2)$.

Summarising the scalars $c(w)$ and $c'(w)$, we can define the middle operator either as the operator $M^{\mathbf{a}}_{\mathbf{A}}$ on $\mathcal{E}_{\mathbf{a}^2}[w]$ or M on $(\mathcal{E}_{\mathbb{YZ}})_{\mathbf{A}^2}[w-2]$ by the formulae

$$f_{\mathbf{a}} \xrightarrow{M^{\mathbf{a}}_{\mathbf{A}}} \begin{pmatrix} 0 \\ (n+w-4)f_{\mathbf{a}} \;\; 0 \\ -2\nabla^p f_{pa^2} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 \\ f_{\mathbf{a}} \;\; - \\ - \end{pmatrix} \xrightarrow{M} \begin{pmatrix} 0 \\ (n+w-4)f_{\mathbf{a}} \;\; 0 \\ -2\nabla^p f_{pa^2} \end{pmatrix} \tag{2.9}$$

respectively, for $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^2}[w]$. (See [21] for an analogous operator on 1–forms.) Using an obvious polynomial continuation, we removed the scalar $n+2w-2$ from both $c(w)$ and $c'(w)$. This does not affect the invariance, see 2.1.4. Clearly the middle operator is a splitting operator for $w \neq 4-n$.

Top operator on $\mathcal{E}_{\mathbf{a}}[w]$. Having the middle operator at hand, we can use the tractor D–operator once more and consider $D_{[A^0}M^{\mathbf{a}}_{\mathbf{A}]}f_{\mathbf{a}}$. In terms of invariant quotient spaces, this is the composition

$$f_{\mathbf{a}} \xrightarrow{\mathbb{Y}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}} \begin{pmatrix} f_{\mathbf{a}} \\ - \;\; - \\ - \end{pmatrix} \xrightarrow{\varepsilon(Y)M\iota(X)} \begin{pmatrix} m(w)f_{\mathbf{a}} \\ - \;\; * \\ - \end{pmatrix} \xrightarrow{\varepsilon(D)\iota(X)} \begin{pmatrix} t(w)f_{\mathbf{a}} \\ * \;\; * \\ * \end{pmatrix} \tag{2.10}$$

where $m(w) = n+w-4$ and $t(w)$ is another scalar (depending on w). Using $oh(\mathbb{Y}) = 1$ and $oh(\varepsilon(Y)M\iota(X)) = oh(\varepsilon(D)\iota(X)) = 0$, we see that oh of the whole composition is 1. Since the top slot of $\mathcal{E}_{[A^0\mathbf{A}^2]}$ has the homogenity $oh(\mathbb{Y}) = 1$, only an algebraic operator on f could be in this slot. It follows from the tensor valence of $\mathbb{Y}_{A^0}{}^{a^1}_{A^1}{}^{a^2}_{A^2}$ that it is a multiple of $f_{\mathbf{a}}$. Similarly, it follows from $oh(\mathbb{Z}) = oh(\mathbb{W}) = 0$ that the $\mathbb{Z}$ and $\mathbb{W}$–slots yield first order operators. Using the tensor valences of $\mathbb{Z}^{a^0}_{A^0}{}^{a^1}_{A^1}{}^{a^2}_{A^2}$ and $\mathbb{W}_{A^0A^1}{}^{a^2}_{A^2}$, these operators can be only $\nabla_{[a^0}f_{\mathbf{a}]}$ and $\nabla^p f_{pa^2}$ up to multiples, respectively. An explicit formula for the top operator on forms is computed in Example 2.1.6.

Middle operator on $\mathcal{E}(2,2)_0[w]$. The (two–dimensional) matrix notation is getting complicated so we shall use only the XYZ–calculus. We cannot use the operator $M^{\mathbf{a}}_{\mathbf{A}}$ or $M^{\mathbf{b}}_{\mathbf{B}}$, constructed above, directly for $f_{\mathbf{ab}}$. But we can use the bottom operator first which yields $f'_{A^0\mathbf{Ab}} = \mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} \in \mathcal{E}_{[A^0\mathbf{A}]\mathbf{b}}[w-1]$. Then we can apply $M^{\mathbf{b}}_{\mathbf{B}}$. (The middle operator is strongly invariant.) We obtain

$$\begin{aligned}
M^{\mathbf{b}}_{\mathbf{B}} f'_{A^0\mathbf{A}\mathbf{b}} &= \left[(n+w-5)\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} - 2\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}} \nabla^{b^1}\right] \mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} \\
&= (n+w-5)\mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} \mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} f_{\mathbf{ab}} - 2\mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} \mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}} \nabla^{b^1} f_{\mathbf{ab}} \qquad (2.11)\\
&= \mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} \left[(n+w-5)\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} - 2\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}} \nabla^{b^1}\right] f_{\mathbf{ab}}
\end{aligned}$$

where the first equality is just the formula (2.9) rewritten in the XYZ–calculus. The essential part of the second equality is that we can commute ∇^{b^1} and $\mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}$. This follows from a direct computation (see Example 2.1.2) which uses properties characterizing the bundle $E(2,2)_0[w]$. We need the trace–freeness and Weyl tensor symmetries (in particular the fact that the skew–symmetrization $[b^1a^1a^2]$ vanishes for $f_{\mathbf{ab}}$). Therefore we can define the operator $M^{\mathbf{b}}_{\mathbf{B}}$ on $\mathcal{E}(2,2)_0[w]$ by

$$M^{\mathbf{b}}_{\mathbf{B}} f_{\mathbf{ab}} = \left[(n+w-5)\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} - 2\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}} \nabla^{b^1}\right] f_{\mathbf{ab}} \in \mathcal{E}_{\mathbf{aB}}[w-2].$$

Now we can continue and apply $M^{\mathbf{a}}_{\mathbf{A}}$ once again. This puts $f_{\mathbf{ab}}$ to the $\mathbb{ZZ}$–slot of $\mathcal{E}_{\mathbf{AB}}[w-4]$ and there is no other projecting part. The latter fact follows from a direct computation which we describe in detail:

$$\begin{aligned}
M^{\mathbf{ab}}_{\mathbf{AB}} f_{\mathbf{ab}} :=& M^{\mathbf{a}}_{\mathbf{A}}(M^{\mathbf{b}}_{\mathbf{B}} f_{\mathbf{ab}}) = \left[(n+w-6)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} - 2\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}} \nabla^{a^1}\right] (M^{\mathbf{b}}_{\mathbf{B}} f_{\mathbf{ab}}) \\
=& \left[(n+w-6)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} - 2\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}} \nabla^{a^1}\right] \left((n+w-5)\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} - 2\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}} \nabla^{b^1}\right) f_{\mathbf{ab}} \\
=& \Big[(n+w-6)(n+w-5)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} - 2(n+w-6)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla^{b^1} \\
& - 2(n+w-5)\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}(\nabla^{a^1}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}) - 2(n+w-5)\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}\nabla^{a^1} \\
& + 4\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}(\nabla^{a^1}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}})\nabla^{b^1} + 4\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla^{a^1}\nabla^{b^1}\Big] f_{\mathbf{ab}}.
\end{aligned}$$

Looking at formulae (1.49), we see that $(\nabla^{a^1}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}) f_{\mathbf{ab}} = -2P^{a^1b^1}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}} f_{\mathbf{ab}}$ because the second term vanishes due to trace–freeness. Similarly,

$$(\nabla^{a^1}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}})\nabla^{b^1} f_{\mathbf{ab}} = \mathbb{Z}^{a^1\,b^2}_{B_1B_2}\nabla^{b^1} f_{a^1a^2b^1b^2} = \mathbb{Z}^{b^1\,b^2}_{B_1B_2}\nabla^{a^1} f_{b^1a^2a^1b^2} = \frac{1}{2}\mathbb{Z}^{b^1\,b^2}_{B_1B_2}\nabla^{a^1} f_{\mathbf{ab}},$$

cf. 1.7 for the last equality. Now we can finish the previous computation:

$$\begin{aligned} M^{\mathbf{a}\,\mathbf{b}}_{\mathbf{AB}} f_{\mathbf{ab}} =& \Big[(n+w-6)(n+w-5)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} - 2(n+w-6)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla^{b^1} \\ &+ 4(n+w-5)\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}P^{a^1b^1} - 2(n+w-5)\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}\nabla^{a^1} \\ &+ 2\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}\nabla^{a^1} + 4\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla^{a^1}\nabla^{b^1}\Big] f_{\mathbf{ab}} \\ =& \Big[(n+w-6)(n+w-5)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} \\ &- 2(n+w-6)\left(\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla^{b^1} + \mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}\nabla^{a^1}\right) \\ & 4\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\left(\nabla^{a^1}\nabla^{b^1} + (n+w-5)P^{a^1b^1}\right)\Big] f_{\mathbf{ab}} \in \mathcal{E}_{\mathbf{AB}}[w-4]. \end{aligned} \tag{2.12}$$

The result is the complete conformally invariant middle operator for the bundle $E(2,2)_0[w]$. This is a splitting operator if the scalars $n+w-6$ and $n+w-5$ are nonzero.

$M^{\mathbf{ab}}_{\mathbf{AB}} f_{\mathbf{ab}}$ is clearly trace free on the tractor indices. Although $M^{\mathbf{ab}}_{[\mathbf{AB}]} f_{\mathbf{ab}} = 0$, the tractor indices do not, in general, satisfy Weyl tensor symmetries. Details are in Example 2.1.3. Here we only note that if we replace $\nabla^{a^1}\nabla^{b^1}$ by $\nabla^{(a^1}\nabla^{b^1)}$ in (2.12) then these symmetries will be satisfied. Then $M^{\mathbf{ab}}_{\mathbf{AB}} f_{\mathbf{ab}}$ will be indecomposable

We can apply the computed middle operators to the Weyl curvature tensor $C_{\mathbf{ab}} \in \mathcal{E}(2,2)_0[2]$. Let us start with $M^{\mathbf{b}}_{\mathbf{B}}$. This yields the tractor curvature Ω:

$$\begin{aligned} M^{\mathbf{b}}_{\mathbf{B}} C_{\mathbf{ab}} =& (n-3)\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} C_{\mathbf{ab}} - 2\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla^{b^1} C_{\mathbf{ab}} \\ =& (n-3)\left[\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} C_{\mathbf{ab}} - 4\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla_{[a^1}P_{a^2]b^2}\right] = (n-3)\Omega_{\mathbf{aB}}. \end{aligned}$$

Here we have used (1.17) i.e. $\nabla^{b^1} C_{a^1a^2b^1b^2} = 2(n-3)\nabla_{[a^1}P_{a^2]b^2}$. Using this relation in the case of the operator $M^{\mathbf{a}\,\mathbf{b}}_{\mathbf{AB}}$, we get

$$\begin{aligned} M^{\mathbf{a}\,\mathbf{b}}_{\mathbf{AB}} C_{\mathbf{ab}} =& (n-4)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}} C_{\mathbf{ab}} \\ &- 4(n-4)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}\nabla_{[a^1}P_{a^2]b^2} - 4(n-4)\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}\nabla_{[b^1}P_{b^2]a^2} \\ &+ 4\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\mathbb{X}^{\dot{\mathbf{b}}}_{\mathbf{B}}B_{a^2b^2} \end{aligned}$$

where $B_{a^2b^2} := (\nabla^{a^1}\nabla^{b^1} + (n-3)P^{a^1b^1})C_{\mathbf{ab}}$ is the *Bach* tensor. The result $W_{\mathbf{AB}} := M^{\mathbf{a}\,\mathbf{b}}_{\mathbf{AB}}C_{\mathbf{ab}}$ is the curvature of the *Fefferman Graham ambient metric* if $n \neq 4$ (up to a scalar multiple), see [35] for more details. Let us note the formula for $W_{\mathbf{AB}}$ implies that the Bach tensor $B_{a^2b^2}$ is invariant in the dimension 4 and the Cotton–York tensor $A_{ca^1a^2} = 2\nabla_{[a^1}P_{a^2]c}$ is invariant in the dimension 3. It is noted in [35] that $W_{\mathbf{AB}}$ satisfies Weyl tensor symmetries on tractor indices i.e. $W_{[\mathbf{A}B^1]B^2} = 0$. This follows from $\nabla^{[a^1}\nabla^{b^1]}C_{a^1a^2b^1b^2} = \frac{1}{2}(n-3)\nabla^p A_{pb^2a^2} = 0$ which can be obtained from (1.17) after some computation (cf. Example 2.1.3).

Let us summarise the results and also problems we face in the general case. First, the case of $\mathcal{E}_{\mathbf{a}^k}[w]$ is not difficult – it is straightforward to rewrite the procedures (2.8) and (2.10) from 2–forms to any valence k. See (2.14) and Example 2.1.6 for the results.

Already the space $\mathcal{E}(k,l)_0[w]$, $k \geq l$ reveals some of the problems we will meet in the general case $\mathcal{E}(p; s_1, \cdots, s_r)_0[w]$, $p \in \{0, \frac{1}{2}\}$ and hints how to solve them. For example, trying to generalise (2.11), we have two choices: either $M^{\mathbf{b}}_{\mathbf{B}}\mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}f_{\mathbf{ab}}$ or $M^{\mathbf{a}}_{\mathbf{A}}\mathbb{X}_{B^0}{}^{\mathbf{b}}_{\mathbf{B}}f_{\mathbf{ab}}$ where $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{B} = \mathbf{B}^l$. We have to be careful here to choose the former because the latter has generally two projecting parts $\mathbb{Z}_{\mathbf{A}}\mathbb{X}_{B^0\mathbf{B}}$ and $\mathbb{X}_{\mathbf{A}}\mathbb{Z}_{B^0\mathbf{B}}$ for $k > l$. Following (2.10), we will construct the top operator T from the middle one using the tractor D–operator but again, it is not clear ab initio whether one should choose $D_{[A^0}M^{\mathbf{a}}_{\mathbf{A}]}M^{\mathbf{b}}_{\mathbf{B}}f_{\mathbf{ab}}$ or $D_{[B^0}M^{\mathbf{a}}_{|\mathbf{A}|}M^{\mathbf{b}}_{\mathbf{B}]}f_{\mathbf{ab}}$. We will see later that the former is the right choice but the proof for the general case (see Theorem 2.1.5) is rather technical.

Another issue is the scalar, depending on w, which appears as a coefficient of the desired projecting part. For example, we have seen in (2.12) that $M^{\mathbf{a}}_{\mathbf{A}}M^{\mathbf{b}}_{\mathbf{B}}$ is a splitting operator on $\mathcal{E}(2,2)_0[w]$ only if $(n+w-5)(n+w-6) \neq 0$. In the general case, we shall construct candidates for splitting operators as compositions of appropriate top and middle operators and collect carefully scalars emerging in all steps of this composition. A very simple example of such a composition follows.

Example 2.1.1. Let us consider the operator $T^{(p)} = D_{A_1} \cdots D_{A_p}$, $p \in \mathbb{N}$ on $\mathcal{E}_{\mathfrak{T}}[w]$ and the TFP–component $pr^p \in TFPC(E_{\mathfrak{A}})$, $\mathfrak{A} := A_1 \cdots A_p$ of the (highest) homogenity $p = hh(E_{\mathfrak{A}})$, see 1.2.6 for the notation. Then pr^p is a projecting part of $T^{(p)}$ i.e. the operator $(pr^p)^* T^{(p)}$ is invariant. However, it sometimes vanishes. Namely, it follows from the term $w(n + 2w - 2)Y_A$ in the formula (1.32) for D_A that $(pr^p)^* T^{(p)} = C \cdot \mathrm{id}$ where

$$C = \prod_{i=1}^{p} (w - i + 1)(n + 2w - 2i).$$

Now consider the scalar $s(p, 0) := w - p + 1$. It is obvious from the form of C that if $s(p, 0) > 0$ then $C \neq 0$ and pr^p is the only TFP–projecting part of $T^{(p)}$. (Note there is a unique irreducible projecting part of $T^{(p)}$ which is nontrivial for $C \neq 0$. This is a component of pr^p.) On the other hand, if $s(p, 0) = 0$ and $\mathfrak{T} = \emptyset$ then $T^{(p)}$ is not a splitting operator. Further, suppose $\widetilde{pr} \in TFPC(E_{\mathfrak{T}})$ is of the highest homogenity $hh(E_{\mathfrak{T}})$ and $f \in E_{\mathfrak{T}}[w]$ satisfies $\widetilde{pr}^* f \neq 0$. Then $T^{(p)} f \in \mathcal{E}_{\mathfrak{A}\mathfrak{T}}[w - p]$ and similarly as above, $\widetilde{pr} pr^p \in TFPC(\mathfrak{A}\mathfrak{T})$ has the highest homogenity $hh(E_{\mathfrak{A}\mathfrak{T}})$ and $(\widetilde{pr} pr^p)^* (T^{(p)} f) \neq 0$ for $s(p, 0) > 0$.

2.1.3 Remark. The middle and top operators constructed until now can be viewed either as operators between quotient bundles or subbundles of tractor bundles. For example, we have

$$T : (\mathcal{E}_{\mathbb{Y}})_{\mathbf{A}^{k+1}}[w] \longrightarrow (\mathcal{E}_{\mathbb{Y}\mathbb{W}})_{\mathbf{A}^{k+1}}[w] \longrightarrow \mathcal{E}_{\mathbf{A}^{k+1}}[w] \quad \text{or}$$

$$T : (\mathcal{E}_{\mathbb{X}})_{\mathbf{A}^{k+1}}[w + 2] \longrightarrow (\mathcal{E}_{\mathbb{X}\mathbb{Z}})_{\mathbf{A}^{k+1}}[w + 1] \longrightarrow \mathcal{E}_{\mathbf{A}^{k+1}}[w],$$

respectively, in the case of the top splitting. See (2.10) for the former case in the matrix notation, the latter would be

$$f_{\mathbf{a}} \xrightarrow{\mathbb{X}_{A^0}{}_{\mathbf{A}}^{\mathbf{a}}} \begin{pmatrix} 0 \\ 0 \;\; 0 \\ f_{\mathbf{a}} \end{pmatrix} \xrightarrow{\iota(D)} \begin{pmatrix} 0 \\ m(w) f_{\mathbf{a}} \;\; 0 \\ * \end{pmatrix} \xrightarrow{\varepsilon(D)} \begin{pmatrix} t(w) f_{\mathbf{a}} \\ * \;\; * \\ * \end{pmatrix}. \tag{2.13}$$

The former view corresponds better to the idea of the splitting operator as an operator which "puts" a tensor section to a given slot of a tractor bundle and then "extends" this (noninvariant) section to an invariant one. But we will prefer the latter approach i.e. we shall build splitting operators via composition of operators between subbundles. They will be invariant as operators between the whole tractor bundles (see $\iota(D)$ and $\varepsilon(D)$ in the last display) and therefore more manageable.

2.1.4. Middle operator for tensor representations. We are going to construct the middle operators as advertised in (2.2) with $r_{n'} \in \mathbb{Z}$. We shall start with the case of k–forms, which is considerably simpler than the general case.

Middle operator on k–forms

We can use an analogue of the construction used for 2–forms in 2.1.2 i.e. consider $D^{A^0}\mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}}$ for a k–form $f_{\mathbf{a}}$. That is, $\mathbf{a} = \mathbf{a}^k$ and $\mathbf{A} = \mathbf{A}^k$. We shall avoid this technical computation and state directly the result which agrees with $D^{A^0}\mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}}$ up to a scalar multiple which vanishes if $n+2(w-k)+2 = 0$.

$$
\begin{aligned}
M^{\mathbf{a}}_{\mathbf{A}} &: \mathcal{E}_{\mathfrak{T}\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{\mathfrak{T}\mathbf{A}^k}[w-k] \\
M^{\mathbf{a}}_{\mathbf{A}} f_{\mathfrak{T}\mathbf{a}} &= \big((n+w-2k)\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} - k\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\nabla^{a_1}\big) f_{\mathfrak{T}\mathbf{a}}
\end{aligned}
\tag{2.14}
$$

Lemma. *The operator (2.14) is conformally invariant.*

Proof. Let us consider a rescaling $\hat{g} = \Omega^2 g$. The transformation of the $\mathbb{Z}$–slot for tractor k–forms is $\widehat{\mathbb{Z}}^{\mathbf{a}^k}_{\mathbf{A}^k} f_{\mathbf{a}^k} = (\hat{\mathbb{Z}}^{\mathbf{a}^k}_{\mathbf{A}^k} + k\Upsilon^{a^1}\dot{\mathbb{X}}^{\mathbf{a}^k}_{\mathbf{A}^k}) f_{\mathfrak{T}\mathbf{a}^k}$ according to (1.47). On the other hand, we have shown $\hat{\nabla}^{b^1} f_{\mathbf{b}^k} = \nabla^{b^1} f_{\mathbf{b}^k} + (n+w-2k)\Upsilon^{b^1} f_{\mathbf{b}^k}$ in (1.19). Thus the invariance of $M^{\mathbf{a}}_{\mathbf{A}} f_{\mathfrak{T}\mathbf{a}}$ follows. □

Middle operator on irreducible tensors

For the remainder of this section, we will consider the general case i.e. the operator on a section $f = f_{\mathfrak{T}\mathbf{a}_1\cdots\mathbf{a}_r} \in \mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w]$. This can be reduced to the middle operator on s_r–forms by use of $r-1$ bottom operators on form indices $\mathbf{a}_1, \ldots, \mathbf{a}_{r-1}$ first. This yields $B^{(r-1)}f \in \mathcal{E}_{\mathfrak{T}\mathfrak{A}\mathbf{a}_r}[w']$ where $w' = w - s + s_r + r - 1$ and $\mathfrak{A} = \mathbf{A}_1 \cdots \mathbf{A}_{r-1}$ is the corresponding system of tractor indices. Now we can apply the (strongly invariant) operator (2.14). We obtain

$$\begin{aligned} &\Big[(n+w'-2s_r)\mathbb{Z}_{\mathbf{A}_r}{}^{\mathbf{a}_r} - s_r \mathbb{X}_{A_r^1 \dot{\mathbf{A}}_r}{}^{\dot{\mathbf{a}}_r} \nabla^{a_r^1}\Big] \mathbb{X}_{A_{r-1}^0 \mathbf{A}_{r-1}}{}^{\mathbf{a}_{r-1}} \cdots \mathbb{X}_{A_1^0 \mathbf{A}_1}{}^{\mathbf{a}_1} f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} \\ =&\mathbb{X}_{A_{r-1}^0 \mathbf{A}_{r-1}}{}^{\mathbf{a}_{r-1}} \cdots \mathbb{X}_{A_1^0 \mathbf{A}_1}{}^{\mathbf{a}_1} \Big[(n+w'-2s_r)\mathbb{Z}_{\mathbf{A}_r}{}^{\mathbf{a}_r} - s_r \mathbb{X}_{A_r^1 \dot{\mathbf{A}}_r}{}^{\dot{\mathbf{a}}_r} \nabla^{a_r^1}\Big] f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r}. \end{aligned} \tag{2.15}$$

Here the equality follows from

$$\big(\nabla^{a_r^1} \mathbb{X}_{A_q^0 \mathbf{A}_q}{}^{\mathbf{a}_q}\big) f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} = \big(\mathbb{Z}_{A_q^0 \mathbf{A}_q}{}^{a_r^1 \mathbf{a}_r} - s_q \boldsymbol{g}^{a_r^1 a_q^1} \mathbb{W}_{A_q^0 \mathbf{A}_q}{}^{\dot{\mathbf{a}}_q}\big) f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} = 0,$$

see (1.49) for the first equality, where $1 \le q \le r-1$ and $\nabla^{a_r^1}$ acts only on the section $\mathbb{X}$. The second equality in the last display follows from the trace freeness of f and Young symmetries $(s_1, \ldots, s_r)$. (Recall this means skew symmetrization over any $s_q + 1$ indices among $\mathbf{a}_q \cdots \mathbf{a}_r$ vanishes.) Summarising, (2.15) yields the conformally invariant *middle operator*

$$\begin{aligned} M_{\mathbf{A}_r}^{\mathbf{a}_r} &: \mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}\mathbf{A}_r}(s_1, \ldots, s_{r-1})_0[w - s_r] \\ M_{\mathbf{A}_r}^{\mathbf{a}_r} f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} &= \Big((n+w-s-s_r+r-1)\mathbb{Z}_{\mathbf{A}_r}{}^{\mathbf{a}_r} - s_r \mathbb{X}_{\mathbf{A}_r}{}^{\dot{\mathbf{a}}_r} \nabla^{a_r^1}\Big) f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r}. \end{aligned} \tag{2.16}$$

Properties of the middle operator

The scalar in the relation (2.16) says immediately for which weights w is $M_{\mathbf{A}_r}^{\mathbf{a}_r}$ a splitting operator on $\mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w]$. Since the middle operator is strongly invariant we can use it repeatedly. Let us suppose we apply the middle operator m times, $1 \le m \le r$. The result, also called the *middle operator*, is the composition

$$\begin{aligned} \big(M^{(m)}\big)_{\mathbf{A}_{\bar{m}} \cdots \mathbf{A}_r}^{\mathbf{a}_{\bar{m}} \cdots \mathbf{a}_r} &:= M_{\mathbf{A}_{\bar{m}}}^{\mathbf{a}_{\bar{m}}} \cdots M_{\mathbf{A}_r}^{\mathbf{a}_r} : \\ \mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w] &\longrightarrow \mathcal{E}_{\mathfrak{T}\mathbf{A}_{\bar{m}} \cdots \mathbf{A}_r}(s_1, \ldots, s_{\bar{m}-1})_0[w - \tilde{s}^{\bar{m}}] \end{aligned} \tag{2.17}$$

where $\bar{m} = r - m + 1$ and $\tilde{s}^{\bar{m}}$ is defined by (1.1). The following Theorem (i) says when this is a splitting operator. Let us recall that contrary to the bottom splitting, the order of M's is now important. That is, every M in the composition $M^{(m)}$ is applied to the shortest (tensor) form index and this order is necessary. (Otherwise M would not be invariant.)

It follows from (2.16) and the definition of the quantity oh in 1.2.6 that $oh(M) = 0$. Therefore also

$$oh(M^{(m)}) = oh(\underbrace{M \cdots M}_{m}) = 0. \tag{2.18}$$

Definition. The middle operator $M^{(m)}$ defined by (2.17) will be called the *middle splitting* or the *middle splitting operator* if this is a splitting operator for $\mathfrak{T} = \emptyset$.

Note if $M^{(m)}$ is the middle splitting then it is a splitting operator for any $\mathfrak{T}$. (This is obvious.) However, for example M_A^a is a splitting operator for $X_B f_a \in \mathcal{E}_{aB}[2-n]$ (i.e. $f_a \in \mathcal{E}_a[2-n-1]$) but M_A^a is not the middle splitting operator on $\mathcal{E}_a[2-n]$.

Theorem (Properties of the middle operator).
Let us consider the middle operator $M^{(m)}$ given by (2.17) and a section $f \in \mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w]$ where $w \in \mathbb{R}$. Let $E_{\mathfrak{A}} := E_{\mathbf{A}_{\bar{m}} \cdots \mathbf{A}_r}$, $\bar{m} = r - m + 1$ and write

$$s(0, m) := n + w - s - s_{\bar{m}} + \bar{m} - 1.$$

(i) The TFP–component $pr(m) := \mathbb{Z}_{\mathbf{A}_{\bar{m}}}^{\mathbf{a}_{\bar{m}}} \cdots \mathbb{Z}_{\mathbf{A}_r}^{\mathbf{a}_r} \in TFPC(E_{\mathfrak{A}})$ of homogenity 0 is a projecting part of $M^{(m)}$ and satisfies

$$\forall pr' \in TFPC(E_{\mathfrak{A}}) : (pr')^* M^{(m)} \neq 0 \Longrightarrow \\ \Longrightarrow \Big[pr' = pr(m)\Big] \vee \Big[h(pr') < h\big(pr(m)\big)\Big]. \tag{2.19}$$

Furthermore $(pr(m))^* M^{(m)} = C \cdot \mathrm{id}$ *and the scalar* C *satisfies the following. If* $s(0,m) > 0$ *then* $C \neq 0$ *and* $M^{(m)}$ *is the middle splitting and if* $s(0,m) = 0$ *then* $C = 0$ *and* $M^{(m)}$ *is not the middle splitting operator.*

(ii) Let us suppose $\widetilde{pr} \in TFPC(E_{\mathfrak{T}})$ *has the highest homogenity i.e.* $h(\widetilde{pr}) = hh(E_{\mathfrak{T}})$*. Then the TFP–component* $pr(m)\widetilde{pr} \in TFPC(E_{\mathfrak{A}\mathfrak{T}})$ *of the homogenity* $hh(E_{\mathfrak{T}})$ *is a projecting part of the section* $M^{(m)}f$ *and satisfies*

$$\forall pr' \in TFPC(E_{\mathfrak{A}\mathfrak{T}}) : (pr')^* M^{(m)} f \neq 0 \Longrightarrow$$
$$\Longrightarrow \Big[pr' = pr(m)\widetilde{pr}\Big] \vee \Big[h(pr') < h\big(pr(m)\widetilde{pr}\big)\Big].$$

Furthermore $\big(pr(m)\widetilde{pr}\big)^* M^{(m)} f = C \cdot \widetilde{pr}^* f$ *and the scalar* C *satisfies the same properties as in (i).*

We can demonstrate the Theorem (i) easily on the matrix form of $M_{\mathbf{A}}^{\mathbf{a}}$ in (2.9). In the notation of the Theorem, $pr = \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}$ and pr' satisfies $(pr')^* M \neq 0$. Therefore $pr' = \mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}$ for $w = 4 - n$ and $pr' \in \{\mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}}, \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\}$ for $w \neq 4 - n$.

Proof. The TFP–component $pr(m)$ is a projecting part of the formula $M^{(m)}$ because $fo\big((pr(m))^* M^{(m)}\big) = oh(M^{(m)}) - h(pr(m)) = 0 - 0 = 0$. It satisfies $(pr(m))^* M^{(m)} = C \cdot \mathrm{id}$ for a scalar C. This is clear for $m = 1$ and in the general case, we can decompose $(pr(m))^*$ into $\mathbb{Z}^* \cdots \mathbb{Z}^*$ and $M^{(m)}$ into $M \cdots M$. (Here every $\mathbb{Z}^*$ is of the form $\mathbb{Z}_{\mathbf{a}_i}^{\mathbf{A}_i}$ for $\bar{m} \leq i \leq r$.)

To prove (2.19), let us make the following observation. We shall denote by $P(q)$, $\bar{m} \leq q \leq r$ the claim that all terms of the formula for $M^{(q)}f$ are of the form

$$\{X, Z, \nabla, P\}^* f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} \tag{2.20}$$

where all tensor indices of Z, ∇ and P are upstairs and contracted with indices in $\mathbf{a}_1, \ldots, \mathbf{a}_r$. (Tractor indices of X and Z are omitted.) Here $\{\ \}^*$ denotes any juxtaposition of the embraced terms. $P(r)$ follows directly from the formula (2.16). Let us use (2.16) repeatedly. Obviously, we need to discuss only the first order term in (2.16) i.e. $\mathbb{X}^{\dot{\mathbf{a}}_r}_{\mathbf{A}_r}\nabla^{a^1_r}$. But it follows from $\nabla^b X_A = Z^b_A$ and $\nabla^b Z^a_A = -g^{ab}Y_A - P^{ab}X_A$ that $P(q) \implies P(q-1)$. (The term $-g^{ab}Y_A$ vanishes because f is trace–free.) This proves $P(\bar{m})$ i.e. that every term of the formula $M^{(m)}$ is of the form (2.20).

According to (2.20), $pr(m)$ is the only TFP–component of homogenity 0 in the formula $M^{(m)}$. Thus (2.19) follows. We need to discuss when $C \neq 0$. Every formula $M^{\mathbf{a}_i}_{\mathbf{A}_i}$, $\bar{m} \leq i \leq r$ is sum of an algebraic term (the $\mathbb{Z}$–slot) and a first order term (the $\mathbb{X}$–slot). Clearly the only way to obtain a $pr(m)$–component in $M^{(m)}$ is to use only the algebraic terms in all M's. (To eliminate any $\mathbb{X}$–term we need the first order term of $M^{\mathbf{a}_i}_{\mathbf{A}_i}$. But the derivative is always associated with a coefficient of $\mathbb{X}$. So these operations cannot result in a term free of $\mathbb{X}$'s.) Hence C is product of scalars of the $\mathbb{Z}$ –slots of all $M^{\mathbf{a}_i}_{\mathbf{A}_i}$, $\bar{m} \leq i \leq r$.

In the case of $M^{\mathbf{a}_r}_{\mathbf{A}_r}$, this scalar is $n + w - s - s_r + r - 1$. $M^{\mathbf{a}_r}_{\mathbf{A}_r}$ changes the weight and removes the last column of the Young diagram. Thus the following application of $M^{\mathbf{a}_{r-1}}_{\mathbf{A}_{r-1}}$ yields the scalar

$$n + (w - s_r) - (s - s_r) - s_{r-1} + (r - 1) - 1 = n + w - s_{r-1} + r - 2.$$

We can continue by induction to show that the application of the last middle operator, $M^{\mathbf{a}_{\bar{m}}}_{\mathbf{A}_{\bar{m}}}$, yields the scalar $n + w - s - s_{\bar{m}} + \bar{m} - 1 = s(0, m)$ where, recall, $\bar{m} = r - m + 1$. Since $s(0, m)$ is the smallest among all the discussed scalars, (i) follows.

The proof of (ii) is analogous to (i), namely for $\mathfrak{T} = \emptyset$ this is exactly (i). The only point here is the (necessary) assumption $h(\widetilde{pr}) = hh(E_{\mathfrak{T}})$. $\square$

Remark. Let us note that also $w \notin \mathbb{Z}$ or $w \in \mathbb{C} \setminus \mathbb{R}$ implies that $C \neq 0$ i.e. $M^{(m)}$ is a splitting operator. This follows from the discussion of the scalars in the proof of Theorem (i) because they can be zero only for $w \in \mathbb{Z}$.

Examples of the middle operator

The simplest case – the middle operator on k–forms – is described at the beginning of this section. Here we describe mainly various middle operators on the space $\mathcal{E}(k,l)_0[w]$ where $n' \geq k \geq l \geq 1$. Sections will be denoted $f = f_{\mathbf{ab}} \in \mathcal{E}(k,l)_0[w]$ and we will use the form indices $\mathbf{a} = \mathbf{a}^k$, $\mathbf{b} = \mathbf{b}^l$ and form tractor indices $\mathbf{A} = \mathbf{A}^k$, $\mathbf{B} = \mathbf{B}^l$. In the formulae below we shall use the scalars

$$c_1 = n + w - 2k - l \quad \text{and} \quad c_2 = n + w - k - 2l + 1.$$

Example 2.1.2. The middle operator (2.16) can be applied to the "shorter" form index $\mathbf{b}$ of $f_{\mathbf{ab}}$. The result is

$$M_{\mathbf{B}}^{\mathbf{b}} : \mathcal{E}(k,l)_0[w] \longrightarrow \mathcal{E}_{\mathbf{a}^k\mathbf{B}^l}[w-l]$$
$$M_{\mathbf{B}}^{\mathbf{b}} f_{\mathbf{ab}} = c_2 \mathbb{Z}_{\mathbf{B}}^{\mathbf{b}} f_{\mathbf{ab}} - l \mathbb{X}_{\mathbf{B}}^{\dot{\mathbf{b}}} \nabla^{b^1} f_{\mathbf{ab}}.$$

Example 2.1.3. Having $M_{\mathbf{B}}^{\mathbf{b}} f_{\mathbf{ab}}$ at hand, we can compute

$$M_{\mathbf{A}}^{\mathbf{a}} M_{\mathbf{B}}^{\mathbf{b}} : \mathcal{E}(k,l)_0[w] \longrightarrow \mathcal{E}_{\mathbf{A}^k\mathbf{B}^l}[w-k-l].$$

This requires some work. The result is

$$\begin{aligned} M_{\mathbf{A}}^{\mathbf{a}} M_{\mathbf{B}}^{\mathbf{b}} f_{\mathbf{ab}} =& c_1 c_2 \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}} \mathbb{Z}_{\mathbf{B}}^{\mathbf{b}} f_{\mathbf{ab}} \\ & - l c_1 \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}} \mathbb{X}_{\mathbf{B}}^{\dot{\mathbf{b}}} \nabla^{b^1} f_{\mathbf{ab}} - k \mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}} \mathbb{Z}_{\mathbf{B}}^{\mathbf{b}} \left[c_2 \nabla^{a^1} f_{\mathbf{ab}} - l \nabla^p f_{b^1 \dot{\mathbf{a}} p \dot{\mathbf{b}}} \right] \\ & + kl \mathbb{X}_{\mathbf{A}}^{\dot{\mathbf{a}}} \mathbb{X}_{\mathbf{B}}^{\dot{\mathbf{b}}} \left[\nabla^{a^1} \nabla^{b^1} f_{\mathbf{ab}} + c_2 P^{a^1 b^1} f_{\mathbf{ab}} \right]. \end{aligned} \tag{2.21}$$

Let us note that $F_{\mathbf{AB}} := M_{\mathbf{A}}^{\mathbf{a}} M_{\mathbf{B}}^{\mathbf{b}} f_{\mathbf{ab}}$ is trace–free and but the tractor indices of $F_{\mathbf{AB}}$ do not, in general, satisfy Young symmetries corresponding to the diagram (k,l). One can compute

$$F_{[\mathbf{A}B^1]\dot{\mathbf{B}}} = \frac{1}{2} k(l-1) \mathbb{X}_{B^1}{}_{\mathbf{A}}^{\mathbf{a}} \mathbb{X}_{\dot{\mathbf{B}}}^{\ddot{\mathbf{b}}} C_{a^1}{}^{rpq} f_{r\dot{\mathbf{a}}pq\ddot{\mathbf{b}}} \tag{2.22}$$

Note that if $f_{\mathbf{ab}} = C_{\mathbf{ab}}$ then actually $F_{\mathbf{AB}}$ satisfies the Young symmetries $(2,2)$ on the tractor indices because the tensor $C_{a^1}{}^{rpq}C_{a^2 rpq}$ is symmetric in indices a^1 and a^2. (Cf. the middle operator on $\mathcal{E}(2,2)_0[w]$ in 2.1.2.)

If $M^{\mathbf{a}}_{\mathbf{A}} M^{\mathbf{b}}_{\mathbf{B}}$ is not a splitting operator on $\mathcal{E}(k,l)_0[w]$ i.e. when $c_1 c_2 = 0$, the possible projecting parts of (2.21) reveal invariant operators on $\mathcal{E}(k,l)_0[w]$. If $c_2 = 0$ then $\nabla^{b^1} f_{\mathbf{ab}}$ will be invariant and if $c_1 = 0$ then $c_2 = k - l + 1$ and the operator $c_2 \nabla^{a^1} f_{\mathbf{ab}} - l \nabla^p f_{[b^1|\dot{\mathbf{a}}p|\dot{\mathbf{b}}]}$ will be invariant. However, a short computation reveals that the latter is just the projection of $\nabla^{a^1} f_{\mathbf{ab}}$ to $\mathcal{E}(k-1,l)_0[w]$ for $k > l$ and vanishes for $k = l$, cf. (1.7). Therefore, if both $k = l$ and $c_1 = 0$ then the bottom slot is invariant. Summarising, (2.21) yields the operators

$$\mathcal{E}(k,l)_0[k{+}2l{-}n{-}1] \longrightarrow \mathcal{E}(k,l{-}1)_0[k{+}2l{-}n{-}3],\ f_{\mathbf{ab}} \mapsto \nabla^{b^1} f_{\mathbf{ab}}$$
$$\mathcal{E}(k,l)_0[2k{+}l{-}n] \longrightarrow \mathcal{E}(k{-}1,l)_0[2k{+}l{-}n{-}2],\ f_{\mathbf{ab}} \mapsto \mathrm{Proj}\nabla^{a^1} f_{\mathbf{ab}},\ k > l$$
$$\mathcal{E}(k,k)_0[3k{-}n] \longrightarrow \mathcal{E}(k{-}1,l)_0[3k{-}n{-}4],\ f_{\mathbf{ab}} \mapsto (\nabla^{(a^1}\nabla^{b^1)} + P^{a^1 b^1}) f_{\mathbf{ab}}$$

where Proj denotes the projection to the target space. In the case of the 2nd order operator, this projection is provided by the symmetrization $\nabla^{(a^1}\nabla^{b^1)}$. The skew symmetrization $\nabla^{[a^1}\nabla^{b^1]}$ would project to another irreducible component, see (2.22). It is a straightforward computation to show directly that these formulae are really independent on the choice of the metric from the conformal class.

Example 2.1.4. The middle operator defined by (2.16) can be applied only to the shortest form index (of tensor indices), in our case $\mathbf{b}^l$. Using the complete middle operator $M^{\mathbf{a}}_{\mathbf{A}} M^{\mathbf{b}}_{\mathbf{B}} f_{\mathbf{ab}}$ one can define a middle operator applicable also to the longer form index $\mathbf{a}^k$ as

$$\check{M}^{\mathbf{a}}_{\mathbf{A}} : \mathcal{E}(k,l)_0[w] \longrightarrow \mathcal{E}_{\mathbf{A}^k \mathbf{b}^l}[w-k]$$
$$\check{M}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} = \mathbb{Z}^{\mathbf{C}}_{\mathbf{b}} M^{\mathbf{a}}_{\mathbf{A}} M^{\mathbf{c}}_{\mathbf{C}} f_{\mathbf{ac}}$$

The conformal invariance follows from the Theorem (i). (In particular, from the fact that $\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}\mathbb{Z}^{\mathbf{b}}_{\mathbf{B}}$ is a projecting part of of $M^{\mathbf{a}}_{\mathbf{A}}M^{\mathbf{c}}_{\mathbf{C}}$.) In the case of general Young symmetries we can define analogously a middle operator applicable to *any* from index: Proposition (i) guaranties that the necessary $\mathbb{Z}$–projections (like $\mathbb{Z}^{\mathbf{C}}_{\mathbf{b}}$ above) are conformally invariant. Using (2.21) it is easy to compute

$$\check{M}^{\mathbf{a}}_{\mathbf{A}}f_{\mathbf{ab}} = c_1c_2\mathbb{Z}^{\mathbf{a}}_{\mathbf{A}}f_{\mathbf{ab}} - k\mathbb{X}^{\dot{\mathbf{a}}}_{\mathbf{A}}\left[c_2\nabla^{a^1}f_{\mathbf{ab}} - l\nabla^p f_{b^1\dot{\mathbf{a}}p\dot{\mathbf{b}}}\right]$$

where $f_{\mathbf{ab}} \in \mathcal{E}(k,l)_0[w]$ and we skew over $[b^1\dot{\mathbf{b}}]$ on the right hand side. Of course, now we can apply again $M^{\mathbf{b}}_{\mathbf{B}}$ to $M^{\mathbf{a}}_{\mathbf{A}}f_{\mathbf{ab}}$ but the result will have generically two projecting parts, in particular in the slots $\mathbb{Z}_{\mathbf{A}}\mathbb{Z}_{\mathbf{B}}$ and $\mathbb{Y}_{\mathbf{A}}\mathbb{X}_{\mathbf{B}}$.

Example 2.1.5. In this example, we look briefly at the middle operator on (density valued) symmetric trace free tensors $\mathcal{E}_{(a_1\ldots a_r)_0}[w]$. To compute the formula for the complete middle operator requires a lot of work but our aim here is to compute only what we will need later for the top operator in Example 2.1.9. This is the operator

$$M^{a_2\cdots a_r}_{A_2\cdots A_r} := M^{a_2}_{A_2}\cdots M^{a_r}_{A_r} : \mathcal{E}_{(a_1\ldots a_r)_0}[w] \longrightarrow \mathcal{E}_{a_1(A_2\ldots A_r)_0}[w-r+1]$$

and actually only its three slots of the highest homogenity, see the next display. We can obviously suppose $r \geq 2$ but since the case $r = 2$ is covered by Example 2.1.2, we assume $r \geq 3$. Using repeatedly the formula (2.16), one can compute (or check the conformal invariance of the formula (2.23) below directly) that

$$\begin{aligned} M^{a_2\cdots a_r}_{A_2\cdots A_r}f_{a_1\cdots a_r} = \ & C\Big\{c(c-1)Z^{a_2}_{A_2}\cdots Z^{a_r}_{A_r}f_{a_1\cdots a_r} \\ & - (r-1)(c-1)Z^{a_2}_{(A_2}\cdots Z^{a_{r-1}}_{A_{r-1}}X_{A_r)}\nabla^p f_{a_1\cdots a_{r-1}p} \\ & + \frac{1}{2}(r-1)(r-2)Z^{a_2}_{(A_2}\cdots Z^{a_{r-2}}_{A_{r-2}}X_{A_{r-1}}X_{A_r)}\left(\nabla^p\nabla^q + cP^{pq}\right)f_{a_1\cdots a_{r-2}pq}\Big\} \\ & + \{\text{lower homogenity terms}\} \end{aligned} \tag{2.23}$$

where $f_{a_1 \cdots a_r} \in \mathcal{E}_{(a_1 \dots a_r)_0}[w]$ and the scalars used are $c = n + w - 2$ and

$$C = \begin{cases} 1 & r = 3, \\ \prod_{i=3}^{r-1}(c - i + 1) & r \geq 4. \end{cases} \tag{2.24}$$

(Note it is not too difficult to verify the conformal invariance of the right hand side of (2.23) directly.)

2.1.5. Top operator for tensor representations. Our aim is to construct the top operators advertised in (2.2) with $r_{n'} \in \mathbb{Z}$. If $r = 1$, we shall follow 2.1.2 and define the top operator as the invariant operator $D_{[A^0} M^{\mathbf{a}^k}_{\mathbf{A}^k]} f_{\mathbf{a}^k}$ for a (density valued) k–form $f_{\mathbf{a}^k}$. The general case is complicated and is one of crucial parts of the thesis.

Top operator on irreducible tensors

For the reminder of this section, we will consider the section in the general form i.e. $f = f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} \in \mathcal{E}_{\mathfrak{T}}(s_1, \dots, s_r)_0[w]$. The case $r = 1$ (i.e. p–forms) is discussed above so we can assume $r \geq 2$. We shall define actually two possibilities $\tilde{T}$ and $\tilde{\tilde{T}}$ for the *top operator*:

$$\begin{aligned}
&\tilde{T}_{A_1^0 \mathbf{A}_1}{}^{\mathbf{a}_1}, \tilde{\tilde{T}}_{A_1^0 \mathbf{A}_1}{}^{\mathbf{a}_1} : \mathcal{E}_{\mathfrak{T}}(s_1, \dots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}[A_1^0 \mathbf{A}_1]}(s_2, \dots, s_r)_0[w - s_1 - 1] \\
&\tilde{T}_{A_1^0 \mathbf{A}_1}{}^{\mathbf{a}_1} f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} = \mathrm{Proj} \circ \mathbb{Z}_{\mathbf{a}_r}^{\mathbf{B}_r} \cdots \mathbb{Z}_{\mathbf{a}_2}^{\mathbf{B}_2} D_{[A_1^0} M^{\mathbf{a}_1}_{\mathbf{A}_1]} M^{\mathbf{b}_2}_{\mathbf{B}_2} \cdots M^{\mathbf{b}_r}_{\mathbf{B}_r} f_{\mathfrak{T}\mathbf{a}_1 \mathbf{b}_2 \cdots \mathbf{b}_r} \\
&\tilde{\tilde{T}}_{A_1^0 \mathbf{A}_1}{}^{\mathbf{a}_1} f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} = \mathrm{Proj} \circ \mathbb{Z}_{\mathbf{a}_r}^{\mathbf{B}_r} \cdots \mathbb{Z}_{\mathbf{a}_3}^{\mathbf{B}_3} \mathbb{Y}^{B_2^0 \mathbf{B}_2}{}_{\mathbf{a}_2} D_{[A_1^0} M^{\mathbf{a}_1}_{\mathbf{A}_1]} \mathbb{X}_{B_2^0 \mathbf{B}_2}{}^{\mathbf{b}_2} M^{\mathbf{b}_3}_{\mathbf{B}_3} \cdots M^{\mathbf{b}_r}_{\mathbf{B}_r} f_{\mathfrak{T}\mathbf{a}_1 \mathbf{b}_2 \cdots \mathbf{b}_r}
\end{aligned} \tag{2.25}$$

where Proj denotes the corresponding projection to the target space of $\tilde{T}$ and $\tilde{\tilde{T}}$. (That is, Proj is a projection on tensor indices.)

Before we discuss invariance of the top operator, let us observe the following properties of the space $\mathcal{E}_{|\mathbf{a}^p \mathbf{b}^q|_0}$ where $p, q \geq 1$ and the subscript 0 indicates the trace–free part on the enclosed indices. First observe

$$\mathcal{E}_{|\mathbf{a}^p \mathbf{b}^q|_0} \text{ nontrivial } \Longleftrightarrow p + q \leq n. \tag{2.26}$$

Consider $\mathcal{E}(s_1, s_2)_0 \subseteq \mathcal{E}_{|\mathbf{a}^p\mathbf{b}^q|_0}$ where we allow the range $n \geq s_1 \geq s_2$. Then $\mathcal{E}(s_1, s_2)_0$ is nontrivial if and only if $s_1 + s_2 \leq n$ [26]. This proves "$\Longleftarrow$". To show the second implication, recall that symmetrization of any triple of indices of $\mathcal{E}_{|\mathbf{a}^p\mathbf{b}^q|_0}$ vanishes. Therefore irreducible components of $\mathcal{E}_{|\mathbf{a}^p\mathbf{b}^q|_0}$ are of the form $\mathcal{E}(s_1, s_2)_0$, $n \geq s_1 \geq s_2$ such that $s_1+s_2 = p+q$. Since $\mathcal{E}(s_1, s_2)_0 \neq 0$ if and only if $s_1 + s_2 \leq n$, "$\Longrightarrow$" follows.

In the Lemma below, we shall need the following property: the mapping

$$\begin{gathered}\chi : \mathcal{E}_{|\mathbf{a}^p\mathbf{b}^q|_0} \longrightarrow \mathcal{E}_{\mathbf{a}^{p+1}\mathbf{b}^{q+1}}[2], \quad (\chi\tilde{f})_{\mathbf{a}^{p+1}\mathbf{b}^{q+1}} = \boldsymbol{g}_{a^{p+1}b^{q+1}}\tilde{f}_{\mathbf{a}^p\mathbf{b}^q} \\ \text{vanishes for } p+q=n\end{gathered} \tag{2.27}$$

where we skew over $[a^{p+1}\mathbf{a}^p]$ and $[b^{q+1}\mathbf{b}^q]$ in $\boldsymbol{g}_{a^{p+1}b^{q+1}}\tilde{f}_{\mathbf{a}^p\mathbf{b}^q}$. The proof follows easily from the trace of $\chi\tilde{f}$. It is straightforward to compute

$$\boldsymbol{g}^{a^{p+1}b^{q+1}}(\chi\tilde{f})_{\mathbf{a}^{p+1}\mathbf{b}^{q+1}} = \frac{n-p-q}{(p+1)(q+1)}\tilde{f}_{\mathbf{a}^p\mathbf{b}^q}.$$

Hence $\boldsymbol{g}^{a^{p+1}b^{q+1}}$ provides an inversion (up to a scalar multiple) of χ if $p+q < n$ and χ is an injection in this case. However if $p+q = n$ then $\chi\tilde{f}$ is trace–free and hence zero according to (2.26).

Lemma. *(i) The operator $\tilde{T}$ is conformally invariant.*
(ii) The operator $\tilde{\tilde{T}}$ is conformally invariant if $s_2 = n' = \frac{n}{2}$ and n is even.

Proof. To simplify the notation, we shall suppose $\mathfrak{T} = \emptyset$ but the proof for $\mathfrak{T} \neq \emptyset$ is formally the same. (We shall comment upon the assumption $\mathfrak{T} = \emptyset$ briefly at the end of the proof.) Much of the notation and several observation are drawn from 1.2.6. The proof, although rather long and technical, is composed of several simple steps based on Lemma 1.3.8 and Proposition 1.3.8 as well as 1.2.6. Concerning the latter, note the volume form is not used in the definitions of $\tilde{T}$ and $\tilde{\tilde{T}}$, see (2.25).

(i) Using the notation $\mathfrak{A} := [A_1^0\mathbf{A}_1]$, $\mathfrak{B} := \mathbf{B}_2 \cdots \mathbf{B}_r$, $\mathfrak{b} := \mathbf{b}_2 \cdots \mathbf{b}_r$ and $\mathfrak{a} := \mathbf{a}_2 \cdots \mathbf{a}_r$ for systems of indices, a part of the formula (2.25) for $\tilde{T}$ is the invariant operator

$$\Phi^{\mathbf{a}_1\mathfrak{b}}_{\mathfrak{A}\mathfrak{B}} = D_{[A^0_1} M^{\mathbf{a}_1}_{\mathbf{A}_1]} M^{\mathbf{b}_2}_{\mathbf{B}_2} \cdots M^{\mathbf{b}_r}_{\mathbf{B}_r} : \mathcal{E}(s_1, \dots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{A}\mathfrak{B}}[w - s - 1].$$

We shall consider $\Phi_{\mathfrak{A}\mathfrak{B}}$ also as a tractor formula (recall M and D are given by formulae) and use the notation from 1.2.6 for Φ. We need to show the projection $\mathbb{Z}^{\mathbf{B}_r}_{\mathbf{a}_r} \cdots \mathbb{Z}^{\mathbf{B}_2}_{\mathbf{a}_2} \Phi_{\mathfrak{A}\mathbf{B}_2 \cdots \mathbf{B}_r}$ is invariant. Considering the $\mathbb{Z}$–terms as the TFP–component $(\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}} := \mathbb{Z}^{\mathbf{a}_2}_{\mathbf{B}_2} \cdots \mathbb{Z}^{\mathbf{a}_r}_{\mathbf{B}_r} \in TFPC(E_{\mathfrak{B}})$, this means to show $\big((\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}}\big)^* \Phi_{\mathfrak{B}}$ is invariant i.e. that $(\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}}$ is a projecting part of the formula Φ. Since $h\big((\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}}\big) = 0$, it is sufficient to show

$$\forall pr_{\mathfrak{B}} \in TFPC(E_{\mathfrak{B}}): \ h(pr_{\mathfrak{B}}) > 0 \Longrightarrow (pr_{\mathfrak{B}})^* \Phi = 0,$$

see Lemma 1.2.6. (Recall $(pr_{\mathfrak{B}})^*\Phi = 0$ means that the operator $(pr_{\mathfrak{B}})^*\Phi$ vanishes.) It turns out that in treating the formula Φ, it is easier to consider TFP–components of the whole bundle $E_{\mathfrak{A}\mathfrak{B}}$. From this point of view, our aim is to show that

$$\forall pr_{\mathfrak{A}\mathfrak{B}} \in TFPC(E_{\mathfrak{A}\mathfrak{B}}): \ h_{\mathfrak{B}}(pr_{\mathfrak{A}\mathfrak{B}}) > 0 \Longrightarrow (pr_{\mathfrak{A}\mathfrak{B}})^* \Phi = 0 \tag{2.28}$$

which is obviously equivalent to the previous display and thus proves (i).

Let us summarise relations between the quantities h, v and fo of TFP–components of $E_{\mathfrak{A}\mathfrak{B}}$ and the formula Φ we shall need later. Henceforth we shall denote elements of $TFPC(E_{\mathfrak{A}\mathfrak{B}})$ simply by pr i.e. without attached indices. Every $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$ satisfies

$$|h(pr) - h_{\mathfrak{B}}(pr)| \ \leq \ 1 \tag{2.29}$$

$$v(pr) - v_{\mathfrak{B}}(pr) \ \leq \ s_1 + 1 \tag{2.30}$$

$$v_{\mathfrak{B}}(pr) \ \leq \ (s - s_1) - |h_{\mathfrak{B}}(pr)| \tag{2.31}$$

and if the formula $pr^*\Phi$ is a nontrivial expression (i.e. a nontrivial formal sum of ‘words’, see Definition 1.2.6, page 54) then moreover

$$fo(pr^*\Phi) \ = \ 1 - h(pr). \tag{2.32}$$

Let us comment upon these relations briefly. Recall $E_{\mathfrak{A}} = E_{[A_1^0 \mathbf{A}_1]}$ is a form tractor bundle where $\mathbf{A}_1 = \mathbf{A}_1^{s_1}$ and, by the definitions of the homogenity and valence in 1.2.6, we have $h(pr) = h_{\mathfrak{A}}(pr) + h_{\mathfrak{B}}(pr)$ where $h_{\mathfrak{A}}(pr) \in \{0, \pm 1\}$ and $v(pr) = v_{\mathfrak{A}}(pr) + v_{\mathfrak{B}}(pr)$ where $v_{\mathfrak{A}}(pr) \leq s_1 + 1$. This proves (2.29) and (2.30), respectively. The relation (2.31) is just Proposition 1.2.6 for the bundle $E_{\mathfrak{B}}$ where $|\mathfrak{B}| = s - s_1$. Finally, note $oh(\Phi_{\mathfrak{A}\mathfrak{B}}) = oh(D_{[A_1^0} M_{\mathbf{A}_1]}) + oh(M_{\mathfrak{B}}) = 1 + 0 = 1$ (with omitted tensor indices) because $oh(D) = 1$ and $oh(M) = 0$. Now (2.32) follows from the relation $h(pr) + fo(pr^* \Phi) = oh(\Phi)$ for a nontrivial formula $pr^* \Phi$.

Let us go back to (2.28) which we want to prove. We have observed $oh(\Phi) = 1$ in the previous paragraph. Therefore

$$\forall pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}}) : \quad h(pr) \geq 2 \Longrightarrow pr^* \Phi = 0 \tag{2.33}$$

$$\forall pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}}) : \quad h_{\mathfrak{B}}(pr) \geq 3 \Longrightarrow pr^* \Phi = 0 \tag{2.34}$$

where (2.34) follows from (2.33) using (2.29). From (2.34) and (2.33), it remains to show the conditions $h_{\mathfrak{B}}(pr) \in \{1, 2\}$ and $h(pr) \leq 1$ imply $pr^* \Phi = 0$. Thus we have to consider the following cases: if $h_{\mathfrak{B}}(pr) = 2$ then (2.29) and $h(pr) \leq 1$ require $(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)) = (-1, 2)$ and if $h_{\mathfrak{B}}(pr) = 1$ then there are two possibilities $(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)) = (0, 1)$ and $(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)) = (-1, 1)$ also using (2.29) and $h(pr) \leq 1$. (In other words, we are using $h_{\mathfrak{A}}(pr) \in \{-1, 0, 1\}$. Recall $h(pr) = h_{\mathfrak{A}\mathfrak{B}}(pr) = h_{\mathfrak{A}}(pr) + h_{\mathfrak{B}}(pr)$ by definition.) Summarising, we are going to show

$$\forall pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}}) : \big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) \in \begin{matrix} (-1,2) & & (0,1) \\ & \diagdown \quad \diagup & \\ & (-1,1) & \end{matrix} \Longrightarrow pr^* \Phi = 0. \tag{2.35}$$

Here rows of the lattice correspond to cases with the same homogenity (either 1 or 0) and (a, b) is connected with (c, d) if $(a, b) < (c, d)$. We use the following ordering on pairs: we say $(a,b) < (c,d)$ if $a \le c$, $b \le d$ and at least one inequality is sharp. We shall discuss these three cases in the lattice (2.35) separately.

(a) Assume $\big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) = (-1,2)$, $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$. That is, $h(pr) = 1$ hence (2.33) together with Lemma 1.2.6 (ii) shows that pr is a projecting part of Φ. Thus the operator $pr^*\Phi$ is invariant and $fo(pr^*\Phi) = 0$ using (2.32). The formal order 0 shows that all terms in $pr^*\Phi f$ involve only f and the conformal metric $\boldsymbol{g}$. Write pr with indices as $pr^{\mathfrak{c}}_{\mathfrak{A}\mathfrak{B}}$ for an appropriate system $\mathfrak{c}$ of tensor indices. Since Φf has no free tensor indices, $\mathfrak{c}$ is the system of free indices in $(pr^*\Phi f)_{\mathfrak{c}} = (pr^*)^{\mathfrak{A}\mathfrak{B}}_{\mathfrak{c}}\Phi_{\mathfrak{A}\mathfrak{B}}f$. In particular, these free indices are covariant. Since only $\boldsymbol{g}$'s are used in $pr^*\Phi$, we can suppose *all* indices of $pr^*\Phi f$ are covariant and free. (That is, there are no contractions.) Let us discuss their number i.e. the valence of pr. Clearly $h_{\mathfrak{A}}(pr) = -1$ implies $v_{\mathfrak{A}}(pr) = s_1$ and this, together with (2.31) yields the inequality in

$$v(pr) := |\mathfrak{c}| = v_{\mathfrak{A}}(pr) + v_{\mathfrak{B}}(pr) \le s_1 + [(s - s_1) - 2] = s - 2.$$

Denoting the (tensor) valence of $pr^*\Phi f$ by $s' := v(pr)$, we have shown $s' = |\mathfrak{c}| \le s - 2$ where s is the (tensor) valence of f. Since $fo(pr^*\Phi) = 0$, it follows from Lemma 1.3.8 (ii), page 75, that the operator $pr^*\Phi$ vanishes.

(b) Assume $\big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) = (0,1)$, $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$. As in (a), pr is a projecting part of Φ (using $h(pr) = 1$, (2.33) and Lemma 1.2.6 (ii)) and $fo(pr^*\Phi) = 0$ (using (2.32)). The projection is again $(pr^*\Phi f)_{\mathfrak{c}}$ with a covariant system of free indices $\mathfrak{c}$. The valence $v_{\mathfrak{A}}(pr) \in \{s_1 \pm 1\}$ together with (2.31) yields the inequality in

$$v(pr) = |\mathfrak{c}| = v_{\mathfrak{A}}(pr) + v_{\mathfrak{B}}(pr) \le s_1 + 1 + [(s - s_1) - 1] = s.$$

We shall apply Lemma 1.3.8 to the operator $pr^*\Phi$. If the inequality is sharp, Lemma 1.3.8 (ii) shows the operator $pr^*\Phi$ vanishes. Suppose $v(pr) = s$. This requires $v_{\mathfrak{A}}(pr) = s_1 + 1$ and we skew over $s_1 + 1$ indices in $\mathfrak{c}$. In other words, this means $s_1 < s_1' := s_1 + 1$ in the notation of Lemma 1.3.8 (iii). Thus the operator $pr^*\Phi$ vanishes as well.

(c) Finally assume $\big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) = (-1, 1)$, $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$. That is, $h(pr) = 0$. We cannot use the same reasoning as above to conclude that pr is a projecting part of Φ. However, this is satisfied, nevertheless. The point is the following. Although we do not have $(\widetilde{pr})^*\Phi = 0$ for $h(\widetilde{pr}) > 0$ (to use Lemma 1.2.6), (2.35) and (2.34) together with (a) and (b) yield a (weaker) statement

$$\forall \widetilde{pr} \in TFPC(E_{\mathfrak{A}\mathfrak{B}}) : \ \big(h_{\mathfrak{A}}(\widetilde{pr}), h_{\mathfrak{B}}(\widetilde{pr})\big) > (-1, 1) \Longrightarrow \widetilde{pr}^*\Phi = 0.$$

Recall $\widetilde{pr}^*\Phi = 0$ means that the operator $\widetilde{pr}^*\Phi$ vanishes. (The formula $\widetilde{pr}^*\Phi$ may be nontrivial.) Now it is straightforward to modify Lemma 1.2.6 and its proof to show that from this it follows that $pr^*\Phi$ is invariant. That is, pr is a projecting part of Φ. But from (2.32), $fo(pr^*\Phi) = 1$ and from this it follows $pr^*\Phi$ satisfies assumptions of Proposition 1.3.8, page 75. Therefore the operator $pr^*\Phi$ vanishes.

(ii) Let us suppose that n is even, $r \geq 2$ and $s_1 = s_2 = n' = \frac{n}{2}$. We shall follow the proof of (i) but the steps (a), (b) and (c) corresponding to the lattice as in (i) will be more complicated. We will need also the observation (2.27).

We will use slightly different systems of indices $\mathfrak{A} := [A_1^0 \mathbf{A}_1]$, $\mathfrak{B} := [B_2^0 \mathbf{B}_2]\mathbf{B}_3 \cdots \mathbf{B}_r$, $\mathfrak{b} := \mathbf{b}_2 \cdots \mathbf{b}_r$ and $\mathfrak{a} := \mathbf{a}_2 \cdots \mathbf{a}_r$. The invariant operator Φ is now given by the formula

$$\Phi_{\mathfrak{A}\mathfrak{B}}^{\mathbf{a}_1 \mathfrak{b}} = D_{[A_1^0} \underbrace{M_{\mathbf{A}_1]}^{\mathbf{a}_1} \mathbb{X}_{B_2^0 \mathbf{B}_2}{}^{\mathbf{b}_2} \overbrace{M_{\mathbf{B}_3}^{\mathbf{b}_3} \cdots M_{\mathbf{B}_r}^{\mathbf{b}_r}}^{M^{(r-2)}}}_{\Phi'} : \mathcal{E}(s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{A}\mathfrak{B}}[w - s],$$

cf. the formula (2.25) for $\tilde{\tilde{T}}$. We will also need the operators (formulae) Φ' and $M^{(r-2)}$ marked on the display.

Considering Φ as a tractor formula, we need to show the projection $\mathbb{Z}_{\mathbf{a}_r}^{\mathbf{B}_r} \cdots \mathbb{Z}_{\mathbf{a}_3}^{\mathbf{B}_3} \mathbb{Y}^{B_2^0 \mathbf{B}_2}_{\mathbf{a}_2} \Phi_{[A_1^0 \mathbf{A}_1][B_2^0 \mathbf{B}_2]\mathbf{B}_3 \cdots \mathbf{B}_r} = \big((\mathbb{X}\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}}\big)^* \Phi_{\mathfrak{B}}$ is invariant where $(\mathbb{X}\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}} := \mathbb{X}_{B_2^0 \mathbf{B}_2}{}^{\mathbf{a}_2} \mathbb{Z}_{\mathbf{B}_3}^{\mathbf{a}_3} \cdots \mathbb{Z}_{\mathbf{B}_r}^{\mathbf{a}_r} \in TFPC(E_{\mathfrak{B}})$. That is, we need to show that $(\mathbb{X}\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}}$ is a projecting part of the formula Φ. Since this has the homogenity $h\big((\mathbb{X}\mathbb{Z}\cdots\mathbb{Z})^{\mathfrak{a}}_{\mathfrak{B}}\big) = -1$, it is sufficient to show

$$\forall pr_{\mathfrak{B}} \in TFPC(E_{\mathfrak{B}}): \ h(pr_{\mathfrak{B}}) > -1 \Longrightarrow (pr_{\mathfrak{B}})^* \Phi = 0,$$

see Lemma 1.2.6. As we prefer to consider TFP–components of the whole bundle $E_{\mathfrak{A}\mathfrak{B}}$, we shall prove the equivalent property

$$\forall pr_{\mathfrak{A}\mathfrak{B}} \in TFPC(E_{\mathfrak{A}\mathfrak{B}}): \ h_{\mathfrak{B}}(pr_{\mathfrak{A}\mathfrak{B}}) > -1 \Longrightarrow (pr_{\mathfrak{A}\mathfrak{B}})^* \Phi = 0. \tag{2.36}$$

Since the bundle $E_{\mathfrak{A}}$ is the same as in (i), the TFP–components of $E_{\mathfrak{A}\mathfrak{B}}$ satisfy (2.29) and (2.30). But we have to modify (2.32) and (2.31) because $|\mathfrak{B}| = s - s_1 + 1$ and $oh(\Phi) = oh(D_{[A_1^0} M_{\mathbf{A}_1]}) + oh(\mathbb{X}_{[B_2^0 \mathbf{B}_2]}) + oh(M^{(r-2)}) = 1 - 1 + 0 = 0$, respectively. Cf. discussion on (2.32) and (2.31) in the proof of (i) above. Summarising, every $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$ now satisfies

$$v_{\mathfrak{B}}(pr) \ \leq \ (s - s_1 + 1) - |h_{\mathfrak{B}}(pr)| \tag{2.37}$$

$$fo(pr^* \Phi) \ = \ -h(pr) \quad \text{if } pr^*\Phi \text{ is a nontrivial formula.} \tag{2.38}$$

We want to prove (2.36). Following (i), we get

$$\forall pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}}): \ \ h(pr) \geq 1 \Longrightarrow pr^*\Phi = 0 \tag{2.39}$$

$$\forall pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}}): \ \ h_{\mathfrak{B}}(pr) \geq 2 \Longrightarrow pr^*\Phi = 0 \tag{2.40}$$

where (2.39) follows from (2.38) (recall $pr^*\Phi \neq 0$ means $fo(pr^*\Phi) \geq 0$) and (2.40) follows from (2.39) and $h_{\mathfrak{A}}(pr) \in \{\pm 1, 0\}$. Using these and (2.29), it remains to show the conditions $h_{\mathfrak{B}}(pr) \in \{0,1\}$ and $h(pr) \le 0$ imply $pr^*\Phi = 0$. Thus we have the following cases: if $h_{\mathfrak{B}}(pr) = 1$ then (2.29) and $h(pr) \le 0$ require $(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)) = (-1,1)$ and if $h_{\mathfrak{B}}(pr) = 0$ then there are two possibilities $(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)) = (0,0)$ and $(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)) = (-1,0)$ also using (2.29) and $h(pr) \le 1$. Summarising, we are going to show

$$\forall pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}}) : \big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) \in \begin{array}{ccc} (0,0) & & (-1,1) \\ & \diagdown \;\; \diagup & \\ & (-1,0) & \end{array} \implies pr^*\Phi = 0. \tag{2.41}$$

Here the lattice is formed in the same way as in (i) and we use the same ordering for pairs $(a,b) < (c,d)$.

Before we discuss the three cases in the lattice (2.41) separately, we need the following property which follows from the construction of the middle operator on $\mathcal{E}(n', n')_0[w']$, page 90. (Here w' is an arbitrary weight.) Using the notation $\dot{\mathfrak{B}} := \mathbf{B}_3 \cdots \mathbf{B}_r$ and similarly $\dot{\mathfrak{b}} := \mathbf{b}_3 \cdots \mathbf{b}_r$ and $\dot{\mathfrak{a}} := \mathbf{a}_3 \cdots \mathbf{a}_r$, the formula Φ' (a part of Φ) satisfies

$$\Phi'{}^{\mathbf{a}_1\dot{\mathfrak{b}}}_{\mathbf{A}_1\dot{\mathfrak{B}}} = M^{\mathbf{a}_1}_{\mathbf{A}_1} \mathbb{X}_{B_2^0\mathbf{B}_2}{}^{\mathbf{b}_2} (M^{(r-2)})^{\dot{\mathfrak{b}}}_{\dot{\mathfrak{B}}} = \mathbb{X}_{B_2^0\mathbf{B}_2}{}^{\mathbf{b}_2} M^{\mathbf{a}_1}_{\mathbf{A}_1} (M^{(r-2)})^{\dot{\mathfrak{b}}}_{\dot{\mathfrak{B}}}$$

where $M^{\mathbf{a}_1}_{\mathbf{A}_1}$ on the left–hand side is the formula for the middle operator on $\mathcal{E}_{\mathfrak{B}}(n')[w - s + n' + 1]$ and $M^{\mathbf{a}_1}_{\mathbf{A}_1}$ on the right–hand side the formula for the middle operator on $\mathcal{E}_{\dot{\mathfrak{B}}}(n', n')_0[w - s + n]$. (That is, the middle operator on $\mathcal{E}_{\dot{\mathfrak{B}}}(n', n')_0[w - s + n]$ is constructed using this relation, cf. page 90.) The right–hand side shows every $pr' \in TFPC(E_{\mathbf{A}_1\mathfrak{B}})$ such that $(pr')^*\Phi'$ is a nontrivial formula, involves the factor $\mathbb{X}_{B_2^0\mathbf{B}_2}{}^{\mathbf{a}_2}$. Further, since $\Phi'{}^{\mathbf{a}_1\dot{\mathfrak{b}}}_{\mathbf{A}_1\dot{\mathfrak{B}}} = \mathbb{X}_{B_2^0\mathbf{B}_2}{}^{\mathbf{b}_2} (M^{(r-1)})^{\mathbf{a}_1\dot{\mathfrak{b}}}_{\mathbf{A}_1\dot{\mathfrak{B}}}$, it follows from Theorem 2.1.4 (i) that $(pr')^*\Phi'$ nontrivial requires $h_{\mathbf{A}_1\dot{\mathfrak{B}}}(pr') \le 0$. Summarising the last two observations, $(pr')^*\Phi'$ nontrivial implies $h_{\mathfrak{B}}(pr') = h_{B_2^0\mathbf{B}_2}(pr') + h_{\dot{\mathfrak{B}}}(pr') \le -1$. (We have used $h_{\dot{\mathfrak{B}}}(pr') \le h_{\mathbf{A}_1\dot{\mathfrak{B}}}(pr')$ here.) Moreover, the equality can happen only for

TFP–components

$$pr_1(r-1) \;:=\; \mathbb{Z}^{\mathbf{a}_1}_{\mathbf{A}_1}\mathbb{X}_{B^0_2\mathbf{B}_2}{}^{\mathbf{b}_2}\mathbb{Z}^{\mathbf{b}_3}_{\mathbf{B}_3}\cdots\mathbb{Z}^{\mathbf{b}_r}_{\mathbf{B}_r} \in TFPC(E_{\mathbf{A}_1\mathfrak{B}}) \qquad (2.42)$$

$$pr_2(r-2) \;:=\; \mathbb{X}^{\dot{\mathbf{a}}_1}_{\mathbf{A}_1}\mathbb{X}_{B^0_2\mathbf{B}_2}{}^{\mathbf{b}_2}\mathbb{Z}^{\mathbf{b}_3}_{\mathbf{B}_3}\cdots\mathbb{Z}^{\mathbf{b}_r}_{\mathbf{B}_r} \in TFPC(E_{\mathbf{A}_1\mathfrak{B}}) \qquad (2.43)$$

of the $\mathfrak{B}$–homogenity -1. Summarising, the formula Φ' satisfies

$$\forall pr' \in TFPC(E_{\mathbf{A}_1\mathfrak{B}}) : (pr')^*\Phi' \text{ nontrivial} \Longrightarrow$$
$$\Longrightarrow [pr' = pr_1(r-1)] \vee [pr' = pr_2(r-1)] \vee [h(pr') < -1]. \quad (2.44)$$

Note these TFP–components pr, $pr_1(r-1)$ etc. are formally terms in Φ' or Φ and shall be referred as terms. Also recall the definition of the tractor D–operator (1.32): $D_{A^0_1}$ has the terms $Y_{A^0_1}$, $Z^a_{A^0_1}\nabla_a$ and $X_{A^0_1}(\Delta + w'P)$ up to scalar multiples. (We will not need their explicit value or the value of the scalar w'.) Now we can start the discussion on the lattice (2.41).

(a) Assume $\big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) = (0,0)$, $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$. That is, $h(pr) = 0$ hence (2.39) together with Lemma 1.2.6 shows that pr is a projecting part of Φ. Thus the operator $pr^*\Phi$ is invariant and $fo(pr^*\Phi) = 0$ using (2.32). Using the same argument as in (a) in the proof of (i), we can suppose all indices of $(pr^*\Phi)f$ are downstairs and free. Let us look at the valence of pr. Clearly $h_{\mathfrak{A}}(pr) = 0$ implies $v_{\mathfrak{A}}(pr) = \{s_1+1, s_1-1\}$ and this, together with (2.37) yields the inequality

$$v(pr) = v_{\mathfrak{A}}(pr) + v_{\mathfrak{B}}(pr) \leq s_1 + 1 + (s + 1 - s_1) = s + 2.$$

If $v(pr) < s$ or $v(pr) = s+1$ then the operator $pr^*\Phi$ vanishes due to Lemma 1.3.8 (ii). Hence it remains to discuss two possibilities $v(pr) \in \{s, s+2\}$.

- Assume $v(pr) = s$. As in (i), we shall apply Lemma 1.3.8 to the operator $pr^*\Phi$. Recall $v_{\mathfrak{A}}(pr) = \{s_1+1, s_1-1\}$. If $v_{\mathfrak{A}}(pr) = s_1+1$ then $s_1 < s'_1 = n'+1$ in the notation of Lemma 1.3.8 (iii) hence $pr^*\Phi$

vanishes. If $v_{\mathfrak{A}}(pr) = s_1 - 1$ then $v_{\mathfrak{A}}(pr) = s - s_1 + 1 = |\mathfrak{B}|$. Therefore pr on $\mathfrak{B}$–indices is of the form $\mathbb{Z} \cdots \mathbb{Z}$. In particular, the term $\mathbb{Z}_{B_2^0 \mathbf{B}_2}^{b_2^0 \mathbf{b}_2}$ appears in pr. Since $\mathbf{b}_2 = \mathbf{b}_2^{n'}$, $[b_2^0 \mathbf{b}_2]$ requires $n' + 1 = \frac{n}{2} + 1$ skewed indices. In other words, $s_2 < s_2' = n' + 1$ in the notation of Lemma 1.3.8 (iii) hence the operator $pr^* \Phi$ vanishes.

- Assume $v(pr) = s + 2$. This case is more complicated – we will not use Lemma 1.3.8 but the observation (2.27) instead. Since $v(pr) = s + 2 = |\mathfrak{A}\mathfrak{B}|$, clearly

$$pr = \mathbb{Z}_{A_1^0 \mathbf{A}_1} \mathbb{Z}_{B_2^0 \mathbf{B}_2} \mathbb{Z}_{\mathbf{B}_3} \cdots \mathbb{Z}_{\mathbf{B}_r} \tag{2.45}$$

where we have omitted tensor indices. Therefore we can use only the Z–term in $D_{A_1^0}$ to obtain the term corresponding to pr in Φ. That is, we have just one derivative at disposal. Now consider how to obtain the term pr in the formula Φ from terms in the formula Φ'. Since one derivative can change the homogenity by at most one and $h_{\mathfrak{B}}(pr) = 0$, the relation (2.44) says we must apply the derivative to one of the terms $pr_1(r-1)$ or $pr_2(r-2)$ in such a way that the $\mathfrak{B}$–homogenity increases. But considering the whole ($\mathbf{A}_1\mathfrak{B}$–)homogenity, we see we cannot obtain pr from the latter. Hence it remains to consider $pr_1(r-1)$.

Comparing (2.42) with (2.45) we immediately see one has to apply the derivative to the $\mathbb{X}$–term in $pr_1(r-1)$. The result is

$$\mathbb{Z}_{A_1^0 \mathbf{A}_1}^{a_1^0 \mathbf{a}_1} (\nabla_{a_1^0} \mathbb{X}_{B_2^0 \mathbf{B}_2}{}^{\mathbf{b}_2}) M^{(r-2)} = \underbrace{\mathbb{Z}_{A_1^0 \mathbf{A}_1}^{a_1^0 \mathbf{a}_1} \mathbb{Z}_{B_2^0 \mathbf{B}_2}^{b_2^0 \mathbf{b}_2} \boldsymbol{g}_{a_1^0 b_2^0}}_{\chi} M^{(r-2)} + \text{“lvt”}$$

up to a scalar multiple, where “lvt” stands for “lower valence terms”. (“Terms” here are formally TFP–components so we can consider their (tensor) valence. “Lower valence” means the valence smaller than $s+2$.) But χ on the right hand side is just the mapping from (2.27) which vanishes because $s_1 + s_2 = 2n' = n$. Therefore the operator $pr^*\Phi$ vanishes also.

(b) Assume $\big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) = (-1, 1)$, $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$. That is, $h(pr) = 0$ hence (2.39) together with Lemma 1.2.6 shows that pr is a projecting part of Φ. Thus the operator $pr^*\Phi$ is invariant and $fo(pr^*\Phi) = 0$ using (2.32). As in (a), we can suppose all indices of $(pr^*\Phi)f$ are downstairs and free. Let us look at the valence of pr. Clearly $h_{\mathfrak{A}}(pr) = -1$ implies $v_{\mathfrak{A}}(pr) = s_1$ and this, together with (2.37) yields the inequality in

$$v(pr) = v_{\mathfrak{A}}(pr) + v_{\mathfrak{B}}(pr) \leq s_1 + (s - s_1) = s.$$

If $v(pr) < s$ then $pr^*\Phi = 0$ due to Lemma 1.3.8 (ii). Thus it remains to discuss the case $v(pr) = s$. This clearly implies $v_{\mathfrak{B}}(pr) = s - s_1$.

Now consider how to obtain the term pr in the formula Φ from terms in the formula Φ'. Since one derivative can change the homogenity by at most one and $h_{\mathfrak{B}}(pr) = 1$, the relation (2.44) says we need at least two derivatives to obtain pr in Φ. That is, to obtain the pr in Φ we must use the Laplacian in the $X_{A_1^0}$–term of $D_{A_1^0}$. Moreover, we must apply both derivatives either to $pr_1(r-1)$ or to $pr_2(r-2)$ and only to $\mathfrak{B}$–indices in these two terms. But the latter will vanishes after the skew–symmetrization $[A_1^0\mathbf{A}_1]$. Therefore it remains to consider $pr_1(r-1)$.

Looking at $pr_1(r–1)$ in (2.42), there is one $\mathbb{X}$–factor and several $\mathbb{Z}$–factors on $\mathfrak{B}$–indices. So we have three possibilities of how to apply the Laplacian on $\mathfrak{B}$–indices to increase the $\mathfrak{B}$–homogenity by two, see below. Recall $v(pr) = s$ and $v_{\mathfrak{B}}(pr_1(r-1)) = v_{\mathfrak{B}}(pr) = s - s_1$.

- If we apply both derivatives to $\mathbb{Z}$–terms in $pr_1(r-1)$ and the $\mathfrak{B}$–homogenity increases by two, the $\mathfrak{B}$–valence necessarily decreases by 2 thus $v(pr) = s - 2$. Hence the operator $pr^*\Phi$ vanishes using Lemma

1.3.8 (ii). (Recall $fo(pr^*\Phi) = 0$ guarantees no curvature terms can appear after application of the derivatives.)

- If we apply one derivative to the $\mathbb{X}$–term and the second one to one of the $\mathbb{Z}$–terms in $pr_1(r-1)$, the resulting pr has to involve either the term $\mathbb{Z}_{B_2^0\mathbf{B}_2}^{b_2^0\,\mathbf{b}_2}$ or the term $\mathbb{W}_{B_2^0\mathbf{B}_2}^{\,\dot{\mathbf{b}}_2}$. The first case requires skewing over $\frac{n}{2}+1$ indices $[b_2^0\mathbf{b}_2]$ hence $pr^*\Phi$ vanishes. (In other words, $s_2 < s_2' = n'+1$ in the notation of Lemma 1.3.8 (iii).) In the second case, clearly $v(pr) = s-2$ hence the operator $pr^*\Phi$ vanishes using Lemma 1.3.8 (ii).

- If we apply both derivatives to the $\mathbb{X}$–terms in $pr_1(r-1)$, we will get only one term increasing the $\mathfrak{B}$–homogenity by two, in particular

$$\Delta\mathbb{X}^k = -(n-2k)\mathbb{Y}^k + \text{lht}$$

  where lht stands for "lower homogenity terms", see (1.50). But $k = s_1 = n'$ in our case hence the top slot on the right hand side vanishes. Therefore the operator $pr^*\Phi$ vanishes.

(c) Finally assume $\big(h_{\mathfrak{A}}(pr), h_{\mathfrak{B}}(pr)\big) = (-1, 0)$, $pr \in TFPC(E_{\mathfrak{A}\mathfrak{B}})$. That is, this is the lower possibility in the lattice (2.41). We have shown in (a) and (b) that both higher possibilities, as TFP–components of Φ, yield trivial operators. Now the same reasoning as in (c) of the part (i) reveals pr is a projecting part of Φ. But $h(pr) = -1$ and $fo(pr^*\Phi) = 1$ using (2.38). Hence the operator $pr^*\Phi$ satisfies assumptions of Proposition 1.3.8 and therefore vanishes.

We have proved the Theorem for $\mathfrak{T} = \emptyset$ but as we have not needed to consider anything about $\mathfrak{T}$, the same proof clearly applies for any $\mathfrak{T}$. Let us note we have proved triviality of many operators using Lemma 1.3.8 and Proposition 1.3.8 throughout the proof. All of them have been given by formulae of the formal order 0 or 1 and curvature terms have not appeared. Hence the same reasoning can be used if $\mathfrak{T} \neq \emptyset$. $\square$

Properties of the top operator

We defined two candidates for the top operator so we need to decide when to use $\tilde{T}$ and when $\tilde{\tilde{T}}$, see (2.25). We shall define the *top operator* T as

$$T = \begin{cases} \tilde{T} & s_2 < \frac{n}{2} \\ \tilde{\tilde{T}} & s_2 = \frac{n}{2}. \end{cases} \tag{2.46}$$

In particular, $T = \tilde{T}$ for n odd. This choice is actually not always necessary. But in certain cases where $s_1 = s_2 = \frac{n}{2}$, $\tilde{T}$ is not a splitting operator whereas $\tilde{\tilde{T}}$ is. This is easy to see from Examples 2.1.7 and 2.1.8. (The latter discusses the issue in details.)

Further we need to know properties of composition of the top operators. (Recall they are strongly invariant so we can compose them.) Let us suppose we apply the top operator t times, $1 \leq t \leq r$. That is, in the odd dimensional case we apply $\tilde{T}$ only, and in the even dimensional case we apply $\tilde{\tilde{T}}$ to $r_{n'} - 1$ (longest) form indices of the valence $n' = \frac{n}{2}$ first and $\tilde{T}$ next. So we use $\tilde{\tilde{T}}$ q–times where

$$q := \begin{cases} \min\{t, \max\{r_{n'} - 1, 0\}\} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Summarizing, the result, also called the *top operator*, is the composition

$$\begin{aligned} \left(T^{(t)}\right)_{A^0_t \mathbf{A}_t \cdots A^0_1 \mathbf{A}_1}^{\mathbf{a}_t \cdots \quad \mathbf{a}_t} &:= \tilde{T}_{A^0_t \mathbf{A}_t}^{\mathbf{a}_t} \cdots \tilde{T}_{A^0_{q+1} \mathbf{A}_{q+1}}^{\mathbf{a}_{q+1}} \tilde{\tilde{T}}_{A^0_q \mathbf{A}_q}^{\mathbf{a}_q} \cdots \tilde{\tilde{T}}_{A^0_1 \mathbf{A}_1}^{\mathbf{a}_1} : \\ \mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w] &\longrightarrow \mathcal{E}_{\mathfrak{T}[A^0_1 \mathbf{A}_1] \cdots [A^0_t \mathbf{A}_t]}(s_{t+1}, \ldots, s_r)_0[w - s^t - t] \end{aligned} \tag{2.47}$$

The following Theorem says when this is a splitting operator. Let us note that the formula for every T in the composition $T^{(t)}$ is applied to the longest available (tensor) form index. Here and below, every $\tilde{T}$ and $\tilde{\tilde{T}}$ in the composition $T^{(t)}$ is referred as 'T'.

Recall that the formulae for both operators $\tilde{T}$ and $\tilde{\tilde{T}}$ involve several middle operators and one tractor D–operator. Recall also that $oh(D) = 1$ and $oh(M) = 0$ hence $oh(T) = 1$. Therefore

$$oh(T^{(t)}) = oh(\underbrace{T \cdots T}_{t}) = t. \tag{2.48}$$

Definition. The top operator $T^{(t)}$ will be called *top splitting* or *top splitting operator* if this is a splitting operator for $\mathfrak{T} = \emptyset$.

Recall if $T^{(t)}$ is a splitting for $\mathfrak{T} = \emptyset$ then it is a splitting for any $\mathfrak{T}$. See also $M^{(m)}$ and a (similar) note for Definition 2.1.4.

The explicit formulae of the top operators $\tilde{T}$ on $\mathcal{E}(s_1)[w]$ and $\tilde{\tilde{T}}$ on $\mathcal{E}(\frac{n}{2}, \frac{n}{2})_0[w]$ are computed in Examples 2.1.6 and 2.1.8 below. We shall need these explicit computations in the following theorem.

Theorem (Properties of the top operator).
Let us consider the top operator $T^{(t)}$ given by the relation (2.47) and a section

$$f \in \mathcal{E}_{\mathfrak{T}}(s_1, \ldots, s_r)_0[w] = \mathcal{E}_{\mathfrak{T}}\{r_1, \ldots, r_{n'}\}_0[w],$$

where $w \in \mathbb{R}$. Let $E_{\mathfrak{A}} := E_{[A_1^0 \mathbf{A}_1] \cdots [A_t^0 \mathbf{A}_t]}$ and consider the scalar

$$s(t,0) := \begin{cases} w - s - t + s_t + 1 & 0 < t < r_{n'},\ n \text{ even, or } t > r_{n'},\ n \in \mathbb{N} \\ w - s - t + \frac{n}{2} & t = r_{n'},\ n \text{ even, or } t \leq r_{n'},\ n \text{ odd.} \end{cases} \tag{2.49}$$

(i) The TFP–component $pr^t := \mathbb{Y}_{A_1^0 \mathbf{A}_1}^{\mathbf{a}_1} \cdots \mathbb{Y}_{A_t^0 \mathbf{A}_t}^{\mathbf{a}_t} \in TFPC(E_{\mathfrak{A}})$ of the homogenity $t = hh(E_{\mathfrak{A}})$ is a projecting part of the operator $T^{(t)}$ and satisfies

$$\forall pr' \in TFPC(E_{\mathfrak{A}})\colon (pr')^* T^{(t)} \neq 0 \Longrightarrow \big[pr' = pr^t\big] \vee \big[h(pr') < h\big(pr^t\big)\big]. \tag{2.50}$$

Furthermore $(pr^t)^* T^{(t)} = C \cdot \mathrm{id}$ *and the scalar* C *satisfies the following. If* $s(t,0) > 0$ *then* $C \neq 0$ *and* $T^{(t)}$ *is the top splitting; if* $s(t,0) = 0$ *then* $C = 0$ *and* $T^{(t)}$ *is not the top splitting operator.*

(ii) Let us suppose $\widetilde{pr} \in TFPC(E_{\mathfrak{T}})$ *has the highest homogenity i.e.* $h(\widetilde{pr}) = hh(E_{\mathfrak{T}})$*. Then the TFP–component* $pr^t \widetilde{pr} \in TFPC(E_{\mathfrak{A}\mathfrak{T}})$ *of the homogenity* $t + hh(E_{\mathfrak{T}}) = hh(E_{\mathfrak{A}\mathfrak{T}})$ *is a projecting part of the section* $T^{(t)} f$ *and satisfies*

$$\forall pr' \in TFPC(E_{\mathfrak{A}\mathfrak{T}})\colon (pr')^* T^{(t)} f \neq 0 \Longrightarrow \big[pr' = pr^t \widetilde{pr}\big] \vee \big[h(pr') < h\big(pr^t \widetilde{pr}\big)\big].$$

Furthermore $\big(pr^t \widetilde{pr}\big)^* T^{(t)} f = C \cdot \widetilde{pr}^* f$ *and where the scalar* C *satisfies the same property as in (i).*

Proof. The TFP–component pr^t is a projecting part of $T^{(t)} f$ because its homogenity is equal to the highest homogenity in $E_{\mathfrak{A}}$ i.e. $hh(E_{\mathfrak{A}}) = t$. Such a TFP–component is unique and so (2.50) follows. Since $oh(T^{(t)}) = t = h(pr^t)$, we get $fo((pr^t)^* T^{(t)}) = 0$ and $(pr^t)^* T^{(t)} = C \cdot \mathrm{id}$. This is clear for $t = 1$ and in the general case, we can decompose $(pr^t)^*$ into the form factors (i.e. projectors from form tractors to forms) and $T^{(t)}$ into $T \cdots T$. To show $C \neq 0$ for $s(t,0) > 0$ we need a detailed analysis of the scalars which appear in the top slot of the formula for $T^{(t)}$.

The composition of $i \leq t$ top operators $T^{(i)}$ is of the form

$$\begin{aligned} T^{(i)} &= \mathbb{Z}^{\mathbf{B}_r} \cdots \mathbb{Z}^{\mathbf{B}_{i+1}} \tilde{T}_{A_i^0 \mathbf{A}_i} \overbrace{M_{\mathbf{B}_{i+1}} \cdots M_{\mathbf{B}_r}}^{M^{(r-i)}} T^{(i-1)} \quad \text{or} \\ T^{(i)} &= \mathbb{Z}^{\mathbf{B}_r} \cdots \mathbb{Z}^{\mathbf{B}_{i+2}} \tilde{\tilde{T}}_{A_i^0 \mathbf{A}_i} \underbrace{M_{\mathbf{B}_{i+2}} \cdots M_{\mathbf{B}_r}}_{M^{(r-i-1)}} T^{(i-1)} \end{aligned} \tag{2.51}$$

where the tensor indices are omitted, see (2.25). The operators $\tilde{T}_{A_i^0 \mathbf{A}_i}$ and $\tilde{\tilde{T}}_{A_i^0 \mathbf{A}_i}$ in the previous display are given by explicit formulae from Examples 2.1.6 or 2.1.8, respectively. To prove (i) we need to show that the middle operator $M^{(r-i)}$ or $M^{(r-i-1)}$ in (2.51) and also $\tilde{T}$ or $\tilde{\tilde{T}}$ therein are splitting operators, and that this is satisfied for every $i \leq t$. Then the $\mathbb{Z}$–projections cannot kill the section they are applied to, because its top slot recovers f up to a nonzero multiple. (The top slot is $\mathbb{Y}_{A_i^0 \mathbf{A}_i} \mathbb{Z}_{\mathbf{B_{i+j}}} \cdots \mathbb{Z}_{\mathbf{B_r}}$, $j \in \{1, 2\}$.)

Let us denote the conformal weight before the application of $\tilde{T}$ and $\tilde{\tilde{T}}$ by $\tilde{w}_i$ and $\tilde{\tilde{w}}_i$, respectively. That is $\tilde{w}_i$ is the weight of $M^{(r-i)}T^{(i-1)}f$ and $\tilde{\tilde{w}}_i$ is the weight of $M^{(r-i-1)}T^{(i-1)}f$. Looking at the formulae (2.54) and (2.57), $\tilde{T}$ and $\tilde{\tilde{T}}$ in (2.51) are splitting operators if and only if

$$\begin{aligned} &\tilde{w}_i\big(n + \tilde{w}_i - 2s_i\big)\big(n + 2(\tilde{w}_i - s_i) - 2\big) \neq 0 \text{ and} \\ &\big(\tilde{\tilde{w}}_i - n'\big)\big(\tilde{\tilde{w}}_i - n' + 1\big) \neq 0 \text{ for } n \text{ even,} \end{aligned} \tag{2.52}$$

respectively. First note this is clearly satisfied if $\tilde{w}_i, \tilde{\tilde{w}}_i \notin \mathbb{AW}$, see (1.61), page 67. This is equivalent to $w \notin \mathbb{AW}$ which also guarantees the middle operator in (2.51) is actually a splitting operator, see Remark 2.1.4. Thus we have proved that $T^{(t)}$ is a splitting operator for $w \notin \mathbb{AW}$.

Henceforth we will assume $w \in \mathbb{AW}$. We shall show below that $s(t, 0) > 0$ implies

$$\begin{array}{lll} \tilde{w}_i > 1 & s_i = n',\ n \text{ even} & \\ \tilde{w}_i > \dfrac{1}{2} & s_i = n',\ n \text{ odd} \qquad \text{or} \qquad & \tilde{\tilde{w}}_i > n',\ n \text{ even} \\ \tilde{w}_i > 0 & s_i < n' & \end{array} \tag{2.53}$$

where the left hand side concerns $\tilde{w}_i$ and the right hand side concerns $\tilde{\tilde{w}}_i$. Recall $s_i \leq n' = \lfloor \frac{n}{2} \rfloor$ in (2.52). Using this, (2.52) follows immediately from the last display. (Recall (2.52) means $\tilde{T}$ and $\tilde{\tilde{T}}$ in (2.51) are splitting operators.) Also, (2.53) implies that the middle operator in (2.51) is a splitting operator as follows. In the case of $\tilde{T}$, it is the operator $M^{(r-i)}$ applied to the space $\mathcal{E}_{\mathfrak{T}'}(s_i, \ldots, s_r)_0[\tilde{w}_i + \tilde{s}^{i+1}]$ and the condition from Theorem 2.1.4 (i) is satisfied because

$$n + \left(\tilde{w}_i + \tilde{s}^{i+1}\right) - \tilde{s}^i - s_{i+1} + 2 - 1 = n + \tilde{w}_i - s_i - s_{i+1} + 1 > 0 \quad \text{for} \quad \tilde{w}_i > 0.$$

Similarly, in the case of $\tilde{\tilde{T}}$ the middle operator $M^{(r-i-1)}$ in (2.51) is applied to $\mathcal{E}_{\mathfrak{T}'}(s_i, \ldots, s_r)_0[\tilde{\tilde{w}}_i + \tilde{s}^{i+2}]$ and the condition from Theorem 2.1.4 (i) is also satisfied because

$$n + \left(\tilde{\tilde{w}}_i + \tilde{s}^{i+2}\right) - \tilde{s}^i - s_{i+2} + 3 - 1 = n + \tilde{\tilde{w}}_i - s_i - s_{i+1} - s_{i+2} + 2 > 0 \quad \text{for} \quad \tilde{\tilde{w}}_i > n'.$$

It remains to prove that $s(t, 0) > 0$ implies (2.53). The ith top operator is either $\tilde{T}$ or $\tilde{\tilde{T}}$ and we will consider both cases separately.

(a) Assume the ith top operator is of the form (2.51) with $\tilde{T}$. That is, the formula for $\tilde{T}_{A_i^0 \mathbf{A}_i}{}^{\mathbf{a}_i}$ is applied to the ith longest column of the Young diagram. The choice $\tilde{T}$ in even dimensions means that $s_{i+1} < \frac{n}{2}$, see (2.46). Therefore, we assume either $i \geq r_{n'}$ for n even or n is odd.

- If $s(t, 0) = w - s - t + s_t + 1$ then either $i \leq t < r_{n'}$ for n even or $t > r_{n'}$ according to (2.49). But the former case requires the ith top operator is $\tilde{\tilde{T}}$ (cf. (2.46)) so we can suppose $t > r_{n'}$. Since $\tilde{w}_i$ is the weight of $M^{(r-i)} T^{(i-1)} f$ we get

$$\tilde{w}_i = w - s - (i - 1) + s_i \geq w - s - t + s_t + 1 = s(t, 0) > 0$$

  using $i \leq t$ and $s_i \geq s_t$ in the first inequality and $s(t, 0) > 0$ in the second. To discuss the two stronger inequalities in (2.53), assume $i \leq r_{n'}$ (i.e. $s_i = n'$). This, together with $i \geq r_{n'}$ for n even above means $i = r_{n'}$ for n even. Using $i \leq t$ and $t > r_{n'}$ (see above), we obtain $i < t$ in both dimensions. Then both inequalities in the last display are sharp. Thus (2.53) follows as we suppose $w \in \mathbb{AW}$.

- If $s(t,0) = w - s - t + \frac{n}{2}$ then $i \le t \le r_{n'}$ i.e. $s_i = n'$. We get

$$\tilde{w}_i = w - s - (i-1) + s_i \ge w - s - t + n' + 1 = \begin{cases} s(t,0) + 1 > 1 & n \text{ even} \\ s(t,0) + \frac{1}{2} > \frac{1}{2} & n \text{ odd} \end{cases}$$

using $i \le t$ and $s_i = n'$ in the first inequality and $s(t,0) > 0$ in the second.

(b) Now assume the ith top operator is of the form (2.51) with $\tilde{\tilde{T}}$. That is, $i < r_{n'}$ i.e. $s_i = s_{i+1} = n'$ and n is even. According to (2.53), we need to show $\tilde{\tilde{w}}_i > n'$.

- If $s(t,0) = w - s - t + s_t + 1$ then $t \ne r_{n'}$. We get

$$\begin{aligned} \tilde{\tilde{w}}_i &= w - s - (i-1) + \overbrace{s_i + s_{i+1}}^{n} = n + w - s - i + 1 \ge \\ &\ge n + (w - s - t + s_t + 1) - s_t = n + s(t,0) - s_t \ge n' + s(t,0) > n' \end{aligned}$$

using $i \le t$ in the first equality, $s_t \le n'$ in the second and $s(t,0) > 0$ in the third.

- If $s(t,0) = w - s - t + \frac{n}{2}$ then $t = r_{n'}$. Using the same arguments as in the previous display, we get

$$\begin{aligned} \tilde{\tilde{w}}_i &= w - s - (i-1) + \overbrace{s_i + s_{i+1}}^{n} = n + w - s - i + 1 \ge \\ &\ge n' + (w - s - t + n') + 1 = n' + s(t,0) + 1 > n'. \end{aligned}$$

We have proved that if $s(t,0) > 0$ then $T^{(t)}$ is a splitting operator. If $s(t,0) = 0$, the choice $i := t$ yields $\tilde{w}_i = 0$ or $\tilde{\tilde{w}}_i = n$ because the inequalities in the displays above are satisfied as equalities. (Note the case corresponding to the previous display requires $i < t = r_{n'}$. Hence the choice $i := t$ excludes the last display.) Then $\tilde{T}$ or $\tilde{\tilde{T}}$ in (2.51) are not splitting operators, cf. the scalars in (2.52), thus $C = 0$ in Theorem (i). Summarizing, if $s(t,0) = 0$ then $T^{(t)}$ is not a splitting operator.

(ii) This is analogous to (i) above, in particular for $\mathfrak{T} = \emptyset$ this is exactly (i). The only point here is the (necessary) assumption $h(\widetilde{pr}) = hh(E_{\mathfrak{T}})$. (Cf. Theorem (2.16) (i) and (ii)). $\square$

Remark. 1. Let us note that without use of the top operator $\tilde{\tilde{T}}$, the stronger condition $w - s + \frac{n}{2} - 1 > 0$ would be necessary for all $1 \leq t \leq r_{n'}$ and both parities of dimensions. This will be important later.

2. We are interested mainly in admissible weights $w \in \mathbb{AW}$ because these cases admit operators from the pattern. But we have shown in the proof that if $w \notin \mathbb{AW}$ then $T^{(t)}$ is the top splitting operator. Similarly, if $w \in \mathbb{C} \setminus \mathbb{R}$ then $T^{(t)}$ is always the top splitting.

Examples of the top operator

Computation of explicit formulae for the top operator by hand is much more difficult than for the middle operator and is not, in general, manageable. (But note both these operators, given by tractor formulae, are in a form loadable into a computer.) We shall demonstrate these formulae on spaces $\mathcal{E}(k)[w]$ and $\mathcal{E}(k,l)_0[w]$, $n' \geq k \geq l \geq 1$. All of them can be obtained using the calculus developed in 1.2.5 and formulae for middle operators from Examples in 2.1.4. To simplify the notation, we will often omit the superscript indicating the valence of (tractor) form indices i.e. we shall write $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{A}$ instead of $\mathbf{a}^k$, $\mathbf{b}^l$ and $\mathbf{A}^k$.

Example 2.1.6. Let us start with k–forms i.e. with a section $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[w]$. Using the formula (2.16), page 90 for the middle operator $M^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}}$ on k–forms, one can compute that the top operator

$$T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} : \mathcal{E}_{\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{[A^0\mathbf{A}^k]}[w-k-1]$$
$$T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}} = D_{[A^0} M^{\mathbf{a}}_{\mathbf{A}]} f_{\mathbf{a}}$$

is given by the formula

$$\begin{aligned} T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}} =& (n+2(w-k)-2)\Big[w(n+w-2k)\mathbb{Y}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{a}} \\ &+ (n+w-2k)\mathbb{Z}^{a^0}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} \nabla_{a^0} f_{\mathbf{a}} + kw\mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}}_{\mathbf{A}} \nabla^{a^1} f_{\mathbf{a}} \Big] \\ &- \mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} \Big[(n+w-2k)(\triangle + (w-k)P) f_{\mathbf{a}} \\ &- k(n+2(w-k))\big(\nabla_{a^1}\nabla^p + (n+w-2k)P_{a^1}{}^p\big) f_{p\dot{\mathbf{a}}} \Big]. \end{aligned} \tag{2.54}$$

Let us look at which invariant operators can we extract directly from this formula. There are three possibilities for w which kill the top slot. Choices $w=0$ and $w=2k-n$ yield the exterior derivative and its formal adjoint in the $\mathbb{Z}$– and $\mathbb{W}$–slot, respectively. If $w=k+1-\frac{n}{2}$, the bottom slot will be invariant. So we have the 2nd order (long) operator

$$\mathcal{E}_{\mathbf{a}^k}[k+1-n/2] \longrightarrow \mathcal{E}_{\mathbf{a}^k}[k-1-n/2]$$
$$f_{\mathbf{a}} \mapsto (\frac{n}{2}-k+1)\Big[\Delta + (1-\frac{n}{2})P\Big] f_{\mathbf{a}} - 2k(k+1)\Big[\nabla_{a^1}\nabla^p + (\frac{n}{2}-k+1)P_{a^1}{}^p\Big] f_{p\dot{\mathbf{a}}}$$

where we skew over $[a^1\dot{\mathbf{a}}]$ on the right hand side.

Example 2.1.7. The top operator for $f_{\mathbf{ab}} = f_{\mathfrak{a}^k\mathfrak{b}^l} \in \mathcal{E}(k,l)_0[w]$ requires more computation. Recall we have two versions, see (2.25). In this Example, we shall consider the operator $\tilde{T}$. Using the formula for the complete middle operator $M^{\mathbf{a}}_{\mathbf{A}} M^{\mathbf{b}}_{\mathbf{B}} f_{\mathbf{ab}}$ from Example 2.1.3 one can compute that the top operator

$$\tilde{T}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} : \mathcal{E}(k,l)_0[w] \longrightarrow \mathcal{E}_{[A^0\mathbf{A}^k]\mathbf{b}^l}[w-k-1]$$
$$\tilde{T}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} = \mathbb{Z}^{\mathbf{C}}_{\mathbf{b}} D_{[A^0} M^{\mathbf{a}}_{\mathbf{A}]} M^{\mathbf{c}}_{\mathbf{C}} f_{\mathbf{ac}}$$

is given by the formula

$$
\begin{aligned}
\tilde{T}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} &= d(w-l)c_1c_2\mathbb{Y}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} \\
&+ dc_1\mathbb{Z}_{A^0}^{a_0}{}^{\mathbf{a}}_{\mathbf{A}}\left[c_2\nabla_{a^0}f_{\mathbf{ab}} - l\boldsymbol{g}_{a^0b^1}\nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}}\right] \\
&+ dk(w-l)\mathbb{W}_{A^0}{}^{\dot{\mathbf{a}}}_{\mathbf{A}}\left[c_2\nabla^{a^1}f_{\mathbf{ab}} - l\nabla^p f_{b^1\dot{\mathbf{a}}p\dot{\mathbf{b}}}\right] \\
&+ \mathbb{X}_{A_0}{}^{\mathbf{a}}_{\mathbf{A}}\Bigg[-c_1c_2\left[\Delta + (w-k-l)P\right]f_{\mathbf{ab}} \\
&\qquad + k(d+2)\left[c_2\left(\nabla_{a^1}\nabla^p + c_1P^p_{a^1}\right)f_{p\dot{\mathbf{a}}\mathbf{b}} - l\nabla_{a^1}\nabla^p f_{b^1\dot{\mathbf{a}}p\dot{\mathbf{b}}}\right] \\
&\qquad + 2lc_1\left[\nabla_{b^1}\nabla^p + c_2P^p_{b^1}\right]f_{\mathbf{a}p\dot{\mathbf{b}}} \\
&\qquad - kl(d+2)\boldsymbol{g}_{a^1b^1}\left[\nabla^p\nabla^p + c_2P^{pq}\right]f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}}\Bigg]
\end{aligned}
\tag{2.55}
$$

where we have used the scalars

$$c_1 = n+w-2k-l, \quad c_2 = n+w-k-2l+1 \quad \text{and} \quad d = n+2(w-k-l)-2$$

and we skew over $[b^1\dot{\mathbf{b}}]$ on the right hand side. Let us note that this formula simplifies a bit if $k = l$.

The formula (2.55) reveals directly 5 operators from the pattern for the weights w which kill the top slot. In the tensor formulae below, Proj denotes the projection to the corresponding target space. We obtain three first order operators

$$
\begin{gathered}
\mathcal{E}(k,l)_0[l] \longrightarrow \mathcal{E}(k+1,l)_0[l], \ f_{\mathbf{ab}} \mapsto \mathrm{Proj}\nabla_{[a^1}f_{\mathbf{a}]\mathbf{b}}, \ l \neq \frac{n}{2} \\
\mathcal{E}(k,l)_0[k+2l-n-1] \longrightarrow \mathcal{E}(k,l-1)_0[k+2l-n-3], \ f_{\mathbf{ab}} \mapsto \nabla^{b^1}f_{\mathbf{ab}} \\
\mathcal{E}(k,l)_0[2k+l-n] \longrightarrow \mathcal{E}(k-1,l)_0[2k+l-n-2], \ f_{\mathbf{ab}} \mapsto \mathrm{Proj}\nabla^{a^1}f_{\mathbf{ab}}, \ k > l.
\end{gathered}
$$

The first one appears in the $\mathbb{Z}$–slot (see (2.58) for the restriction $l \neq \frac{n}{2}$), the third one in the $\mathbb{W}$–slot and the second in both these slots. All these operators, together with

$$\mathcal{E}(k,k)_0[3k-n] \longrightarrow \mathcal{E}(k-1,l)_0[3k-n-4], \ f_{\mathbf{ab}} \mapsto (\nabla^{(a^1}\nabla^{b^1)} + P^{a^1b^1})f_{\mathbf{ab}}$$

which appears in the bottom slot, are short ones. (Cf. Example 2.1.3.) A more interesting choice is $d = 0$ i.e. $w = -\frac{n}{2} + k + l + 1$, $c_1 = \frac{n}{2} - k + 1$ and $c_2 = \frac{n}{2} - l + 2$ which reveals the long operator

$$\mathcal{E}(k,l)_0[k+l+1-\frac{n}{2}] \longrightarrow \mathcal{E}(k,l)_0[k+l-1-\frac{n}{2}],\ l \neq \frac{n}{2}$$

$$\begin{aligned} f_{\mathbf{ab}} \mapsto \mathrm{Proj}\Big\{ &-(k-1-\frac{n}{2})(l-2-\frac{n}{2})\Big[\Delta + (1-\frac{n}{2})\Big] f_{\mathbf{ab}} - 2kl\nabla_{a^1}\nabla^p f_{b^1\dot{\mathbf{a}}p\dot{\mathbf{b}}} \\ &-2k(l-2-\frac{n}{2})\Big[\nabla_{a^1}\nabla^p - (k-1-\frac{n}{2})P_{a^1}{}^p\Big] f_{p\dot{\mathbf{a}}\mathbf{b}} \\ &-2l(k-1-\frac{n}{2})\Big[\nabla_{b^1}\nabla^p - (l-2-\frac{n}{2})P_{b^1}{}^p\Big] f_{\mathbf{a}p\dot{\mathbf{b}}}\Big\} \end{aligned}$$

where we skew over $[a^1\dot{\mathbf{a}}]$ and $[b^1\dot{\mathbf{b}}]$ on the right hand side before the projection Proj to $\mathcal{E}(k,l)_0[w-2]$. (See (2.58) for the restriction $l \neq \frac{n}{2}$.) Let us note we can get rid of the term $\nabla_{a^1}\nabla^p f_{b^1\dot{\mathbf{a}}p\dot{\mathbf{b}}}$ with (skew) form indices $\mathbf{a}$ and $\mathbf{b}$. Clearly

$$\begin{aligned} (k+1)\nabla_{[a^1}\nabla^p f_{b^1\dot{\mathbf{a}}]p\dot{\mathbf{b}}} &= \nabla_{b^1}\nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}} - k\nabla_{[a^1}\nabla^p f_{|b^1|\dot{\mathbf{a}}]p\dot{\mathbf{b}}} \\ &\in \mathcal{E}(k+1,l-1)[w-2]. \end{aligned} \tag{2.56}$$

Now skewing over $[b^1\dot{\mathbf{b}}]$ we obtain a term still living in $\mathcal{E}(k+1,l-1)[w-2]$ so we can subtract (an appropriate multiple of) this term from the formula for the long operator above.

Example 2.1.8. The case $\mathcal{E}(n',n')_0[w]$ in the dimension $n = 2n'$ is more involved. We will demonstrate why the top operator $\tilde{T}$ from Example 2.1.7 is not sufficient in this case. Let us consider the version $\tilde{\tilde{T}}$ i.e.

$$\tilde{\tilde{T}}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} : \mathcal{E}(n',n')_0[w] \longrightarrow \mathcal{E}_{[A^0\mathbf{A}^{n'}]\mathbf{b}^{n'}}[w-n'-1]$$

$$\tilde{\tilde{T}}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} = \mathbb{Y}^{C^0}{}^{\mathbf{C}}_{\mathbf{b}} D_{[A^0} M^{\mathbf{a}}_{\mathbf{A}]} \mathbb{X}_{C^0}{}^{\mathbf{c}}_{\mathbf{C}} f_{\mathbf{ac}} = \mathbb{Y}^{C^0}{}^{\mathbf{C}}_{\mathbf{b}} T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} \mathbb{X}_{C^0}{}^{\mathbf{c}}_{\mathbf{C}} f_{\mathbf{ac}}$$

first. Recall all form indices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{A}$ are of the valence $n' = \frac{n}{2}$ now. After some computation and using the formula for $T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}$ from Example 2.1.6

one gets the result

$$
\begin{aligned}
\tilde{\tilde{T}}_{A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} = (w-n'+1)\Big\{ 2(w-n')\Big[& (w-n'+1)\mathbb{Y}_{A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} \\
& + \mathbb{Z}^{a^0}_{A^0\mathbf{A}}{}^{\mathbf{a}} \nabla_{a^0} f_{\mathbf{ab}} + n'\mathbb{W}_{A^0\mathbf{A}}{}^{\dot{\mathbf{a}}} \nabla^{a^1} f_{\mathbf{ab}} \Big] \\
- \mathbb{X}_{A_0\mathbf{A}}{}^{\mathbf{a}} \Big[& (\triangle + (w-n)P) f_{\mathbf{ab}} \\
& - n\big(\nabla_{a^1}\nabla^p + (w-n'+1)P^p_{a^1}\big) f_{p\dot{\mathbf{a}}\mathbf{b}} \Big] \Big\}.
\end{aligned} \tag{2.57}
$$

Note removing the outer scalar $w - n' + 1$ clearly does not affect the invariance or the set of weights w for which is $\tilde{\tilde{T}}$ the top splitting. Thus we can *define* $\tilde{\tilde{T}}$ by the part of the formula (2.57) embraced by $\{\}$. This can simplify a bit any further computation.

Now let us compare $\tilde{\tilde{T}}$ with $\tilde{T}$ from Example 2.1.7 where $k = l = n'$ and $n = 2n'$. One can compute

$$
\begin{aligned}
\tilde{T}_{A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} = & \frac{(w-n')(w-n'-1)}{w-n'+1} \tilde{\tilde{T}}_{A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} \\
& + n'(w-n')\mathbb{X}_{A_0\mathbf{A}}{}^{\mathbf{a}} \left[(n'-1) C_{a^1a^2}{}^{pq} f_{pq\ddot{\mathbf{a}}\mathbf{b}} + n C_{a^1}{}^{p}{}_{b^1}{}^{q} f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}} \right]
\end{aligned} \tag{2.58}
$$

where we skew over the indices $[b^1\dot{\mathbf{b}}]$ on the right hand side. This computation is actually quite tedious. It is based on the following fact: Consider a tensor $F_{a^0\mathbf{a}b^0\mathbf{a}} \in \mathcal{E}_{[a^0\mathbf{a}][b^0\mathbf{a}]}[w]$ and its trace $\tilde{F} = \boldsymbol{g}^{a^0b^0} F_{a^0\mathbf{a}b^0\mathbf{a}}$. Using (2.27), a moment of thinking reveals that the trace–free part of $\tilde{F}$ vanishes. That is, $\tilde{F}$ is a pure trace. Applying this fact to $F = \nabla_{a^0}\nabla_{b^0} f_{\mathbf{ab}}$ and $F = P_{a^0b^0} f_{\mathbf{ab}}$, one can compute

$$
\begin{aligned}
\Delta f_{\mathbf{ab}} &= n' \left(\nabla^p \nabla_{a^1} f_{p\dot{\mathbf{a}}\mathbf{b}} + \nabla_{b^1}\nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}} - n'\boldsymbol{g}_{a^1b^1}\nabla^p\nabla^q f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}} \right) \\
P f_{\mathbf{ab}} &= n' \left(P^p_{a^1} f_{p\dot{\mathbf{a}}\mathbf{b}} + P^p_{b^1} f_{\mathbf{a}p\dot{\mathbf{b}}} - n'\boldsymbol{g}_{a^1b^1} P^{pq} f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}} \right)
\end{aligned} \tag{2.59}
$$

where we skew over the indices $[a^1\dot{\mathbf{a}}]$ and $[b^1\dot{\mathbf{b}}]$. Using these, it is a matter of a direct computation to establish the relation (2.58) between $\tilde{T}$ and $\tilde{\tilde{T}}$.

Now we can easily see the flaw of the operator $\tilde{T}$ – this is not a splitting operator for $w = n' + 1$ whereas $\tilde{\tilde{T}}$ is. This is important because a splitting operator $\mathcal{E}(n',n')_0[n'+1] \longrightarrow \mathcal{E}_{\mathbf{a}^{n'}[A^0\mathbf{A}^{n'}]}[0]$ with the projecting part in the top slot is just the gBGG splitting operator which we need.

As before, the formula (2.57) provides some operators from the pattern. The projection to the $\mathbb{Z}$– or $\mathbb{W}$–slot for $w = n'-1$ yields the operator $f_{\mathbf{ab}} \mapsto \nabla^{a^1} f_{\mathbf{ab}}$. For $w = n'$, the projection to the $\mathbb{X}$–slot is invariant. However, this correspond to a middle position in the (even dimensional) pattern (i.e. n'_Y or n'_X) hence there is supposed to be no long operator. Indeed, skewing over $[b^1\dot{\mathbf{b}}]$ in (2.56) and using the property (1.7), we obtain

$$\nabla_{a^1}\nabla^p f_{p\dot{\mathbf{a}}\mathbf{b}} - \nabla_{b^1}\nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}} \in \mathcal{E}(n'+1,n'-1)[w-2]$$

in the $\mathbb{X}$–slot, where we skew over $[a^1\dot{\mathbf{a}}]$ and $[b^1\dot{\mathbf{b}}]$ on the left hand side. Using this and (2.59), an easy computation reveals the invariant operator in the bottom slot of (2.57) for $w = n'$, projected to $\mathcal{E}(n',n')_0[n'-2]$, yields only curvature terms. Note the projection $\mathcal{E}(n'+1,n'-1)_0[n'-2] \cong \mathcal{E}(n'-1,n'-1)_0[n'-4]$ yields the short operator $f_{\mathbf{ab}} \mapsto (\nabla^{a^1}\nabla^{b^1} + P^{a^1b^1})f_{\mathbf{ab}}$.

Remark. 1. Putting $k = l = 0$ in Examples 2.1.6 and 2.1.7, we recover the formula for the tractor D–operator up to a scalar multiples $n+w$ and $(n+w)(n+w+1)$, respectively. For $n' \geq k \geq l \geq 1$ and any dimension n, the top operator on $\mathcal{E}(k)[w]$ and $\mathcal{E}(k,l)_0[w]$ can be viewed as a generalisation of the tractor D–operator to these spaces.

2. Using (2.58), a moment of thinking reveals we can obtain a curved modification of the formula for $\tilde{\tilde{T}}$ from $\tilde{T}$ by a “weight continuation” argument. The geometric construction of $\tilde{\tilde{T}}$ avoids this reasoning.

Example 2.1.9. In the last example, we shall look at the top operator on the space of (density–valued) trace–free symmetric r–tensors $\mathcal{E}_{(a_1\cdots a_r)_0}[w]$. We compute the formula for $T = T^{(1)}$ which puts just one tensor index to the top slot of $E_{[A^0A^1]}[w-2]$. We can suppose $r \geq 3$ because the cases $r = 1$

and $r = 2$ are covered by Examples 2.1.6 and 2.1.7. Following the general construction of the top operator (2.25), our top operator is

$$T_{A^0}{}^{a^1}_{A^1} : \mathcal{E}_{(a^1a_2\cdots a_r)_0}[w] \longrightarrow \mathcal{E}_{[A^0A^1](a_2\cdots a_r)_0}[w-2]$$
$$T_{A^0}{}^{a^1}_{A^1} f_{a^1a_2\cdots a_r} = \mathbb{Z}^{B_2}_{a_2} \cdots \mathbb{Z}^{B_r}_{a_r} D_{[A^0} M^{a^1}_{A^1]} M^{b_1\cdots b_r}_{B_2\cdots B_r} f_{a^1b_2\cdots b_r}$$

for $f_{a^1a_2\cdots a_r} \in \mathcal{E}_{(a^1a_2\cdots a_r)_0}[w]$. Since the operator $D_{[A^0} M^{a^1}_{A^1]}$ given by (2.54) is of the second order we actually need only the slots of $M^{b_1\cdots b_r}_{B_2\cdots B_r} f_{a^1b_2\cdots b_r}$ of the homogenity 0, −1 and −2. These are computed in Example 2.1.5. Now a tedious (but manageable) computation reveals

$$\begin{aligned}
T_{A^0}{}^{a^1}_{A^1} f_{a^1a_2\cdots a_r} = \tilde{C}\Big\{ &(c-1)\big(n+2(w-r)-2\big)\Big[c(w-r+1)\mathbb{Y}_{A^0}{}^{a^1}_{A^1} f_{a^1a_2\cdots a_r} \\
&\quad + \mathbb{Z}^{a^0\,a^1}_{A^0A^1}\big[c\nabla_{a^0} f_{a^1a_2\cdots a_r} - (r-1)\boldsymbol{g}_{a^0a_2}\nabla^p f_{a^1pa_3\cdots a_r}\big] \\
&\quad + (w-r+1)\mathbb{W}_{A^0A^1}\nabla^p f_{pa_2\cdots a_r}\Big] \\
- \mathbb{X}_{A^0}{}^{a^1}_{A^1}\Big[&c(c-1)\big(\Delta + (w-r)P\big) f_{a^1a_2\cdots a_r} \\
&- 2(c-1)(r-1)\big[\nabla_{a_2}\nabla^p + cP_{a_2}{}^p\big] f_{a^1pa_3\cdots a_r} \\
&- \big(n+2(w-r)\big)(c-1)\big[\nabla_{a^1}\nabla^p + cP_{a^1}{}^p\big] f_{pa_2\cdots a_r} \\
&+ \big(n+2(w-r)\big)(r-1)\boldsymbol{g}_{a^1a_2}\big[\nabla^p\nabla^q + cP^{pq}\big] f_{pqa_3\cdots a_r} \\
&+ (r-1)(r-2)\boldsymbol{g}_{a^2a^3}\big[\nabla^p\nabla^q + cP^{pq}\big] f_{a^1pqa_4\cdots a_r}\Big]\Big\}
\end{aligned}$$

where the indices $a_2\cdots a_r$ are symmetrized and we use the scalars $c = n + w - 2$ and $\tilde{C} = \prod_{i=3}^{r}(c-i+1)$. A short computation shows the right hand side is trace free on the tensor indices $a_2\cdots a_r$.

The last display reveals several operators from the pattern. Beside short operators, the weight w satisfying $n+2(w-r)-2=0$ yields the long operator

$$
\begin{aligned}
\mathcal{E}_{(a_1\cdots a_r)_0}[r-\frac{n}{2}+1] &\longrightarrow \mathcal{E}_{(a_1\cdots a_r)_0}[r-\frac{n}{2}-1] \\
f_{a_1a_2\cdots a_r} \mapsto \mathrm{Proj}\Big\{ &\big(r+\frac{n}{2}-1\big)\big[\Delta+(1-\frac{n}{2})P\big]f_{a_1a_2\cdots a_r} \\
&- 2r\big[\nabla_{a_1}\nabla^p+(r+\frac{n}{2}-1)P_{a_1}{}^p\big]f_{pa_2\cdots a_r}\Big\}
\end{aligned}
$$

where Proj denotes projection the target space. Let us note we can use this formula for any $r\geq 1$.

2.1.6. Generalisation to spinor representations. We shall work with spinor bundles and their sections of the form $f=f_{\mathfrak{T}\lambda\mathbf{a}_1\cdots\mathbf{a}_r}=f_{\mathfrak{T}\mathbf{a}_1\cdots\mathbf{a}_r}\in\mathcal{E}_{\mathfrak{T}}(\frac{1}{2};s_1,\ldots,s_r)_0[w]$ now. Recall that $r,s\notin\mathbb{N}$ now and we implicitly consider $\lfloor r\rfloor$ in expressions like s_r. We will use the "X,Y"–calculus for the spinor tractor bundle developed in 1.2.4. We often suppress spinor and tractor spinor indices. The bottom operator B^λ_Λ is defined in 2.1.1.

Middle operator for spinor bundles

As in the tensor case (2.16), we define the middle operator for spinors directly by the formula

$$
\begin{aligned}
M^{\mathbf{a}_r}_{\mathbf{A}_r}:\mathcal{E}_{\mathfrak{T}}(\frac{1}{2};s_1,\ldots,s_r)_0[w] &\longrightarrow \mathcal{E}_{\mathfrak{T}\mathbf{A}_r}(\frac{1}{2};s_1,\ldots,s_r-1)_0[w-s_r] \\
M^{\mathbf{a}_r}_{\mathbf{A}_r}f_{\mathfrak{T}\mathbf{a}_1\cdots\mathbf{a}_r} &= \Big((n+w-s-s_r+r-1)\,\mathbb{Z}^{\mathbf{a}_r}_{\mathbf{A}_r}-s_r\mathbb{X}^{\dot{\mathbf{a}}_r}_{\mathbf{A}_r}\nabla^{a^1_r}\Big)f_{\mathfrak{T}\mathbf{a}_1\cdots\mathbf{a}_r}.
\end{aligned}
\tag{2.60}
$$

Lemma. *The operator $M^{\mathbf{a}_r}_{\mathbf{A}_r}$ is conformally invariant.*

Proof. The middle operator (2.16) for tensors is strongly invariant hence the composition $M^{\mathbf{a}_r}_{\mathbf{A}_r}B^\lambda_\Lambda$ is invariant. This is given by the formula

$$
\Big[(n+w-s-s_r+r-1)\mathbb{Z}^{\mathbf{a}_r}_{\mathbf{A}_r}-s_r\mathbb{X}^{\dot{\mathbf{a}}_r}_{\mathbf{A}_r}\nabla^{a^1_r}\Big]Xf_{\mathfrak{T}\mathbf{a}_1\cdots\mathbf{a}_r}. \tag{2.61}
$$

This agrees with the tensor middle operator (2.16) because the (spinor) bottom operator B does not change the weight and $-\lfloor s\rfloor+\lfloor r\rfloor=-s+r$. It remains to show that $\nabla^{a^1_r}$ commutes with X in (2.61). They commute up the term

$$-s_r \mathbb{X}^{\dot{\mathbf{a}}_r}_{\mathbf{A}_r} (\nabla^{a^1_r} X) f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} = -s_r \mathbb{X}^{\dot{\mathbf{a}}_r}_{\mathbf{A}_r} Y \beta^{a^1_r} f_{\mathfrak{T}\mathbf{a}_1 \cdots \mathbf{a}_r} = 0,$$

see (1.38). The last display vanishes because f is "Clifford free", see Table 1.3. $\square$

Properties of the middle operator for spinors

We will continue in a similar way as in 2.1.4. The composition of $m \in \mathbb{N}$, $m \leq r$ middle operators is the operator

$$\begin{gathered} M^{(m)}{}^{\mathbf{a}_{\bar{m}} \cdots \mathbf{a}_r}_{\mathbf{A}_{\bar{m}} \cdots \mathbf{A}_r} := M^{\mathbf{a}_{\bar{m}}}_{\mathbf{A}_{\bar{m}}} \cdots M^{\mathbf{a}_r}_{\mathbf{A}_r} : \\ \mathcal{E}_{\mathfrak{T}}(\frac{1}{2}; s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}\mathbf{A}_{\bar{m}} \cdots \mathbf{A}_r}(\frac{1}{2}; s_1, \ldots, s_{\bar{m}-1})_0[w - \tilde{s}^{\bar{m}}] \end{gathered} \tag{2.62}$$

where $\bar{m} = \lfloor r \rfloor - m + 1$. This has analogous properties as in the tensor case. They are summarised in the proposition below. Finally the quantity oh from 1.2.6 is given, similarly as (2.18), by

$$oh(M^{(m)}) = oh(\underbrace{M \cdots M}_{m}) = 0. \tag{2.63}$$

Definition. The middle operator $M^{(m)}$, defined by (2.62), will be called *middle splitting* or *middle splitting operator* if this is a splitting operator for $\mathfrak{T} = \emptyset$. (Then $M^{(m)}$ is a splitting operator for any $\mathfrak{T}$.)

Top operator for spinors

Also construction of T^λ_Λ follows the tensor case in 2.1.5. If $r = \frac{1}{2}$, we will put $T^\lambda_\Lambda := D^\lambda_\Lambda$. Henceforth suppose $r \geq \frac{3}{2}$. As in 2.1.5, we define two possibilities

$$
\begin{aligned}
\tilde{T}^\lambda_\Lambda, \tilde{\tilde{T}}^\lambda_\Lambda &: \mathcal{E}_{\mathfrak{T}}(\frac{1}{2}; s_1, \dots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{T}\Lambda}(s_1, \dots, s_r)_0[w-1] \\
\tilde{T} f_{\mathfrak{T}\mathbf{a}_1\cdots\mathbf{a}_r} &= \mathrm{Proj} \circ \mathbb{Z}_{\mathbf{a}_r}^{\mathbf{B}_r} \cdots \mathbb{Z}_{\mathbf{a}_1}^{\mathbf{B}_1} D\, M_{\mathbf{B}_1}^{\mathbf{b}_1} \cdots M_{\mathbf{B}_r}^{\mathbf{b}_r} f_{\mathfrak{T}\mathbf{b}_1\cdots\mathbf{b}_r} \\
\tilde{\tilde{T}} f_{\mathfrak{T}\mathbf{a}_1\cdots\mathbf{a}_r} &= \mathrm{Proj} \circ \mathbb{Z}_{\mathbf{a}_r}^{\mathbf{B}_r} \cdots \mathbb{Z}_{\mathbf{a}_2}^{\mathbf{B}_2} \mathbb{Y}^{B_1^0\mathbf{B}_1}_{\quad\mathbf{a}_1} D\, \mathbb{X}_{B_1^0\mathbf{B}_1}^{\quad\mathbf{b}_1} M_{\mathbf{B}_2}^{\mathbf{b}_2} \cdots M_{\mathbf{B}_r}^{\mathbf{b}_r} f_{\mathfrak{T}\mathbf{b}_1\cdots\mathbf{b}_r}
\end{aligned}
\tag{2.64}
$$

where Proj denotes the projection (on tensor indices) to the target space of $\tilde{T}$ and $\tilde{\tilde{T}}$.

To prove invariance of $\tilde{\tilde{T}}$, we need

$$
\begin{gathered}
\chi : \mathcal{E}(\tfrac{1}{2}; k)_0 \longrightarrow \mathcal{E}(\tfrac{1}{2}; k+1)[1], \quad (\chi \tilde{f})_{\mathbf{a}^{k+1}} = \boldsymbol{\beta}_{[a^{k+1}} \tilde{f}_{\mathbf{a}^k]} \\
\text{vanishes for } 2k = n,
\end{gathered}
\tag{2.65}
$$

which is an analogue of (2.27). Assume the complex setting and n even. Let us consider $f_{\mathbf{a}^{n'}} \in \mathcal{E}(\frac{1}{2}; n')_0$ such that $\boldsymbol{\epsilon}_{\mathbf{a}^{n'}}{}^{\mathbf{c}^{n'}} f_{\mathbf{c}^{n'}} = c f_{\mathbf{a}^{n'}}$ for $c \in \mathbb{C}$. Then

$$
\boldsymbol{\epsilon}_{\mathbf{a}^{n'-1}}{}^{c^0\mathbf{c}^{n'}} \boldsymbol{\beta}_{c^0} f_{\mathbf{c}^{n'}} = \boldsymbol{\beta}^{c^0} \boldsymbol{\epsilon}_{\mathbf{a}^{n'-1}c^0}{}^{\mathbf{c}^{n'}} f_{\mathbf{c}^{n'}} = c\boldsymbol{\beta}^{c^0} f_{\mathbf{a}^{n'}c^0} = 0.
$$

because f is "Clifford free". Since $c \neq 0$ and $\boldsymbol{\epsilon}_{\mathbf{a}^{n'-1}}{}^{c^0\mathbf{c}^{n'}}$ induces an isomorphism $\mathcal{E}(\frac{1}{2}, n'+1)[1] \longrightarrow \mathcal{E}(\frac{1}{2}, n'-1)[-1]$, (2.65) follows. The real case follows from the complexification.

Lemma. *(i) The operator $\tilde{T}$ is conformally invariant.*
(ii) The operator $\tilde{\tilde{T}}$ is conformally invariant if $s_1 = n' = \frac{n}{2}$ and n is even.

Proof. We shall follow the proof of Lemma 2.1.5 but since the tractor D–operator for spinors is only of the first order, everything will be easier. Again, it is sufficient to assume $\mathfrak{T} = \emptyset$.

(i) Using the notation $\mathfrak{B} := \mathbf{B}_1 \cdots \mathbf{B}_r$, $\mathfrak{b} := \mathbf{b}_1 \cdots \mathbf{b}_r$ and $\mathfrak{a} := \mathbf{a}_1 \cdots \mathbf{a}_r$ for systems of indices, a part of the formula (2.64) for $\tilde{T}^\lambda_\Lambda$ is the invariant operator

$$
\Phi^{\lambda\ \mathfrak{b}}_{\Lambda\mathfrak{B}} = D^\lambda_\Lambda \underbrace{M_{\mathbf{B}_1}^{\mathbf{b}_1} \cdots M_{\mathbf{B}_r}^{\mathbf{b}_r}}_{\Phi'} : \mathcal{E}(\frac{1}{2}; s_1, \dots, s_r)_0[w] \longrightarrow \mathcal{E}_{\Lambda\mathfrak{B}}[w - s - \frac{1}{2}]. \tag{2.66}
$$

(Note $-s-\frac{1}{2}=-\lfloor s\rfloor-1$.) We will consider $\Phi_{\Lambda\mathfrak{B}}$ as a tractor formula (recall M and D are given by formulae) and use the notation from 1.2.6 for $\Phi_{\Lambda\mathfrak{B}}$. We need to show the projection $\mathbb{Z}_{\mathbf{a}_r}^{\mathbf{B}_r}\cdots\mathbb{Z}_{\mathbf{a}_1}^{\mathbf{B}_1}\Phi_{\Lambda\mathbf{B}_1\cdots\mathbf{B}_r}$ is invariant. Following the same arguments as in Lemma 2.1.5, it is sufficient to show

$$\forall pr_{\Lambda\mathfrak{B}}\in TFPC(E_{\Lambda\mathfrak{B}}):\ h_{\mathfrak{B}}(pr_{\Lambda\mathfrak{B}})>0\Longrightarrow(pr_{\Lambda\mathfrak{B}})^*\Phi=0. \tag{2.67}$$

Clearly $oh(\Phi)=oh(D_\Lambda^\lambda)+oh(M^r)=\frac{1}{2}$, see Example 1.2.6 and (2.63), and $h_\Lambda(pr)\in\{\pm\frac{1}{2}\}$. Also note $h_{\mathfrak{B}}(pr)\in\mathbb{Z}$ for every $pr\in TFPC(E_{\Lambda\mathfrak{B}})$. From this, it follows easily that

$$\forall pr\in TFPC(E_{\Lambda\mathfrak{B}}):\big(h_\Lambda(pr),h_{\mathfrak{B}}(pr)\big)=(-\frac{1}{2},1)\Longrightarrow pr^*\Phi=0 \tag{2.68}$$

implies (2.67) hence proves the Theorem (i). (Let us note the last display plays the same role as (2.35) but now, the "lattice" is trivial and equal to $\{(-\frac{1}{2},1)\}$.)

To prove (2.68), suppose $pr\in TFPC(E_{\Lambda\mathfrak{B}})$ satisfies $\big(h_\Lambda(pr),h_{\mathfrak{B}}(pr)\big)=(-\frac{1}{2},1)$. The operator $pr^*\Phi$ is of the zero formal order and invariant but we cannot use a modification of Lemma 1.3.8 because irreducibility of the space $\mathcal{E}(\frac{1}{2};s_1,\ldots,s_r)_0$ is not guaranteed. We will use an analogue of the reasoning from the proof of Lemma 2.1.5 (ii) i.e. we will describe how a term with pr can appear in the formula Φ.

Let us consider the middle operator $\Phi'^{\mathfrak{b}}_{\mathfrak{B}}:=(M^{(\lfloor r\rfloor)})^{\mathfrak{b}}_{\mathfrak{B}}$ i.e. $\Phi^{\lambda}_{\Lambda\mathfrak{B}}{}^{\mathfrak{b}}=D^\lambda_\Lambda\Phi'^{\mathfrak{b}}_{\mathfrak{B}}$, see (2.66). Since D is of the first order, it can increase the $\mathfrak{B}$–homogenity of TFP–components in $E_{\mathfrak{B}}$ in the formula $\Phi'^{\mathfrak{b}}_{\mathfrak{B}}$ by at most one. (Recall these TFP–components are formally terms of $\Phi'^{\mathfrak{b}}_{\mathfrak{B}}$.) According to Theorem 2.1.4 (i), a nonvanishing TFP–projecting part of $\Phi'^{\mathfrak{b}}_{\mathfrak{B}}$ of the $\mathfrak{B}$–homogenity at least 0 can be only $pr(\lfloor r\rfloor)^{\mathfrak{b}}_{\mathfrak{B}}:=\mathbb{Z}^{\mathbf{b}_1}_{\mathbf{B}_1}\cdots\mathbb{Z}^{\mathbf{b}_r}_{\mathbf{B}_r}\in TFPC(E_{\mathfrak{B}})$. Since its $\mathfrak{B}$–homogenity is 0, the only way how to obtain a projecting part $pr\in TFPC(E_{\Lambda\mathfrak{B}})$ such that the formula $pr^*\Phi$ is nontrivial and $h_{\mathfrak{B}}(pr)=1$, is to apply the first order term $X\boldsymbol{\beta}^p\nabla_p$ of D to $pr(\lfloor r\rfloor)$. But the resulting operator vanishes because

$$\boldsymbol{\beta}^p\nabla_p\mathbb{Z}_{\mathbf{B}_1}^{\mathbf{b}_1}\cdots\mathbb{Z}_{\mathbf{B}_r}^{\mathbf{b}_r}f_{\mathbf{b}_1\cdots\mathbf{b}_r}=-\sum_{i=1}^{\lfloor r\rfloor}(s_i+1)\mathbb{Z}\cdots\mathbb{Z}\mathbb{Y}_{\mathbf{B}_i}^{\dot{\mathbf{b}}_i}\mathbb{Z}\cdots\mathbb{Z}\boldsymbol{\beta}^p f_{\cdots[p\dot{\mathbf{b}}_i]\cdots}$$
$$+\text{“lht”}=\text{“lht”}$$

where “lht” denotes terms (i.e. TFP–components) of the homogenity at most 0, and, recall, f is “Clifford free”. The last display follows from the formula (1.49) for $\mathbb{Z}$ and the Leibnitz rule. (In the ith summand, the form indices with the exception of $\mathbf{B}_i$ and $\dot{\mathbf{b}}_i$ are omitted).

(ii) Assume $r_{n'}\geq\frac{3}{2}$ and n even. The proof is similar to (i) and we emphasise only what is different. The systems of indices are $\mathfrak{B}:=[B_1^0\mathbf{B}_1]\mathbf{B}_2\cdots\mathbf{B}_r$, $\mathfrak{b}:=\mathbf{b}_1\cdots\mathbf{b}_r$ and $\mathfrak{a}:=\mathbf{a}_1\cdots\mathbf{a}_r$ and the analogue of (2.66) is the tractor formula

$$\Phi_{\Lambda\mathfrak{B}}^{\lambda\;\mathfrak{b}}=D_\Lambda^\lambda\underbrace{\mathbb{X}_{B_1^0\mathbf{B}_1}^{\;\;\mathbf{b}_1}M_{\mathbf{B}_2}^{\mathbf{b}_2}\cdots M_{\mathbf{B}_r}^{\mathbf{b}_r}}_{\Phi'}:\mathcal{E}(\frac{1}{2};s_1,\dots,s_r)_0[w]\longrightarrow\mathcal{E}_{\Lambda\mathfrak{B}}[w-s+\frac{1}{2}].$$

Using the notation $(pr_1(\lfloor r-1\rfloor))_{\mathfrak{B}}^{\mathfrak{a}}:=\mathbb{X}_{B_1^0\mathbf{B}_1}^{\;\;\mathbf{a}_1}\mathbb{Z}_{\mathbf{B}_2}^{\mathbf{a}_2}\cdots\mathbb{Z}_{\mathbf{B}_r}^{\mathbf{a}_r}\in TFPC(E_{\mathfrak{B}})$, we need to show $\big(pr_1(\lfloor r-1\rfloor)\big)^*\Phi$ is invariant. Similarly as in (i), it is sufficient to show

$$\forall pr\in TFPC(E_{\Lambda\mathfrak{B}}):\big(h_\Lambda(pr),h_{\mathfrak{B}}(pr)\big)=(-\frac{1}{2},0)\Longrightarrow pr^*\Phi=0.$$

Since $\Phi'=\mathbb{X}M^{\lfloor r-1\rfloor}$, a TFP–projecting part of the $\mathfrak{B}$–homogenity at least -1 which is nonzero for the formula Φ' can be only $pr_1(\lfloor r-1\rfloor)\in TFPC(E_{\mathfrak{B}})$. Recall $pr_1(\lfloor r-1\rfloor)_{\mathfrak{B}}^{\mathfrak{b}}$ is also a term in the formula $\Phi'^{\mathfrak{b}}_{\mathfrak{B}}$. Applying

$\boldsymbol{\beta}^p\nabla_p$ to $pr_1(\lfloor r-1\rfloor)f$, we obtain the operator

$$\begin{aligned}
&\boldsymbol{\beta}^p\nabla_p \mathbb{X}_{B_1^0\mathbf{B}_1}{}^{\mathbf{b}_1}\mathbb{Z}_{\mathbf{B}_2}^{\mathbf{b}_2}\cdots\mathbb{Z}_{\mathbf{B}_r}^{\mathbf{b}_r} f_{\mathbf{b}_1\cdots\mathbf{b}_r} = \\
&\quad = \mathbb{Z}_{B_1^0\mathbf{B}_1}{}^{b_1^0\,\mathbf{b}_1}\mathbb{Z}_{\mathbf{B}_2}^{\mathbf{b}_2}\cdots\mathbb{Z}_{\mathbf{B}_r}^{\mathbf{b}_r}\boldsymbol{\beta}_{b_1^0} f_{\mathbf{b}_1\cdots\mathbf{b}_r} - n'\mathbb{W}_{B_1^0\mathbf{B}_1}{}^{\dot{\mathbf{b}}_1}\mathbb{Z}_{\mathbf{B}_2}^{\mathbf{b}_2}\cdots\mathbb{Z}_{\mathbf{B}_r}^{\mathbf{b}_r}\boldsymbol{\beta}^p f_{p\dot{\mathbf{b}}_1\mathbf{b}_2\cdots\mathbf{b}_r} \\
&\quad\quad - \sum_{i=2}^{\lfloor r\rfloor}(s_i+1)\mathbb{X}\mathbb{Z}\cdots\mathbb{Z}\mathbb{Y}_{\mathbf{B}_i}^{\dot{\mathbf{b}}_i}\mathbb{Z}\cdots\mathbb{Z}\boldsymbol{\beta}^p f_{\cdots[p\dot{\mathbf{b}}_i]\cdots} + \text{“lht”} = \text{“lht”}.
\end{aligned}$$

where “lht” denotes terms of the homogenity at most -1 . The first term on the right hand side vanishes due to (2.65). ((2.65) obviously holds also for tractor valued k–forms.) The remaining terms (with the exception of “lht's”) vanish due to $\boldsymbol{\beta}^p f_{\dots p\dots} = 0$. Thus (ii) follows. □

Properties of the top operator for spinors

Following (2.46), we define *top operator* T as

$$T = \begin{cases} \tilde{T} & s_1 < \frac{n}{2} \\ \tilde{\tilde{T}} & s_1 = \frac{n}{2}. \end{cases} \tag{2.69}$$

In particular, $T = \tilde{T}$ for n odd. Further, let us consider composition of the top operator T_Λ^λ and $\lfloor t\rfloor \in \mathbb{N}$ tensor top operators where $\frac{1}{2} \le t = \lfloor t\rfloor + \frac{1}{2} \le r$. This will be denoted by

$$\begin{aligned}
&\big(T^{(t)}\big)_{A_t^0\mathbf{A}_t\cdots A_1^0\mathbf{A}_1\Lambda}{}^{\mathbf{a}_t\cdots\quad\mathbf{a}_1\,\lambda} = T_{A_t^0\mathbf{A}_t}{}^{\mathbf{a}_t}\cdots T_{A_1^0\mathbf{A}_1}{}^{\mathbf{a}_1}T_\Lambda^\lambda \\
&\mathcal{E}(\frac{1}{2};s_1,\dots,s_r)_0 \longrightarrow \mathcal{E}_{\Lambda\mathfrak{T}[A_1^0\mathbf{A}_1]\cdots[A_t^0\mathbf{A}_t]}(s_{t+1},\dots,s_r)_0[w-s^t-t].
\end{aligned} \tag{2.70}$$

Recall we use the conventions from 1.1.3 i.e. we consider implicitly the integer part $\lfloor t\rfloor$ of t in expressions with non-integer subscript like $\mathbf{a}_t$ or s_r but $s^t = \frac{1}{2} + s^{\lfloor t\rfloor} \notin \mathbb{Z}$ now. The following Theorem says when this is a splitting operator. Let us note the formula for every T in the composition $T^{(\lfloor t\rfloor)}$ is applied to the longest available form index. Finally note that $oh(D_\Lambda^\lambda) = \frac{1}{2}$ together with (2.70) and (2.48) yields

$$oh(T^{(t)}) = \lfloor t\rfloor + \frac{1}{2} = t. \tag{2.71}$$

Definition. The top operator $T^{(t)}$, $t \notin \mathbb{N}$ will be called *top splitting* or *top splitting operator* if this is a splitting operator for $\mathfrak{T} = \emptyset$. (Then $T^{(t)}$ is a splitting operator for any $\mathfrak{T}$.)

Theorem (Properties of the middle and top operator for spinors). *Let us consider the middle operator and the top operator*

$$M^{(m)},\ m \in \{1, 2, \cdots, \lfloor r \rfloor\} \quad \textit{and} \quad T^{(t)},\ t \in \{1/2, 3/2, \cdots, r\}$$

given by the relation (2.62) and (2.70), respectively, and a section $f \in \mathcal{E}_{\mathfrak{T}}(\frac{1}{2}; s_1, \ldots, s_r)_0[w]$, $w \in \mathbb{R}$. *Let*

$$pr(m) := \mathbb{Z}^{\mathbf{a}_{\bar{m}}}_{\mathbf{A}_{\bar{m}}} \cdots \mathbb{Z}^{\mathbf{a}_r}_{\mathbf{A}_r} \in TFPC(E_{\mathfrak{A}}) \textit{ where } \mathfrak{A} := \mathbf{A}_{\bar{m}} \cdots \mathbf{A}_{\lfloor r \rfloor} \textit{ and}$$
$$pr^t := Y^{\lambda}_{\Lambda} \mathbb{Y}_{B^0_1 \mathbf{B}_1}{}^{\mathbf{b}_1} \cdots \mathbb{Y}_{B^0_{\lfloor t \rfloor} \mathbf{B}_{\lfloor t \rfloor}}{}^{\mathbf{b}_{\lfloor t \rfloor}} \in TFPC(E_{\mathfrak{B}}) \textit{ where } \mathfrak{B} := \Lambda \mathbf{B}_1 \cdots \mathbf{B}_{\lfloor t \rfloor}$$

where $\bar{m} = \lfloor r \rfloor - m + 1$, *be TFP–projecting parts of homogenities* 0 *and* t, *respectively. Let us consider the scalar*

$$s(t,0) := \begin{cases} w - s - t + s_{\lfloor t \rfloor} + 1 & 1 < t < r_{n'},\ n \textit{ even, or } t > r_{n'},\ n \in \mathbb{N} \\ w - s - t + \frac{n}{2} + 1 & \frac{1}{2} = t < r_{n'},\ n \textit{ even} \\ w - s - t + \frac{n}{2} & t = r_{n'},\ n \textit{ even, or } t \leq r_{n'},\ n \textit{ odd} \end{cases}$$
$$s(0,m) := n + w - \lfloor s \rfloor - s_{\bar{m}} + \bar{m} - 1.$$

(i) Theorem 2.1.4 (i) will hold if we use $M^{(m)}$, $pr(m)$ *and* $s(0,m)$ *defined above in this Theorem. Theorem 2.1.5 (i) will hold if we use* $T^{(t)}$, pr^t *and* $s(t,0)$ *defined above in this Theorem and also replaced* $\mathfrak{A}$ *by* $\mathfrak{B}$ *defined above in this Theorem.*

(ii) Using the same modifications as in (i), the statements of Theorem 2.1.4 (ii) and Theorem 2.1.5 (ii) are satisfied.

Proof. Recall the construction of the middle operator for spinor bundles $M^{(m)} f = Y^{\Gamma} M^{(m)} X_{\Gamma}$ where $M^{(m)}$ is the middle operator for spinors on the left hand side and the middle operator for tensors on the right hand side. From this it follows we can use directly Theorem 2.1.4. The bottom operator X_Γ lowers parameters r and s by $\frac{1}{2}$ and does not change the weight. Therefore we have to replace s by $s - \frac{1}{2} = \lfloor s \rfloor$ and r by $r - \frac{1}{2} = \lfloor r \rfloor$. This proves all statements of the Theorem concerning the middle operator.

Analogously, the statements for the top operator mostly follow from Theorem 2.1.5. We need only to show that for $\mathfrak{T} = \emptyset$, if $s(t, 0) > 0$ then $T^{(t)}$ is a splitting operator and if $s(t, 0) = 0$ then $T^{(t)}$ is not a splitting operator.

Henceforth we assume $\mathfrak{T} = \emptyset$. We shall discuss the case $t = \frac{1}{2}$ i.e. $T^{(t)} = T^\lambda_\Lambda$ first. There are two possibilities $\tilde{T}$ and $\tilde{\tilde{T}}$ according to (2.64).

- Assume $T^{(1/2)} = \tilde{T}$ i.e. $r = \frac{1}{2}$ or $s_1 < \frac{n}{2}$. This, together with $t = \frac{1}{2}$, guarantees $s(\frac{1}{2}, 0) = w - s - \frac{1}{2} + \frac{n}{2} = w - \lfloor s \rfloor - 1 + \frac{n}{2}$. In other words,

  $$s(\frac{1}{2}, 0) > 0 \Longleftrightarrow w - \lfloor s \rfloor > 1 - \frac{n}{2}.$$

  There are two possibilities. Assume $r = \frac{1}{2}$. Then $T^{(1/2)} = D$ and $s(\frac{1}{2}, 0) = \frac{1}{2}(n+2w-2)$ because $\lfloor s \rfloor = 0$. Thus it follows from (1.39) that $T^{(1/2)}$ is a splitting for $s(\frac{1}{2}, 0) > 0$ and is not a splitting for $s(\frac{1}{2}, 0) = 0$. Now assume $r \geq \frac{3}{2}$. Then $T^{(1/2)} = (pr(\lfloor r \rfloor))^* DM^{(\lfloor r \rfloor)}$ according to (2.64). It follows from the last display that $s(\frac{1}{2}, 0) > 0$ yields $s(0, \lfloor r \rfloor) = n+w-\lfloor s \rfloor - s_1 > 0$. That is, $s(\frac{1}{2}, 0) > 0$ implies that $M^{(\lfloor r \rfloor)}$ is a splitting operator. (We have used Theorem 2.1.4 here.) Let us consider the operator D in $DM^{(\lfloor r \rfloor)}$. According to (1.39), D is a splitting operator if and only if $\frac{n}{2} + w' - 1 \neq 0$ where $w' = w - \lfloor s \rfloor$ is the conformal weight after application of $M^{(\lfloor r \rfloor)}$. It follows from the previous display that D is a splitting for $s(\frac{1}{2}, 0) > 0$ and is not a splitting for $s(\frac{1}{2}, 0) = 0$.

- Assume $T^{(1/2)} = \tilde{\tilde{T}}$ i.e. $r \geq r_{n'} \geq \frac{3}{2}$ and n even. This, together with $t = \frac{1}{2}$, guarantees $s(\frac{1}{2}, 0) = w - s - \frac{1}{2} + \frac{n}{2} + 1 = w - \lfloor s \rfloor + \frac{n}{2}$. In other words,

$$s(\frac{1}{2},0) > 0 \Longleftrightarrow w - \lfloor s \rfloor > -\frac{n}{2}.$$

There are again two possibilities. Assume $r = \frac{3}{2}$. Then $T^{(1/2)} = \mathbb{X}^* D\mathbb{X}$, see (2.64), and $w' = w - \frac{n}{2} + 1$ is the conformal weight after application of $\mathbb{X}$. Then D in $D\mathbb{X}$ is a splitting if and only if $\frac{1}{2}[n+2(w-\frac{n}{2}+1)-2] \neq 0$. Using this and the last display (where $\lfloor s \rfloor = \frac{n}{2}$), the Theorem follows.

Assume $r \geq \frac{1}{2} + 2$. Then $T^{(1/2)} = (pr(\lfloor r-1 \rfloor)\mathbb{X})^* D\mathbb{X}M^{(\lfloor r-1 \rfloor)}$ according to (2.64). It follows from the last display that $s(\frac{1}{2},0) > 0$ yields $s(0,\lfloor r \rfloor) = n + w - \lfloor s \rfloor - s_2 + 1 > 0$. Hence using Theorem 2.1.4, $s(\frac{1}{2},0) > 0$ implies that $M^{(\lfloor r-1 \rfloor)}$ is a splitting operator. Further, using (1.39), the operator D in the formula $D\mathbb{X}M^{(\lfloor r-1 \rfloor)}$ is a splitting operator if and only if $\frac{n}{2} + w' - 1 \neq 0$ where $w' = w - \lfloor s \rfloor + 1$ is the conformal weight after application of $\mathbb{X}M^{(\lfloor r-1 \rfloor)}$, see (2.64) for details. Hence D is a splitting for $s(\frac{1}{2},0) > 0$ and is not a splitting for $s(\frac{1}{2},0) = 0$.

It remains to consider the case $t \geq \frac{3}{2}$. That is, $s(t,0) = w - s - t + s_{\lfloor t \rfloor} + 1$ or $s(t,0) = w - s - t + \frac{n}{2}$. Clearly if all the top operators in the composition $T^{(t)} = T^{(\lfloor t \rfloor)} T^{\lambda}_{\Lambda}$, $\lfloor t \rfloor \geq 1$ are splittings then $T^{(t)}$ is a splitting operator. We shall discuss the "tensor part" $T^{(\lfloor t \rfloor)}$ of $T^{(t)}$ first. To use Theorem 2.1.5 for $T^{(\lfloor t \rfloor)}$, let us denote all quantities concerning the tensor structure *after* application of T^{λ}_{Λ} by primes. That is, $w' = w - 1$, $s' = s - \frac{1}{2}$, $t' = t - \frac{1}{2}$ and $s_{t'} = s_{\lfloor t \rfloor} = s_t$. (The latter is just our convention form 1.1.3.) If we input the primed quantities into (2.49), we will obtain the same scalar $s(t,0)$ because $w' - s' - t' = w - s - t$. Therefore $s(t,0) > 0$ guarantees $T^{(\lfloor t \rfloor)}$ is a splitting operator and $s(t,0) = 0$ means $T^{(\lfloor t \rfloor)}$ is not a splitting.

It remains to prove that if $s(t,0) > 0$ then $s(\frac{1}{2},0) > 0$. The latter inequality guarantees that T^{λ}_{Λ} in $T^{(t)}$ is a splitting operator. (Recall we assume $t \geq \frac{3}{2}$.) Observe $s(\frac{1}{2},0) \geq w - s - \frac{1}{2} + \frac{n}{2}$. There are two possible forms of $s(t,0)$ for $t \geq \frac{3}{2}$, see above. If $s(t,0) = w - s - t + s_{\lfloor t \rfloor} + 1 > 0$ then

$$s(\frac{1}{2},0) \geq w - s - \frac{1}{2} + \frac{n}{2} = s(t,0) + \underbrace{(t - \frac{1}{2})}_{\geq 1} + \underbrace{(\frac{n}{2} - s_{\lfloor t \rfloor})}_{\geq 0} - 1 \geq s(t,0) > 0.$$

Finally in the second case $s(t,0) = w - s - t + \frac{n}{2} > 0$ we obtain directly $s(\frac{1}{2},0) \geq w - s - \frac{1}{2} + \frac{n}{2} > s(t,0) > 0$ because $t \geq \frac{3}{2}$. □

Remark. If $w \in \mathbb{AW}$ (see (1.61)) or $w \in \mathbb{C} \setminus \mathbb{R}$ then $T^{(t)}$, $t \in \frac{1}{2}\mathbb{N}_0$ is always the top splitting operator. This follows the similar property in tensor cases. To verify this for spinors, note the tractor spinor D–operator is not a splitting for $n + 2w - 2 = 0$ (see (1.39)). The latter requires $w \in \mathbb{AW}$.

Examples of middle and top operators for spinor representations

We shall consider the bundles treated in Examples in 2.1.4 and 2.1.5 with an additional spin index ("Clifford free" in the sense of 1.1.3). These are $E(\frac{1}{2};k)_0[w]$ and $E(\frac{1}{2};k,l)_0[w]$ for $n' \geq k \geq l \geq 1$. In the notation for sections, the valences k and l of $\mathbf{a} = \mathbf{a}^k$ and $\mathbf{b} = \mathbf{b}^l$, respectively, will be omitted as well as spinor and tractor spinor indices. That is, we shall use the notation $f_{\mathbf{a}} = f_{\lambda \mathbf{a}^k} \in \mathcal{E}(\frac{1}{2},k)_0[w]$, $f_{\mathbf{ab}} = f_{\lambda \mathbf{a}^k \mathbf{b}^l} \in \mathcal{E}(\frac{1}{2},k,l)_0[w]$, $X = X_\Lambda^\lambda$, $D = D_\Lambda^\lambda$ etc.

Example 2.1.10. The middle operator M for spinor representation is defined by the formula (2.60). This is formally the same as in the tensor case (2.16) as we are omitting spinor indices. Therefore more complicated middle operators are also given by the same formulae in the tensor and spinor cases. In particular, we can use corresponding formulae from examples in 2.1.4 also for the spaces $\mathcal{E}(\frac{1}{2};k)_0[w]$ and $\mathcal{E}(\frac{1}{2};k,l)_0[w]$.

Example 2.1.11. Following (2.64), we have two possibilities for the top operator T on $\mathcal{E}(\frac{1}{2};k)_0[w]$: $\tilde{T}$ and $\tilde{\tilde{T}}$. Let us consider the former one first, i.e.

$k \neq \frac{n}{2}$. Using the definition (2.64), this is

$$\tilde{T} : \mathcal{E}(\frac{1}{2}; k)_0[w] \longrightarrow \mathcal{E}_{\Lambda \mathbf{a}^k}[w-1]$$
$$\tilde{T} f_{\mathbf{a}} = \mathbb{Z}_{\mathbf{a}}^{\mathbf{C}} D M_{\mathbf{C}}^{\mathbf{c}} f_{\mathbf{c}}$$

where $\mathbf{c} = \mathbf{c}^k$. After a short computation, we obtain the formula

$$\begin{aligned} \tilde{T} f_{\mathbf{a}} =& (n + 2(w-k) - 2)(n + w - 2k) Y f_{\mathbf{a}} \\ &+ 2X\big[(n + w - 2k)\boldsymbol{\beta}^p \nabla_p f_{\mathbf{a}} - k\boldsymbol{\beta}_{a^1} \nabla^p f_{p\dot{\mathbf{a}}}\big] \end{aligned} \tag{2.72}$$

with the skew–symmetrization $[a^1 \dot{\mathbf{a}}]$ on the right hand side. The second possibility $\tilde{\tilde{T}}$ requires $k = n' = \frac{n}{2}$ (i.e. n is even) and is defined as

$$\tilde{\tilde{T}} : \mathcal{E}(\frac{1}{2}; n')_0[w] \longrightarrow \mathcal{E}_{\Lambda \mathbf{a}^{n'}}[w-1]$$
$$\tilde{\tilde{T}} f_{\mathbf{a}} = \mathbb{Y}^{C^0 \mathbf{C}}{}_{\mathbf{a}} D \mathbb{X}_{C^0}{}^{\mathbf{c}}_{\mathbf{C}} f_{\mathbf{c}}$$

where $c = \mathbf{c}^{n'}$. The explicit formula is

$$\tilde{\tilde{T}} f_{\mathbf{a}} = 2wY f_{\mathbf{a}} + 2X\boldsymbol{\beta}^p \nabla_p f_{\mathbf{a}}. \tag{2.73}$$

Example 2.1.12. We shall consider only the case $k < \frac{n}{2}$ i.e. the version $\tilde{T}$ in (2.64). Then the top operator T on $\mathcal{E}(\frac{1}{2}; k, l)_0[w]$ is defined as

$$T : \mathcal{E}(\frac{1}{2}; k, l)_0[w] \longrightarrow \mathcal{E}_{\Lambda \mathbf{a}^k \mathbf{b}^l}[w-1]$$
$$T f_{\mathbf{ab}} = \mathbb{Z}_{\mathbf{a}}^{\mathbf{C}} \mathbb{Z}_{\mathbf{b}}^{\mathbf{D}} D M_{\mathbf{C}}^{\mathbf{c}} M_{\mathbf{D}}^{\mathbf{d}} f_{\mathbf{cd}}$$

where $\mathbf{c} = \mathbf{c}^k$ and $\mathbf{d} = \mathbf{d}^l$. Using the formula the middle operator $M_{\mathbf{A}}^{\mathbf{a}} M_{\mathbf{B}}^{\mathbf{b}} f_{\lambda \mathbf{ab}}$, formally the same as (2.21) where $\mathcal{E}(k, l)_0[w]$ is treated, one can compute the formula

$$\begin{aligned} T f_{\mathbf{ab}} =& (n+2(w-k-l)-2)(n+w-l-2k)(n+w-k-2l+1) Y f_{\mathbf{ab}} \\ &+2X\Big[(n+w-l-2k)\big[(n+w-k-2l+1)\beta^p \nabla_p f_{\mathbf{ab}} - l\boldsymbol{\beta}_{b^1} \nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}}\big] \\ &- k(n+w-k-2l+1)\boldsymbol{\beta}_{a^1} \nabla^p f_{p\dot{\mathbf{a}}\mathbf{b}} + kl\boldsymbol{\beta}_{a^1} \nabla^p f_{b^1 \dot{\mathbf{a}} p \dot{\mathbf{b}}}\Big] \end{aligned} \tag{2.74}$$

with the skew–symmetrizations $[a^1\dot{\mathbf{a}}]$ and $[b^1\dot{\mathbf{b}}]$ on the right hand side. Then $Tf_{\mathbf{ab}}$ satisfies the Young symmetries (k,l) which can be checked by a direct computation. (One can also use Proposition 1.3.8 - projection to a subspace different than (k,l) kills the top slot and the bottom slot would yield a 1st order operator invariant for any weight w.)

Remark. Formulae from Examples 2.1.11 and 2.1.12 are valid even for $k = l = 0$ when they recover the tractor D–operator D^λ_Λ for spinors up to scalar multiples $n+w$ and $(n+w)(n+w+1)$, respectively.

Example 2.1.13. In the last example, we shall look at the top operator on the space $\mathcal{E}(\frac{1}{2};1,\dots,1)_0[w]$ with $\lfloor r\rfloor \geq 1$ tensor indices. This is the subspace of $\mathcal{E}_{\lambda(a_1\cdots a_r)_0}[w]$ with sections killed by $\boldsymbol{\beta}^{a_i}$, $1 \leq i \leq \lfloor r\rfloor$. The top operator T which puts the spinor index to the top slot, is constructed in (2.64) by

$$T : \mathcal{E}(\frac{1}{2};1\cdots 1)_0[w] \longrightarrow \mathcal{E}_\Lambda(1\cdots 1)_0[w-1]$$
$$Tf_{a_1\cdots a_r} = \mathbb{Z}_{a_1}^{B_1}\cdots \mathbb{Z}_{a_r}^{B_r} D M_{B_1\cdots B_r}^{b_1\cdots b_r} f_{b_1\cdots b_r}$$

for $f_{a_1\cdots a_r} \in \mathcal{E}(\frac{1}{2};1\cdots 1)_0[w]$. Hence we need to compute the (complete) middle operator

$$M_{A_1\cdots A_r}^{a_1\cdots a_r} = M_{A_1}^{a^1}\cdots M_{A_r}^{a^r} : \mathcal{E}(\frac{1}{2};1\cdots 1)_0[w] \longrightarrow \mathcal{E}_{\lambda\mathbf{A}^1\cdots\mathbf{A}^r}[w-r]$$

first. We shall follow Example 2.1.9 where the corresponding tensor case is treated. Considering that the tractor D–operator D^λ_Λ is of the first order, we need actually only two slots of $M_{A_1\cdots A_r}^{a_1\cdots a_r} f_{a_1\cdots a_r}$ of the highest homogenities. These are 0 and -1. Since the formula for the middle operator (2.60) is formally the same for spaces $\mathcal{E}(\frac{1}{2};1\cdots 1)_0[w]$ and $\mathcal{E}(1\cdots 1)_0[w] = \mathcal{E}_{(a_1\cdots a_r)_0}[w]$, we can use (2.23) from Example 2.1.5. Applying the last middle operator

$M^{a_1}_{A_1}$ to (2.23), one can easily compute

$$M^{a_1\cdots a_r}_{A_1\cdots A_r} f_{a_1\cdots a_r} = \bar{C}\Big\{ cZ^{a_1}_{A_1}\cdots Z^{a_r}_{A_r} f_{a_1\cdots a_r} - rX_{(A_1}Z^{a_2}_{A_2}\cdots Z^{a_r}_{A_r)}\nabla^p f_{pa_2\cdots a_r}\Big\}$$
$$+ \{\text{lower homogenity terms}\} \tag{2.75}$$

where we use the scalars $c = n + w - 2$ and

$$\bar{C} = \begin{cases} 1 & r = 1, \\ \prod_{i=2}^{r}(c - i + 1) & r \geq 2. \end{cases}$$

Let us note that (2.23) requires $r \geq 3$ but (2.75) above is satisfied for $r \geq 1$. (The cases $r = 1$ and $r = 2$ are easily checked using formulae (2.72) and (2.74), respectively, for $k = l = 1$.) Using this, it is not difficult to compute the result

$$T f_{a_1\cdots a_r} = \bar{C}\Big\{ c\big(n + 2(w - r) - 2\big) Y f_{a_1\cdots a_r}$$
$$+ 2X\Big[c\boldsymbol{\beta}^p \nabla_p f_{a_1\cdots a_r} - r\boldsymbol{\beta}_{(a_1}\nabla^p f_{a_2\cdots a_r)p}\Big]\Big\}.$$

2.1.7. Summary: *D*–splitting operator. We shall start with the following (obvious) generalisation of the bottom operator B and the top operator T to spaces without tensor or spinor indices. We define them as (the strongly invariant) operators

$$\begin{aligned} B_A &:= X_A : \mathcal{E}_{\mathfrak{T}}[w] \longrightarrow \mathcal{E}_{A\mathfrak{T}}[w+1] \\ T_A &:= D_A : \mathcal{E}_{\mathfrak{T}}[w] \longrightarrow \mathcal{E}_{A\mathfrak{T}}[w-1]. \end{aligned} \tag{2.76}$$

We will also use the familiar notation $T^{(t)} := T\cdots T$ and $B^{(b)} = B\cdots B$ for composition of $t, b \in \mathbb{N}$ of operators T and B, respectively, defined by (2.76). In the case of the general space $\mathcal{E}\{r_1, \cdots, r_{n'}\}_0[w]$, (2.76) gives rise to the operators

$$T^{(p+r)} := \underbrace{T\cdots T}_{p} T^{(r)} \ \text{ and } \ B^{(p+r)} := \underbrace{B\cdots B}_{p} B^{(r)}, \quad p \in \mathbb{N}_0.$$

Henceforth we shall consider the space $\mathcal{E}\{r_1, \cdots, r_{n'}\}_0[w]$. Let us consider all the top, middle and bottom operators defined until now, and their compositions. (They are strongly invariant.) We can consider any such composition (if it is well–defined) but we prefer the following order. We define the strongly invariant operator $DSplit_b^t(m)$ as follows.

$$DSplit_b^t(m) := B^{(b)}M^{(m)}T^{(t)} \quad \text{on} \quad \mathcal{E}_{\mathfrak{T}}(k; s_1, \cdots, s_r)_0[w], \; k \in \{0, \frac{1}{2}\}. \tag{2.77}$$

The advantage of this order is that if $\mathfrak{T} = \emptyset$ and $DSplit_b^t(m)$ is a splitting operator then Theorem 2.1.4 (ii) shows $DSplit_b^t(m)$ as a unique nontrivial TFP–projecting part. (For example, the order $T^{(t)}M^{(m)}$ does not guarantee this.) It follows from the definitions of T, M and B that the parameters t, m, b must satisfy

$$\begin{aligned}
&\bullet\; t, b \in \frac{1}{2}\mathbb{N}_0, \; r \geq m \in \mathbb{N}_0 \\
&\bullet\; t > 0 \Longrightarrow |r - t| \in \mathbb{N}_0 \;\text{ and }\; t \geq r \Longrightarrow m = 0 \\
&\bullet\; r \in \mathbb{N}_0 \Longrightarrow t, b \in \mathbb{N}_0 \;\text{ and }\; r \notin \mathbb{N}_0, t + b > 0 \Longrightarrow |t - b| \notin \mathbb{N}_0.
\end{aligned} \tag{2.78}$$

Definition. The operator $DSplit_b^t(m)$ for t, m, b satisfying (2.78) is called *D–splitting* or *D–splitting operator* if this is a splitting operator for $\mathfrak{T} = \emptyset$. (Then $DSplit_b^t(m)$ is a splitting operator for any $\mathfrak{T}$.)

Actually we shall need only the following special cases of $DSplit_b^t(m)$ in the subsequent computations. We define

$$\begin{aligned}
DSplit^t(m) &:= DSplit_0^t(m) \quad \text{where} \quad t > r \Longrightarrow m = 0 \\
DSplit_b(m) &:= DSplit_b^0(m) \quad \text{where} \quad b > r \Longrightarrow m = 0.
\end{aligned}$$

(Considering the conditions (2.78), the definition of $DSplit^t(m)$ says only that $b = 0$.) It follows from (2.47), (2.17) and (2.5) that for $k = 0$ and $\mathfrak{T} = \emptyset$, these are operators

$$DSplit^t(m) : \mathcal{E}(k; s_1, \cdots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{A}}(s_{\lfloor t \rfloor+1}, \cdots, s_{\bar{m}-1})_0[w - s^t - \tilde{s}^{\bar{m}} - t]$$
$$DSplit_b(m) : \mathcal{E}(k; s_1, \cdots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathfrak{A}}(s_{\lfloor b \rfloor+1}, \cdots, s_{\bar{m}-1})_0[w - s^b - \tilde{s}^{\bar{m}} + b] \tag{2.79}$$

where $\bar{m} = \lfloor r \rfloor - m + 1$. We put $s_i := 0$ for $r < i \in \mathbb{N}$. Then, in the tensor case

$$\mathfrak{A} = [A_1^0 \mathbf{A}_1] \cdots [A^0_{\lfloor q \rfloor} \mathbf{A}_{\lfloor q \rfloor}] \mathbf{A}_{\bar{m}} \cdots \mathbf{A}_{\lfloor r \rfloor}, \;\; \mathbf{A}_i = \mathbf{A}_i^{s_i} \;\; \text{for} \;\; q \in \{t, b\}. \tag{2.80}$$

In the spinor case i.e. for $k = \frac{1}{2}$, the operators $DSplit^t(m)$ and $DSplit_b(m)$ satisfy (2.79) with $\mathfrak{A}$ replaced by $\Lambda\mathfrak{A}$.

Theorem (Properties of the operator $DSplit$**).** *Let us consider the operator $DSplit^t_b(m)$ on the space $\mathcal{E}_{\mathfrak{T}}(k; s_1, \cdots, s_r)_0[w]$, $k \in \{0, \frac{1}{2}\}$, $w \in \mathbb{R}$ where the parameters t, m and b satisfy (2.78). We put moreover $s_i := 0$ for $r < i \in \mathbb{N}$. Let $s(t, m)$ be the scalar*

$$s(t,m) = \begin{cases} w - s - t + s_{\lfloor t \rfloor} + 1 & \bigvee \begin{cases} t > r_{n'} & n \in \mathbb{N} \\ 1 \leq t < r_{n'} \wedge m + t < r & n \ \text{even} \end{cases} \\ w - s - t + \frac{n}{2} + 1 & \frac{1}{2} = t < r_{n'} \wedge m + t < r \quad n \ \text{even} \\ w - s - t + \frac{n}{2} & \bigvee \begin{cases} \frac{1}{2} \leq t \leq r_{n'} & n \ \text{odd} \\ \frac{1}{2} \leq t = r_{n'} & n \ \text{even} \\ \frac{1}{2} \leq t < r_{n'} \wedge m + t = r & n \ \text{even} \end{cases} \\ n + w - \lfloor s \rfloor - s_{\bar{m}} + \bar{m} - 1 & t = 0 \wedge m \geq 1 \\ 1 & t = m = 0 \end{cases} \tag{2.81}$$

where $\bar{m} = \lfloor r \rfloor - m + 1$ and "$\bigvee\{$" denotes disjunction of the condition on the right. Let us denote by $\mathfrak{A}$ the system of free tractor indices in the *formula for the operator* $DSplit^t_b(m)$*. Then* $E_{\mathfrak{A}}$ *is a TFP–bundle and there is a unique TFP–projecting part* $pr^t_b(m) \in TFPC(E_{\mathfrak{A}})$ *of the homogenity* $h(pr^t_b(m)) = t - b$ *satisfying*

$$\forall pr' \in TFPC(E_{\mathfrak{A}}) : (pr')^* DSplit^t_b(m) \neq 0 \Longrightarrow \\ \Longrightarrow \big[pr' = pr^t_b(m)\big] \vee \big[h(pr') < h\big(pr^t_b(m)\big)\big]. \quad (2.82)$$

Furthermore the operator DSplit satisfies $\big(pr^t_b(m)\big)^* DSplit^t_b(m) = C \cdot \mathrm{id}$ *and the scalar* C *is as follows. If* $s(t,m) > 0$ *then* $C \neq 0$ *and* $DSplit^t_b(m)$ *is the D–splitting operator; if* $s(t,m) = 0$ *then* $C = 0$ *and* $DSplit^t_b(m)$ *is not the D–splitting operator.*

The explicit form of $pr^t_b(m)$ *in the cases (2.79) is as follows. The system of indices* $\mathfrak{A}$ *is of the form (2.80) in the tensor case and with the additional index* Λ *in the spinor one. The TFP–projecting part* $pr^t(m) := pr^t_0(m)$ *of* $DSplit^t(m)$ *is of the form*

$$pr^t(m) = \begin{cases} \mathbb{Y}_{[A^0_1 \mathbf{A}_1]}{}^{\mathbf{a}_1} \cdots \mathbb{Y}_{[A^0_t \mathbf{A}_t]}{}^{\mathbf{a}_t} \mathbb{Z}^{\mathbf{a}_{\bar{m}}}_{\mathbf{A}_{\bar{m}}} \cdots \mathbb{Z}^{\mathbf{a}_r}_{\mathbf{A}_r} \in TFPC(E_{\mathfrak{A}}) & \textit{for } t, r \in \mathbb{N}_0 \\ Y^{\lambda}_{\Lambda} \mathbb{Y}_{[A^0_1 \mathbf{A}_1]}{}^{\mathbf{a}_1} \cdots \mathbb{Y}_{[A^0_t \mathbf{A}_t]}{}^{\mathbf{a}_t} \mathbb{Z}^{\mathbf{a}_{\bar{m}}}_{\mathbf{A}_{\bar{m}}} \cdots \mathbb{Z}^{\mathbf{a}_r}_{\mathbf{A}_r} \in TFPC(E_{\Lambda\mathfrak{A}}) & \textit{for } t, r \notin \mathbb{N}_0 \end{cases}$$

where we consider $\lfloor t \rfloor$ *in* A^0_t*,* $\mathbf{A}_t$*,* $\mathbf{a}_t$ *and* $\lfloor r \rfloor$ *in* $\mathbf{A}_r$*,* $\mathbf{a}_r$*. The TFP–projecting part* $pr_b(m) := pr^0_b(m)$ *of* $DSplit_b(m)$ *is given by* $pr^b(m)$ *with* $\mathbb{Y}$ *and* Y *replaced by* $\mathbb{X}$ *and* X*, respectively.*

Remark. The form of $s(t,m)$ may seem complicated but it can be roughly reformulated for $t \geq 1$ as follows: $s(t,m) = w - s - t + s_t + 1$ or lower by one in some cases if $s_t = n'$.

Proof. The existence of the TFP–projecting part $pr^t_b(m)$, its uniqueness in the sense of (2.82) and the forms of $pr^t(m)$ and $pr_b(m)$ follow from Theorems 2.1.5 (i), 2.1.4 (ii) and 2.1.1 (ii) in tensor cases and the corresponding statements in Theorem 2.1.6 in spinor ones. The homogenity of $pr^t_b(m)$ is obvious.

It remains to show that $s(t,m) > 0$ implies $C \neq 0$ and that $s(t,m) = 0$ implies $C = 0$. This obviously does not depend on b. We shall consider the cases $t = 0$, $t = \frac{1}{2}$, $1 \leq t \leq r$ and $t > r$ separately. Similarly as in these theorems, we assume $\mathfrak{T} = \emptyset$.

I. $t = 0$. In this case, the scalars $s(0,m)$ from Theorem 2.1.4 (i) and Theorem 2.1.6 (i) clearly coincide with the form of $s(0,m)$ in (2.81).

II. $t = \frac{1}{2}$. This means $DSplit_b^t(m) = B^{(b)} M^{(m)} T_\Lambda^\lambda$ where T_Λ^λ is the spinor top operator. It follows from (2.81) that $s(\frac{1}{2},0) = s(\frac{1}{2},m)$ or $s(\frac{1}{2},0) = s(\frac{1}{2},m)+1$.

Assume $s(\frac{1}{2},m) = s(\frac{1}{2},0)$. This excludes the possibility $\frac{1}{2} = t < r_{n'}$, $m+t = r$, n even, see (2.81). Comparing the scalar (2.81) with the scalar from Theorem 2.1.6 for the top operator (denoted also by $s(\frac{1}{2},0)$ therein), we see they agree. This proves the case of the equality and also that if $s(\frac{1}{2},m) > 0$ then T_Λ^λ is a splitting operator. Further we need to know that $M^{(m)}$ is a splitting operator for $s(\frac{1}{2},m) = s(\frac{1}{2},0) > 0$ and $m \geq 1$. Assume $s(\frac{1}{2},0) > 0$. Let us denote the scalar from Theorem 2.1.4 for $M^{(m)}$ applied *after* T_Λ^λ by $s_{s',w'}(0,m)$. Here $w' = w-1$ and $s' = s - \frac{1}{2}$ indicate quantities corresponding to the tensor structure after application of T_Λ^λ. There are two possibilities for $s(\frac{1}{2},0)$. If $s(\frac{1}{2},m) = s(\frac{1}{2},0) = w - s - \frac{1}{2} + \frac{n}{2}$ then

$$s_{s',w'}(0,m) = n + w' - s' - s_{\bar{m}} + \bar{m} - 1 = n + (w-1) - (s - \frac{1}{2}) - s_{\bar{m}} + \bar{m} - 1 =$$
$$= \underbrace{\frac{n}{2} + w - s - \frac{1}{2}}_{s(\frac{1}{2},0)>0} + \underbrace{\frac{n}{2} - s_{\bar{m}}}_{\geq 0} + \underbrace{\bar{m} - 1}_{\geq 0} > 0.$$

Hence $M^{(m)}$ and consequently also $DSplit_b^t(m)$ are splitting operators. The second possibility is $s(\frac{1}{2},m) = s(\frac{1}{2},0) = w - s + \frac{1}{2} + \frac{n}{2}$ can happen only for $m + t = m + \frac{1}{2} < r$, see (2.81), or equivalently $\bar{m} \geq 2$. It is easily check from the last display that again $s_{s',w'}(0,m) > 0$ hence $M^{(m)}$ and $DSplit_b^t(m)$ are splittings.

It remains to discuss the case $s(\frac{1}{2}, 0) = s(\frac{1}{2}, m) + 1$. This can happen only for $\frac{1}{2} < r_{n'}$, n even and $m + t = r$. This means $s_1 = \frac{n}{2}$ and $\bar{m} = 1$. Clearly, if $s(\frac{1}{2}, m) > 0$ then $s(\frac{1}{2}, 0) = w - s + \frac{1}{2} + \frac{n}{2} > 0$. The latter is just the scalar corresponding to T^λ_Λ form Theorem 2.1.6 (denoted by $s(\frac{1}{2}, 0)$ therein) because $\frac{1}{2} < r_{n'}$ and n even. Thus T^λ_Λ is a splitting operator for $s(\frac{1}{2}, m) > 0$. The rest follows immediately from the last display (including the case of the equality).

III. $1 \le t \le r$. Let us consider the operator $T^{(t)}$ in the composition $DSplit^t_b(m) = B^{(b)} M^{(m)} T^{(t)}$ first. This is (is not) the top splitting if $s(t, 0) > 0$ ($s(t, 0) = 0$) where $s(t, 0)$ is given by Theorems 2.1.5 (i) and 2.1.6 (i). This scalar actually coincides with the form (2.81) above.

Now observe from (2.81) that $s(t, 0) = s(t, m)$ with the exception of of the case $t < r_{n'}$, $t + m = r$ and n even. In this case and for $s(t, m) > 0$ we have

$$s(t, 0) = w - s - t + s_{\lfloor t \rfloor} + 1 = \underbrace{w - s - t + \frac{n}{2}}_{s(t,m)} + 1 = s(t, m) + 1 > 0$$

because $1 \le t < r_{n'}$, n even means $s_{\lfloor t \rfloor} = n' = \frac{n}{2}$. Summarising, this shows that if $s(t, m) > 0$ then $T^{(t)}$ is the top splitting. Moreover, with the exception of the case $t < r_{n'}$, $t + m = r$ and n even, if $s(t, m) = 0$ then $T^{(t)}$ is not the top splitting. The latter means $DSplit^t_b(m)$ is not the D–splitting.

It remains to discuss the operator $M^{(m)}$ in $DSplit^t_b(m) = B^{(b)} M^{(m)} T^{(t)}$. The top operator $T^{(t)}$ changes the quantities w, s etc. We denote the new ones, i.e. the quantities corresponding to the tensor indices *after* application of $T^{(t)}$, by primes. We have

$$w' = w - s^t - t \qquad r' = r - t \in \mathbb{N}$$
$$s' = s - s^t \in \mathbb{N} \qquad \bar{m}' = \bar{m} - \lfloor t \rfloor \in \mathbb{N}.$$

(See (2.47) and (2.70) for w', the remaining ones are obvious.) In particular, $w' - s' = w - s - t$ and $\bar{m}' \geq 1$. The scalar for the middle operator from Theorem 2.1.4 is

$$s_{w',s',r'}(0,m) = n + w' - s' - s'_{\bar{m}'} + \bar{m}' - 1 = n + w - s - t - s_{\bar{m}} + \bar{m}' - 1$$

where primed subscripts indicates the relevant changed quantities. (In this notation, the primed Young diagram has columns $(s_{\lfloor t \rfloor + 1}, \cdots, s_{\lfloor r \rfloor})$ hence $s'_{\bar{m}'}$ and $s_{\bar{m}}$ denote the same column.) We need to show that $s(t,m) > 0$ implies $s_{w',s',r'}(0,m) > 0$ and also the case of equality for $t < r_{n'}$, $t + m = r$ and n even. According to (2.81), there are two possibilities for $s(t,m)$.

- $s(t,m) = w - s - t + s_{\lfloor t \rfloor} + 1$. We need to discuss only the inequality in this case so assume $s(t,m) > 0$. Then

$$s_{w',s',r'}(0,m) = \underbrace{w - s - t + s_{\lfloor t \rfloor} + 1}_{s(t,m)>0} + \underbrace{n - s_{\lfloor t \rfloor} - s_{\bar{m}}}_{\geq 0} + \underbrace{\bar{m}' - 1}_{\geq 0} - 1 > 0.$$

  The reason for the inequality ">0" is as follows. There are two possibilities according to the definition of $s(t,m)$: if $t > r_{n'}$ then $n - s_{\lfloor t \rfloor} - s_{\bar{m}} > 0$ and if $m + t < r$ then $\bar{m}' \geq 2$. Hence, the last display follows.

- $s(t,m) = w - s - t + \frac{n}{2}$. We have to consider also the equality in the case $t < r_{n'}$, $t + m = r$ and n even so assume $s(t,m) \geq 0$. Then

$$s_{w',s',r'}(0,m) = \underbrace{w - s - t + \frac{n}{2}}_{s(t,m)\geq 0} + \underbrace{\frac{n}{2} - s_{\bar{m}}}_{\geq 0} + \underbrace{\bar{m}' - 1}_{\geq 0} \geq 0.$$

  Obviously the last inequality is sharp for $s(t,m) > 0$. The case of equality is also clear: firstly, $m + t = r$ means $\bar{m} = \lfloor t \rfloor + 1$ hence $\bar{m}' = 1$, and secondly, $t < r_{n'}$ means $s_{\lfloor t \rfloor} = s_{\lfloor t \rfloor + 1} = s_{\bar{m}} = n' = \frac{n}{2}$ because n is even. Hence if $s(t,m) = 0$ then $s_{w',s',r'}(0,m) = 0$.

IV. $t > r$. This requires $m = 0$ according to (2.78) and $s(t,0) = w - s - t + 1$. The latter clearly implies $s(r,0) \geq s(t,0)$ hence if $s(t,0) > 0$ then the $T^{(r)}$ in $DSplit^t_b(0) = B^{(b)}T^{(t-r)}T^{(r)}$ is the top splitting. It remains to show $s(t,0) > 0$ also implies $T^{(t-r)}$ is the top splitting and the case of the equality. Let us note $t > r$ implies $t - r \in \mathbb{N}$ (see (2.78)) and $T^{(t-r)}$ acts on $\mathcal{E}_{\mathfrak{A}}[w']$ for $w' = w - s - r$ according to (2.47) and (2.70). The system of tractor indices $\mathfrak{A}$ is of the form (2.80) where $q = r$. The form of w' means $s(t,0) = w' - (t-r) + 1$. But this is just the scalar $s(t-r,0)$ from Example 2.1.1 for the weight w' and the Theorem follows. □

Remark. We are interested mainly in admissible weights $w \in \mathbb{AW}$ (see 1.61) because they admit operators from the pattern. But let us note if $w \notin \mathbb{AW}$ or $w \in \mathbb{C} \setminus \mathbb{R}$ then $DSplit^t_b(m)$ is always a splitting operator. This follows from similar properties for the middle and top operators.

2.1.8. Formal adjoints. Consider and a natural bundle V and its dual V^*. The conformal volume form $\boldsymbol{\epsilon}_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^n}[n]$ has the conformal weight n so the pairing

$$\begin{gathered} \langle , \rangle : \mathcal{V} \times \mathcal{V}^*[-n] \longrightarrow \mathcal{E} \\ \langle \varphi, \psi \rangle = \int \varphi\psi \end{gathered} \tag{2.83}$$

where φ and ψ are compactly supported, is well–defined on conformal manifolds. Let us consider an invariant differential operator $L : \mathcal{V} \longrightarrow \mathcal{W}$ between bundles V and W. We define its *formal adjoint* $L^* : \mathcal{W}^*[-n] \longrightarrow \mathcal{V}^*[-n]$ by the relation

$$\langle L\varphi, \psi \rangle = \langle \varphi, L^*\psi \rangle \quad \text{for every } \varphi \in \mathcal{V} \text{ and } \psi \in \mathcal{W}^*[-n]. \tag{2.84}$$

Note that if L is conformally invariant then also L^* is conformally invariant. If V is the bundle $E_{\mathfrak{A}}\{r_1, \ldots, r_{n'}\}[w]$ then the dual V^* is isomorphic to

$E_{\mathfrak{A}}\{r_1, \ldots, r_{n'}\}[-w+2s]$. (Recall s is the overall number of tensor indices, increased by $\frac{1}{2}$ in the presence of a spinor index.) This is due to the conformal metric $\boldsymbol{g}^{ab} \in \mathcal{E}^{(ab)}[-2]$, the spinor metric $\boldsymbol{\varepsilon}^{\lambda\omega} \in \mathcal{E}^{\lambda\omega}[-1]$ and the tractor versions $h^{AB} \in \mathcal{E}^{(AB)}$ and $\boldsymbol{\varepsilon}^{\Lambda\Omega} \in \mathcal{E}^{\Lambda\Omega}$, respectively. Denoting the system of all tensor and spinor indices by $\mathfrak{a}$, the pairing becomes

$$\langle , \rangle : \mathcal{E}_{\mathfrak{A}}(s_1, \ldots, s_r)[w] \times \mathcal{E}_{\mathfrak{A}}(s_1, \ldots, s_r)[-w+2s-n] \longrightarrow \mathcal{E}$$
$$\langle \varphi_{\mathfrak{A}\mathfrak{a}}, \psi_{\mathfrak{A}\mathfrak{a}} \rangle = \int \varphi_{\mathfrak{A}\mathfrak{a}} \psi^{\mathfrak{A}\mathfrak{a}}.$$

The formal adjoint of a splitting operator is an invariant replacement of projection to a slot of a tractor section mentioned in 1.3.6. For example, the projection to the $\mathbb{X}$–slot of $F \in \mathcal{E}_{\mathbf{A}^{k+1}}[w']$ is not, in general, invariant. We can use the top operator T on $\mathcal{E}_{\mathbf{a}^k}[w]$ and the formal adjoint T^* as follows. These are operators

$$T : \begin{pmatrix} * \\ - \; - \\ - \end{pmatrix} \longrightarrow \begin{pmatrix} * \\ * \; * \\ * \end{pmatrix} \quad \text{and} \quad T^* : \begin{pmatrix} * \\ * \; * \\ * \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \; 0 \\ * \end{pmatrix}$$

in the matrix notation, cf. Remark 2.1.3. T is defined for any conformal weight hence also T^* is defined for any conformal weight. Therefore we can suppose that T^* acts on $\mathcal{E}_{\mathbf{A}^{k+1}}[w']$. If T is a splitting operator then $\begin{pmatrix} f \\ - \; - \\ - \end{pmatrix} \xrightarrow{T} \begin{pmatrix} f \\ * \; * \\ * \end{pmatrix}$. The Proposition below says that then T^* is the identity on the bottom slot in the sense $\begin{pmatrix} 0 \\ 0 \; 0 \\ \rho \end{pmatrix} \xrightarrow{T^*} \begin{pmatrix} 0 \\ 0 \; 0 \\ \rho \end{pmatrix}$. In general, T^* depends also on the $\mathbb{Z}$, $\mathbb{W}$ and $\mathbb{Y}$–slots.

Henceforth we shall follow the conventions from Remark 2.1.3 i.e. we shall consider, for example, the top operator as a mapping on $\mathcal{E}_{\mathbf{a}^k}[w]$ (and not on the quotient space $(\mathcal{E}_{\mathbb{Y}})_{\mathbf{A}^{k+1}}[w']$). For a given point on the manifold M, a splitting operator at a point is a linear mapping between vector spaces $\Phi : U_1 \longrightarrow U_1 \oplus U_2$ satisfying $\mathrm{Proj}_{U_1} \circ \Phi = \mathrm{id}_{U_1}$ where Proj_{U_1} denotes the projection to $U_1 \oplus U_2 \longrightarrow U_1$. This has the following simple but important property.

Proposition. *Let us consider a vector space* $U = U_1 \oplus U_2$ *and a linear mapping* $\Phi : U_1 \longrightarrow U$ *such that* $\mathrm{Proj}_{U_1} \circ \Phi = \mathrm{id}_{U_1}$. *Then the dual mapping* $\Phi^* : U^* \longrightarrow U_1^*$ *satisfies the analogous property* $\Phi^*|_{U_1^*} = \mathrm{id}_{U_1^*}$.

Proof. Let $\alpha \in U_1^*$ and $w \in U_1$. Let us consider the projection $\iota = \mathrm{Proj}_{U_1} : U \longrightarrow U_1$ and the dual mapping $\iota^* : U_1^* \hookrightarrow U^*$. The definition of the dual mapping for Φ means $\langle \Phi^*|_{U_1^*}(\alpha); w\rangle = \langle \Phi^* \circ \iota^*(\alpha); w\rangle = \langle \iota^*(\alpha); \Phi(w)\rangle = \langle \alpha; \mathrm{Proj}_{U_1} \circ \Phi w\rangle = \langle \alpha; w\rangle$ because $\mathrm{Proj}_{U_1} \circ \Phi = \mathrm{id}_{U_1}$. From this, the proposition follows. □

It is straightforward to transform a formula for a differential operator $L : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ into a formula of $L^* : \mathcal{V}_2^*[-n] \longrightarrow \mathcal{V}_1^*[-n]$. Clearly $(L_1 + L_2)^* = L_1^* + L_2^*$ and it follows immediately from (2.84) that also $(L_1 L_2)^* = L_2^* L_1^*$. Therefore to derive the formula for L^* from a formula for L we need to know only formal adjoints of the derivative ∇_a and of tensorial actions of sections with various upper and lower indices.

Let us start with the covariant derivative $\nabla_a : \mathcal{V} \longrightarrow \mathcal{E}_a \otimes \mathcal{V}$. It follows from (2.83) by integration by parts that the formal adjoint is $(\nabla_a)^* = -\nabla^b : \mathcal{E}_b \otimes \mathcal{V}^*[-n+2] \longrightarrow \mathcal{V}^*[-n]$.

Let us consider a section $S_{\mathfrak{a}\mathfrak{A}}{}^{\mathfrak{b}\mathfrak{B}}$ of weight w and any systems $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{A}$, $\mathfrak{B}$ of tensor and tractor indices, respectively. (Note we have excluded spinor and tractor spinor indices here.) Then S can be considered as an operator $S_{\mathfrak{a}\mathfrak{A}}{}^{\mathfrak{b}\mathfrak{B}} : \mathcal{E}_{\mathfrak{b}\mathfrak{B}} \longrightarrow \mathcal{E}_{\mathfrak{a}\mathfrak{A}}[w]$. Using $\boldsymbol{g}$ and h, the formal adjoint is

$$S^* = S^{\mathfrak{a}\mathfrak{A}}{}_{\mathfrak{b}\mathfrak{B}} : \mathcal{E}_{\mathfrak{a}\mathfrak{A}}[-n - w + 2|\mathfrak{a}|] \longrightarrow \mathcal{E}_{\mathfrak{a}\mathfrak{A}}[-n + 2|\mathfrak{b}|]$$

where $|\mathfrak{a}|$ and $|\mathfrak{b}|$ are numbers of indices in $\mathfrak{a}$ and $\mathfrak{b}$, respectively.

We shall discuss spinor in the complex setting where we know the tractor spinor metric (1.43) explicitly. We will need only formal adjoints of $Y_\Lambda^\lambda \in \mathrm{Hom}\,(\mathcal{E}_\lambda[1], \mathcal{E}_\Lambda)$ and $X_\Lambda^\lambda \in \mathrm{Hom}\,(\mathcal{E}_\lambda, \mathcal{E}_\Lambda)$. It follows from (2.84) that $(Y^*)_\lambda^\Lambda \in$

Hom $(\mathcal{E}_\Lambda, \mathcal{E}_\lambda)$ and $(X^*)^\Lambda_\lambda \in$ Hom $(\mathcal{E}_\Lambda, \mathcal{E}_\lambda[1])$. More accurately, we should write e.g. $(Y^*)^\Lambda_\lambda \in$ Hom $(\mathcal{E}_\Lambda[-n], \mathcal{E}_\lambda[-n])$ but $Y^\lambda_\Lambda : \mathcal{E}_\lambda[w+1] \longrightarrow \mathcal{E}_\Lambda[w]$ for any conformal weight w. Let as emphasize that the star here denotes formal adjoints and *not* projections corresponding to TFP–components in the sense of 1.2.6. Using the (tractor) spinor metric (1.43) with the convention (1.40), we obtain $(X^*)^\Lambda_\lambda = X^\Lambda_\lambda$ and $(Y^*)^\Lambda_\lambda = Y^\Lambda_\lambda$. They satisfy

$$Y^*(X) = \text{id}, \quad X^*(Y) = (-1)^{n'+1}\text{id} \quad \text{and} \quad (X\boldsymbol{\beta}_a)^*(Y) = -\boldsymbol{\beta}_a \tag{2.85}$$

and $Y^*(Y) = X^*(X) = (X\boldsymbol{\beta}_a)^*(X) = 0$. The first two statements in the display have been mentioned in 1.2.4, the last one follows from the second because $\boldsymbol{\beta}^*_a = (-1)^{n'}\boldsymbol{\beta}_a$.

Examples of formal adjoints

We review Examples from 2.1.4, 2.1.5 and 2.1.6 and derive formulae for formal adjoints of the middle and top operators therein. The conformal weight in the formulae for formal adjoints shall denoted by w'. That is, if for example $\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} : \mathcal{E}_{\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{[A^0\mathbf{A}^k]}[w-k+1]$ is the bottom splitting then its formal adjoint is the (invariant) projection $\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} : \mathcal{E}_{[A^0\mathbf{A}^k]}[w'] \longrightarrow \mathcal{E}_{\mathbf{a}^k}[w'+k+1]$ where $w' = -n-w+k-1$. Let us note that the formal adjoints in the examples below are strongly invariant because the middle and top operators are strongly invariant.

I. Middle operators on tensors

Example 2.1.14. The middle operator $M^{\mathbf{a}_r}_{\mathbf{A}_r}$ and the formal adjoint $M^*{}^{\mathbf{A}_r}_{\mathbf{a}_r}$ are operators

$$M^{\mathbf{a}_r}_{\mathbf{A}_r} : \mathcal{E}(s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}(s_1, \ldots, s_{r-1})_{0\mathbf{A}^{s_r}_r}[w-s_r]$$
$$M^*{}^{\mathbf{A}_r}_{\mathbf{a}_r} : \mathcal{E}(s_1, \ldots, s_{r-1})_{0\mathbf{A}^{s_r}_r}[w'] \longrightarrow \mathcal{E}(s_1, \ldots, s_r)_0[w'+s_r]$$

where $\mathbf{a}_r = \mathbf{a}_r^{s_r}$ and $\mathbf{A}_r = \mathbf{A}_r^{s_r}$. $M_{\mathbf{A}_r}^{\mathbf{a}_r}$ is given by the formula (2.16) in Section 2.1.4. Since $w' = -n - w + 2s - s_r$ where $s = \sum_{i=1}^r s_i$, the scalar $n + w - s - s_r + r - 1$ from (2.16) is equal to $-w' + s - 2s_r + r - 1$. We obtain the formula

$$M^{*\mathbf{A}_r}_{\ \ \mathbf{a}_r} F_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{A}_r} = \text{Proj}\left[(-w'+s-2s_r+r-1)\mathbb{Z}^{\mathbf{A}_r}_{\mathbf{a}_r} + s_r \nabla_{a_r^1} \mathbb{X}^{\mathbf{A}_r}_{\dot{\mathbf{a}}_r}\right] F_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{A}_r}$$

where $F_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{A}_r} \in \mathcal{E}(s_1, \ldots, s_{r-1})_{0\,\mathbf{A}_r^{s_r}}[w']$. Here Proj denotes the projection on $\mathcal{E}(s_1, \ldots, s_r)_0[w' + s_r]$ and $\nabla_{a_r^1}$ acts on $\mathbb{X}^{\mathbf{A}_r}_{\dot{\mathbf{a}}_r} F_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \dot{\mathbf{A}}_r}$. If we want to work directly with the relevant slots of F i.e. if we consider F in the form

$$F_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{A}_r} = \mathbb{Y}^{\dot{\mathbf{a}}_r}_{\mathbf{A}_r} \sigma_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \dot{\mathbf{a}}_r} + \mathbb{Z}^{\mathbf{a}_r}_{\mathbf{A}_r} \mu_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{a}_r} + \{\text{remaining slots}\},$$

the formula for M^* becomes

$$M^{*\mathbf{A}_r}_{\ \ \mathbf{a}_r} F_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{A}_r} = \text{Proj}\left[(-w'+s-2s_r+r-1)\mu_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \mathbf{a}_r} + \nabla_{a_r^1} \sigma_{\mathbf{a}_1 \cdots \mathbf{a}_{r-1} \dot{\mathbf{a}}_r}\right].$$

The operator $M_{\mathbf{A}_r}^{\mathbf{a}_r}$ acts on $\mathcal{E}(s_1, \ldots, s_r)_0[w]$ which means $s_{r-1} \geq s_r$. Hence if we want to apply M^* to the space $\mathcal{E}(s_1, \ldots, s_{r-1})_{0\,\mathbf{A}_r^{s_r}}[w']$ we have to ensure $s_{r-1} \geq s_r$. (If this is not satisfied one has to use formal adjoint of another (middle) operator, cf. Example 2.1.4.) Finally, the formula for M^* simplifies for k–forms i.e. when $r = 1$ and $s_1 = k$. We obtain

$$M^{*\mathbf{A}}_{\ \ \mathbf{a}} F_{\mathbf{A}} = -\left[(w' + k)\mu_{\mathbf{a}} - \nabla_{a^1} \sigma_{\dot{\mathbf{a}}}\right]$$

for a section $F_{\mathbf{A}} = \mathbb{Y}^{\dot{\mathbf{a}}}_{\mathbf{A}} \sigma_{\dot{\mathbf{a}}} + \mathbb{Z}^{\mathbf{a}}_{\mathbf{A}} \mu_{\mathbf{a}} + \{\mathbb{W}, \mathbb{X}\text{–slots}\} \in \mathcal{E}_{\mathbf{A}^k}[w']$ where $\mathbf{a} = \mathbf{a}^k$ and $\mathbf{A} = \mathbf{A}^k$ and we skew over $[a^1 \dot{\mathbf{a}}]$ on the right hand side.

Example 2.1.15. The complete middle operator $M^{\mathbf{a}\mathbf{b}}_{\mathbf{A}\mathbf{B}} = M^{\mathbf{a}}_{\mathbf{A}} M^{\mathbf{b}}_{\mathbf{B}}$ on the space $\mathcal{E}(k, l)_0[w]$, $n' \geq k \geq l \geq 1$ from Example 2.1.3, and the formal adjoint of this operator, are

$$M^{\mathbf{a}\mathbf{b}}_{\mathbf{A}\mathbf{B}} : \mathcal{E}(k, l)_0[w] \longrightarrow \mathcal{E}_{\mathbf{A}^k \mathbf{B}^l}[w - k - l]$$

$$M^{*\mathbf{A}\mathbf{B}}_{\ \ \mathbf{a}\mathbf{b}} : \mathcal{E}_{\mathbf{A}\mathbf{B}}[w'] \longrightarrow \mathcal{E}(k, l)_0[w' + k + l]$$

where $\mathbf{a} = \mathbf{a}^k$, $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{b} = \mathbf{b}^l$, $\mathbf{B} = \mathbf{B}^l$. Sections $F_{\mathbf{AB}} \in \mathcal{E}_{\mathbf{A}^k\mathbf{B}^l}[w']$ are of the form

$$F_{\mathbf{AB}} = \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\mathbb{Y}_{\mathbf{B}}^{\dot{\mathbf{b}}}\sigma_{\dot{\mathbf{a}}\dot{\mathbf{b}}} + \mathbb{Y}_{\mathbf{A}}^{\dot{\mathbf{a}}}\mathbb{Z}_{\mathbf{B}}^{\mathbf{b}}\xi_{\dot{\mathbf{a}}\mathbf{b}} + \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\mathbb{Y}_{\mathbf{B}}^{\dot{\mathbf{b}}}\tilde{\xi}_{\mathbf{a}\dot{\mathbf{b}}} + \mathbb{Z}_{\mathbf{A}}^{\mathbf{a}}\mathbb{Z}_{\mathbf{B}}^{\mathbf{b}}\mu_{\mathbf{ab}} + \{\text{remaining slots}\}.$$

Since $w' = -n - w + k + l$, the scalars from Example 2.1.3 are $c_1 = w' - k$ and $c_2 = w' - l + 1$. Clearly the projections $\mathbb{Z}_{\mathbf{a}}^{\mathbf{A}}\mathbb{Z}_{\mathbf{b}}^{\mathbf{B}}$, $\mathbb{Z}_{\mathbf{a}}^{\mathbf{A}}\mathbb{X}_{\dot{\mathbf{b}}}^{\mathbf{B}}$, $\mathbb{X}_{\dot{\mathbf{a}}}^{\mathbf{A}}\mathbb{Z}_{\mathbf{b}}^{\mathbf{B}}$ and $\mathbb{X}_{\dot{\mathbf{a}}}^{\mathbf{A}}\mathbb{X}_{\dot{\mathbf{b}}}^{\mathbf{B}}$, applied to $F_{\mathbf{AB}}$, yield $\mu_{\mathbf{ab}}$, $\frac{1}{l}\tilde{\xi}_{\mathbf{a}\dot{\mathbf{b}}}$, $\frac{1}{k}\xi_{\dot{\mathbf{a}}\mathbf{b}}$ and $\frac{1}{kl}\sigma_{\dot{\mathbf{a}}\dot{\mathbf{b}}}$, respectively. Therefore one can transform the formula (2.21) for $M_{\mathbf{AB}}^{\mathbf{a}\,\mathbf{b}}$ into the formula

$$\begin{aligned} M^{*}{}^{\mathbf{AB}}_{\mathbf{a}\,\mathbf{b}}F_{\mathbf{AB}} =&\mathrm{Proj}\Big\{(w'-k)(w'-l+1)\mu_{\mathbf{ab}} + \Big[(w'-l+1)\nabla_{a^1}\xi_{\dot{\mathbf{a}}\mathbf{b}} - l\nabla_{b^1}\xi_{\dot{\mathbf{a}}a^1\dot{\mathbf{b}}}\Big] \\ &+ (w'-k)\nabla_{b^1}\tilde{\xi}_{\mathbf{a}\dot{\mathbf{b}}} + \Big[\nabla_{b^1}\nabla_{a^1}\sigma_{\dot{\mathbf{a}}\dot{\mathbf{b}}} + (w'-l+1)P_{a^1b^1}\sigma_{\dot{\mathbf{a}}\dot{\mathbf{b}}}\Big]\Big\} \end{aligned}$$

where we skew over $[a^1\dot{\mathbf{a}}]$ and $[b^1\dot{\mathbf{b}}]$ and Proj denotes the projection into $\mathcal{E}(k,l)_0[w'+k+l]$. Let us note the term $\nabla_{b^1}\xi_{\dot{\mathbf{a}}a^1\dot{\mathbf{b}}}$ is the formal adjoint of $\nabla^p f_{b^1\dot{\mathbf{a}}p\dot{\mathbf{b}}}$ for $f_{\mathbf{ab}} \in \mathcal{E}(k,l)_0[w]$ from the formula (2.21). Indeed, relabelling indices and integration by parts yields

$$\int \xi^{\dot{\mathbf{a}}\mathbf{b}}(\nabla^p f_{b^1\dot{\mathbf{a}}p\dot{\mathbf{b}}}) = \int \xi^{\dot{\mathbf{a}}a^1\dot{\mathbf{b}}}(\nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}}) = \int -(\nabla^{b^1}\xi^{\dot{\mathbf{a}}a^1\dot{\mathbf{b}}})f_{\mathbf{ab}}. \tag{2.86}$$

II. Top operators on tensors

Formulae for the top operator T on $\mathcal{E}(k)[w] = \mathcal{E}_{\mathbf{a}^k}[w]$ and $\mathcal{E}(k,l)_0[w]$, $n' \geq k \geq l \geq 1$ are derived in 2.1.5. As therein, we shall use the abbreviations $\mathbf{a} = \mathbf{a}^k$, $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{b} = \mathbf{b}^l$, $\mathbf{B} = \mathbf{B}^l$.

Example 2.1.16. The top operator $T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}$ on (density valued) forms computed in Example 2.1.6 and the formal adjoint $T^{*A^0\mathbf{A}}_{\quad\;\mathbf{a}}$ are operators

$$\begin{aligned} T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} &: \mathcal{E}_{\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{[A^0\mathbf{A}^k]}[w-k-1] \\ T^{*A^0\mathbf{A}}_{\quad\;\mathbf{a}} &: \mathcal{E}_{[A^0\mathbf{A}^k]}[w'] \longrightarrow \mathcal{E}_{\mathbf{a}^k}[w'+k-1] \end{aligned}$$

where $w' = -n - w + k + 1$. Using the formula (2.54) for $T_{A^0\mathbf{A}}{}^{\mathbf{a}}$, we obtain the formula

$$T^{*A^0\mathbf{A}}{}_{\mathbf{a}} F_{A^0\mathbf{A}} = -\frac{n+2w'}{k+1}\Big[(n+w'-k-1)(w'+k-1)\rho_{\mathbf{a}} + (k+1)(w'+k-1)\nabla^p \mu_{p\mathbf{a}} - (n+w'-k-1)\nabla_{a^1}\nu_{\dot{\mathbf{a}}}\Big] + \frac{1}{k+1}\Big[(w'+k-1)\big(\Delta-(n+w'-1)P\big)\sigma_{\mathbf{a}} - k(n+2w'-2)\big(\nabla_{a^1}\nabla^p - (w'+k-1)P_{a^1}^p\big)\sigma_{p\dot{\mathbf{a}}}\Big]$$

for the formal adjoint where we skew over indices $[a^1\dot{\mathbf{a}}]$ on the right hand side and the section $F_{A^0\mathbf{A}} \in \mathcal{E}_{[A^0\mathbf{A}^k]}[w']$ is of the form

$$F_{A^0\mathbf{A}} = \mathbb{Y}_{A^0\mathbf{A}}{}^{\mathbf{a}}\sigma_{\mathbf{a}} + \mathbb{Z}_{A^0\mathbf{A}}^{a^0\,\mathbf{a}}\mu_{a^0\mathbf{a}} + \mathbb{W}_{A^0\mathbf{A}}{}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}} + \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\rho_{\mathbf{a}}\,.$$

Example 2.1.17. Formulae for the top operators on $\mathcal{E}(k,l)_0[w]$ computed in Examples 2.1.7 and 2.1.8 are more complicated hence we can expect the same for their formal adjoints. According to (2.25), these are the operators

$$\tilde{T}_{A^0\mathbf{A}}{}^{\mathbf{a}}, \tilde{\tilde{T}}_{A^0\mathbf{A}}{}^{\mathbf{a}} : \mathcal{E}(k,l)_0[w] \longrightarrow \mathcal{E}_{[A^0\mathbf{A}^k]b^l}[w-k-1]$$
$$\tilde{T}^{*A^0\mathbf{A}}{}_{\mathbf{a}}, \tilde{\tilde{T}}^{*A^0\mathbf{A}}{}_{\mathbf{a}} : \mathcal{E}_{[A^0\mathbf{A}^k]b^l}[w'] \longrightarrow \mathcal{E}(k,l)_0[w'+k-1]$$

where $w' = -n - w + k + 2l + 1$. Let us consider the operator $\tilde{T}_{A^0\mathbf{A}}{}^{\mathbf{a}}$ first. The scalars in the formula (2.55) in Example 2.1.7 now are

$$c_1 = -w' - k + l + 1, \quad c_2 = -w' + 2, \quad d = -n - 2w' + 2l \quad \text{and}$$
$$\tilde{d} := w - l = -n - w' + k + l + 1.$$

We shall consider $F_{A^0\mathbf{A}\mathbf{b}} \in \mathcal{E}_{[A^0\mathbf{A}^k]\mathbf{b}^l}[w'+k-1]$ in the form

$$F_{A^0\mathbf{A}\mathbf{b}} = \mathbb{Y}_{A^0\mathbf{A}}{}^{\mathbf{a}}\sigma_{\mathbf{a}\mathbf{b}} + \mathbb{Z}_{A^0\mathbf{A}}^{a^0\,\mathbf{a}}\mu_{a^0\mathbf{a}\mathbf{b}} + \mathbb{W}_{A^0\mathbf{A}}{}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}\mathbf{b}} + \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\rho_{\mathbf{a}\mathbf{b}}\,. \tag{2.87}$$

Then the formula (2.55) for $\tilde{T}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}$ yields

$$\begin{aligned}
\tilde{T}^{*A^0}{}^{\mathbf{A}}_{\mathbf{a}} F_{A^0\mathbf{A}\mathbf{b}} = \frac{1}{k+1}\mathrm{Proj}\Big\{ & d\tilde{d}c_1c_2\,\rho_{\mathbf{ab}} + d\tilde{d}\Big[c_2\nabla_{a^1}\nu_{\dot{\mathbf{a}}\mathbf{b}} - l\nabla_{b^1}\nu_{\dot{\mathbf{a}}a^1\dot{\mathbf{b}}}\Big] \\
& - (k+1)dc_1\Big[c_2\nabla^p\mu_{p\mathbf{ab}} - l\nabla_{b^1}\mu^p{}_{\mathbf{a}p\dot{\mathbf{b}}}\Big] \\
+ \Big[-c_1c_2\big(\Delta + (\tilde{d}-k)P)\big)\sigma_{\mathbf{ab}} & + 2lc_1\big(\nabla_{b^1}\nabla^p + c_2P^p_{b^1}\big)\sigma_{\mathbf{a}p\dot{\mathbf{b}}} \qquad (2.88) \\
+ k(d+2)\big[c_2\big(\nabla_{a^1}\nabla^p & + c_1P^p_{a^1}\big)\sigma_{p\dot{\mathbf{a}}\mathbf{b}} - l\nabla_{b^1}\nabla^p\sigma_{p\dot{\mathbf{a}}a^1\dot{\mathbf{b}}}\big] \\
- kl(d+2)\big[\nabla_{b^1}\nabla_{a^1} & + c_2P_{a^1b^1}\big]\sigma^p{}_{\dot{\mathbf{a}}p\dot{\mathbf{b}}}\Big]\Big\}
\end{aligned}$$

where we skew over $[a^1\dot{\mathbf{a}}]$ and $[b^1\dot{\mathbf{b}}]$ and Proj denotes the projection to $\mathcal{E}(k,l)_0[w'+k-1]$. Note we have used the relation (2.86) with $\xi_{\dot{\mathbf{a}}\mathbf{b}}$ replaced by $\nu_{\dot{\mathbf{a}}\mathbf{b}}$.

The formula for $\tilde{\tilde{T}}$ from Example 2.1.8 is less complicated. Recall $k = l = n'$ and we are in the dimension $n = 2n'$ now. The weight w' is now $w' = -w + n + 1$ thus $w - n' = -w' + 1$. The formula (2.57) for $\tilde{\tilde{T}}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}$ yields

$$\begin{aligned}
\tilde{\tilde{T}}^{*A^0}{}^{\mathbf{A}}_{\mathbf{a}} F_{A^0\mathbf{A}\mathbf{b}} = \frac{w'-2}{n'+1}\mathrm{Proj}\Big\{ & 2(w'-1)\Big[-(w'-2)\rho_{\mathbf{ab}} - (n'+1)\nabla^p\mu_{p\mathbf{ab}} + \nabla_{a^1}\nu_{\dot{\mathbf{a}}\mathbf{b}}\Big] \\
& + \Big[\big(\Delta - (w'+n'-1)P\big)\sigma_{\mathbf{ab}} - n\big(\nabla_{a^1}\nabla^p - (w'-2)P^p_{a^1}\big)\sigma_{p\dot{\mathbf{a}}\mathbf{b}}\Big]\Big\}
\end{aligned}$$

where we skew over $[a^1\dot{\mathbf{a}}]$ and Proj denotes the projection to $\mathcal{E}(n',n')_0[w'+n'-1]$. Recall we use $F_{A^0\mathbf{A}\mathbf{b}}$ from (2.87).

III. Top operators for spinors

Formulae for the spinor top operator T^λ_Λ on $\mathcal{E}(\frac{1}{2};k)_0[w]$ and $\mathcal{E}(\frac{1}{2};k,l)_0[w]$, $n' \geq k \geq l \geq 1$ are computed in 2.1.6. As usually, we shall use the abbreviations $\mathbf{a} = \mathbf{a}^k$, $\mathbf{A} = \mathbf{A}^k$ and $\mathbf{b} = \mathbf{b}^l$, $\mathbf{B} = \mathbf{B}^l$. In the complex setting, we use (2.85) to compute formulae for T^*. We cannot do this in the real case, however both formulae below are actually invariant for both scalars. (This is easy to check using (1.37)). Hence in the real case, the formulae below can play the role of formal adjoints in all subsequent computations.

Example 2.1.18. We will consider only the version $\tilde{T}$ from (2.64) which is nontrivial only for $k < \frac{n}{2}$. (The version $\tilde{\tilde{T}}$ is similar.) For $k < \frac{n}{2}$, the top operator T on $\mathcal{E}(\frac{1}{2}; k)_0[w]$, computed in Example 2.1.11, and the formal adjoint T^* are the operators

$$T^\lambda_\Lambda : \mathcal{E}(\frac{1}{2}; k)_0[w] \longrightarrow \mathcal{E}_{\Lambda \mathbf{a}^k}[w-1]$$
$$T^{*\Lambda}_{\ \lambda} : \mathcal{E}_{\Lambda \mathbf{a}^k}[w'] \longrightarrow \mathcal{E}(\frac{1}{2}; k)_0[w']$$

where $w' = -n - w + 2k + 1$. Therefore $n + 2(w-k) - 2 = -(n + 2(w'-k))$ and $n + w - 2k = -(w' - 1)$. From (2.72), we obtain the formula

$$T^* F_{\mathbf{a}} = \mathrm{Proj}\Big\{ \big(n + 2(w'-k)\big)(w'-1)\rho_{\lambda \mathbf{a}} \\ - 2\boldsymbol{\beta}^p \big[(w'-1)\nabla_p \sigma_{\mathbf{a}} + k \nabla_{a^1} \sigma_{p\dot{\mathbf{a}}}\big]\Big\}$$

where Proj denotes the projection to the target space of T^* and we consider the section $F_{\mathbf{a}}$ in the form

$$F_{\mathbf{a}} = Y\sigma_{\mathbf{a}} + X\rho_{\mathbf{a}} \in \mathcal{E}_{\Lambda \mathbf{a}}[w'].$$

Example 2.1.19. Assume $k < \frac{n}{2}$. The top operator T on $\mathcal{E}(\frac{1}{2}; k, l)_0[w]$, computed in Example 2.1.12, and the formal adjoint T^* are the operators

$$T^\lambda_\Lambda : \mathcal{E}(\frac{1}{2}; k, l)_0[w] \longrightarrow \mathcal{E}_\Lambda(k, l)_0[w-1]$$
$$T^{*\Lambda}_{\ \lambda} : \mathcal{E}_\Lambda(k, l)_0[w'] \longrightarrow \mathcal{E}(\frac{1}{2}; k, l)_0[w']$$

where $w' = -n - w + 2k + 2l + 1$. Therefore the scalars used in (2.74) are now

$$n + 2(w-k-l) - 2 = -\big(n + 2(w-k-l)\big), \quad n + w - l - 2k = -(w' - l - 1),$$
$$\text{and} \quad n + w - k - 2l + 1 = -(w' - k - 2).$$

Now from (2.74), we can compute the formula

$$
\begin{aligned}
T^* F_{\mathbf{ab}} = -\mathrm{Proj}\Big\{ \big(n + 2(w' - k - l)\big)(w' - l - 1)(w' - k - 2)\rho_{\mathbf{ab}} \\
- 2\boldsymbol{\beta}^p \Big[(w' - k - 2)\big[(w' - l - 1)\nabla_p \sigma_{\mathbf{ab}} + k \nabla_{a^1} \sigma_{p\dot{\mathbf{a}}\mathbf{b}}\big] \\
+ l(w' - l)\nabla_{b^1} \sigma_{\mathbf{a}p\dot{\mathbf{b}}}\Big]\Big\}
\end{aligned}
$$

where Proj denotes the projection to the target space of T^* and we consider the section $F_{\mathbf{ab}}$ in the form

$$F_{\mathbf{ab}} = Y\sigma_{\mathbf{ab}} + X\rho_{\mathbf{ab}} \in \mathcal{E}_\Lambda(k, l)_0[w'].$$

2.1.9. Variations of the middle and top operators. The operator *DSplit* defined in 2.1.7 turns out to be a good choice for the purpose of this thesis but there are many (splitting) operators with the projecting part different from *DSplit*. Here we review some useful modifications of the middle and top operators defined in 2.1.4 and 2.1.5, respectively. We shall consider only the tensor bundle $V = E(s_1, \cdots, s_r)_0[w]$ but a generalisation to spinors is straightforward.

1. *Versions of M and T applicable to any form index.* Whereas the bottom operator B can be applied to an arbitrary form index $\mathbf{a}_1, \cdots, \mathbf{a}_r$ of V, the formula for the middle operator M, defined by (2.16), can be applied only to the "shortest" form index $\mathbf{a}_r = \mathbf{a}_r^{s_r}$ of V. Similarly, the formula for T, defined by (2.25), can be applied only to the longest one $\mathbf{a}_1 = \mathbf{a}_1^{s_1}$. Following Example 2.1.4, we can construct the operator $\check{M}_{\mathbf{A}_i}^{\mathbf{a}_i}$ (where $\mathbf{A}_i$ and $\mathbf{a}_i$ have the valence s_i) for an arbitrary $i \in \{1, \ldots, r\}$. That is,

$$
\begin{aligned}
&\check{M}_{\mathbf{B}_i}^{\mathbf{a}_i} : \mathcal{E}(s_1, \ldots, s_r)_0[w] \longrightarrow \mathcal{E}_{\mathbf{B}_i}(s_1, \ldots, \hat{s}_i, \ldots, s_r)_0[w - s_i] \\
&\check{M}_{\mathbf{B}_i}^{\mathbf{a}_i} f_{\mathbf{a}_1, \ldots, \mathbf{a}_r} = \mathrm{Proj}\, \mathbb{Z}_{\mathbf{a}_{i+1}}^{\mathbf{B}_{i+1}} \cdots \mathbb{Z}_{\mathbf{a}_r}^{\mathbf{B}_r} M_{\mathbf{B}_i}^{\mathbf{b}_i} \cdots M_{\mathbf{B}_r}^{\mathbf{b}_r} f_{\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}\mathbf{b}_i \ldots \mathbf{b}_r}
\end{aligned}
\tag{2.89}
$$

where Proj is the corresponding tensor projection, $\hat{}$ indicates the missing form index and $M_{\mathbf{B}_i}^{\mathbf{b}_i}$ on the right hand side is given by (2.1.4). Using Proposition 2.1.4, one can easily derive the conditions for $\check{M}_{\mathbf{B}_i}^{\mathbf{a}_i}$ to be a splitting operator. Clearly an analogous procedure yields a top (splitting) operator $\check{T}_{A^0_i \mathbf{A}_i}{}^{\mathbf{a}_i}$ on V for any $i \in \{1, \dots, r\}$.

The tractor D–operator D_A plays the role of the top operator on $\mathcal{E}_{\mathfrak{B}}[w]$. A special case of the previous construction for top operators is the operator on forms $f_{\mathbf{a}} \mapsto \mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} D_B T_{A^0\mathbf{A}}{}^{\mathbf{b}} f_{\mathbf{b}}$ for $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[w]$ where the form indices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{A}$ are of the valence $k \geq 1$. See [21] for the special case with $k = 1$. For higher valences, this operator can be simplified by an appropriate (invariant) curvature modification of the X_B slot. After some computation, we obtain the (strongly invariant) operator

$$\check{T}_B : \mathcal{E}_{\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{B\mathbf{a}^k}[w-1]$$

given by the formula

$$\begin{aligned}
\check{T}_B f_{\mathbf{a}} =& (n+2(w-k)-2)\Big\{ w(w-k-1)(n+w-2k) Y_B f_{\mathbf{a}} \\
&+ Z_B^b \Big[w(n+w-2k) \nabla_{\{b} f_{\mathbf{a}\}_0} + \frac{kw(w-k-1)}{n-k+1} \boldsymbol{g}_{ba^1} \nabla^p f_{p\dot{\mathbf{a}}} \\
&+ (w-k-1)(n+w-2k) \nabla_{[b} f_{\mathbf{a}]} \Big] \Big\} \qquad (2.90) \\
&- X_B \Big\{ (w-1)(n+w-2k) \big(\Delta + (w-k)P\big) f_{\mathbf{a}} \\
&\quad k(n-2k) \big(\nabla_{a^1} \nabla^p + (n+w-2k) P_{a^1}{}^p \big) f_{p\dot{\mathbf{a}}} \Big\}
\end{aligned}$$

where we skew over $[a^1 \dot{\mathbf{a}}]$ and $\{b\mathbf{a}\}_0$ denotes the projection to $\mathcal{E}(k,1)_0[w]$ of $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{E}_b[w]$. (See Section 3.2 for details.)

2. *Alternative middle operator* $\tilde{M}$. In (2.1), we omitted a candidate for the splitting on $\mathcal{E}_{\mathbf{a}^k}[w]$, namely $\tilde{M} : f \mapsto \begin{pmatrix} 0 \\ 0 \; f \\ * \end{pmatrix}$. This operator can be easily obtained from M using the volume forms $\boldsymbol{\epsilon}_{\mathbf{a}^n}$ and $\boldsymbol{\epsilon}_{\mathbf{A}^{n+2}}$ as the composition

$$\begin{aligned}
\tilde{M}_{\mathbf{A}^{k+2}}^{\mathbf{a}^k} f_{\mathbf{a}^k} &: \mathcal{E}_{\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{\mathbf{A}^{k+2}}[w-k] \\
\tilde{M}_{\mathbf{A}^{k+2}}^{\mathbf{a}^k} f_{\mathbf{a}^k} &= \boldsymbol{\epsilon}_{\mathbf{A}^{k+2}}{}^{\mathbf{B}^{n-k}} M_{\mathbf{B}^{n-k}}^{\mathbf{b}^{n-k}} \boldsymbol{\epsilon}_{\mathbf{b}^{n-k}}{}^{\mathbf{c}^k} f_{\mathbf{c}^k}.
\end{aligned}$$

Let us note the volume forms are actually not necessary for $\tilde{M}$, see the explicit formulae in the Example below. (See [11] for another construction of

$\tilde{M}$.) It can be generalised to the bundle V: the middle operator (2.16) gives rise to $\tilde{M}_{A'_1A^0_1\mathbf{A}_1}{}^{\mathbf{a}_1}$ (where $\mathbf{a}_1 = \mathbf{a}_1^{s_1}$ i.e. the "longest" form index in V) and $\check{M}^{\mathbf{a}_i}_{\mathbf{A}_i}$ defined by (2.89) yields $\tilde{M}_{A'_iA^0_i\mathbf{A}_i}{}^{\mathbf{a}_i}$ for any $i \in \{1,\dots,r\}$.

Example. We shall demonstrate formulae for $\tilde{M}$ on the spaces $\mathcal{E}_{\mathbf{a}^k}[w]$ and $\mathcal{E}(k,l)_0[w]$, $k \geq l$. Using the notation $\mathbf{a} = \mathbf{a}^k$, $\mathbf{A} = \mathbf{A}^k$, $\mathbf{b} = \mathbf{b}^l$ and $\mathbf{B} = \mathbf{B}^l$, it is easy to compute directly or even easier to extract from the slots $\mathbb{Y}$ and $\mathbb{Z}$ of (2.54) and (2.55), that

$$\begin{aligned}\tilde{M}_{A'A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{a}} &= w\mathbb{W}_{A'A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{a}} + \mathbb{X}_{A'A^0\mathbf{A}}{}^{a^0\mathbf{a}} \nabla_{a^0} f_{\mathbf{a}} \\ &\in \mathcal{E}_{[A'A^0\mathbf{A}]}[w-k] \cong \mathcal{E}_{\mathbf{A}^{k+2}}[w-k]\end{aligned}$$

for $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[w]$. Analogously, we obtain

$$\begin{aligned}\tilde{M}_{A'A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} &= (w-l)c\mathbb{W}_{A'A^0\mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} + \mathbb{X}_{A'A^0\mathbf{A}}{}^{a^0\mathbf{a}} \left[c\nabla_{a^0} f_{\mathbf{ab}} - l\boldsymbol{g}_{a^0b^1}\nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}}\right] \\ &\in \mathcal{E}_{[A'A^0\mathbf{A}]\mathbf{b}}[w-k] \cong \mathcal{E}_{\mathbf{A}^{k+2}\mathbf{b}}[w-k]\end{aligned}$$

for $f_{\mathbf{ab}} \in \mathcal{E}(k,l)_0[w]$. Here $c_l = n + w - k - 2l + 1$ and we skew over $[b^1\dot{\mathbf{b}}]$ on the right hand side. Note it is really sufficient to look at slots of (2.54) and (2.55) because $\tilde{M}$ is of the first order and determined by a formula for M hence no curvature modification in the slot $\mathbb{X}_{A'A^0\mathbf{A}}{}^{a^0\mathbf{a}}$ can appear.

3. *Middle operator for a part of form indices.* Let us "divide" the form index $\mathbf{a}^k$ into $\mathbf{a}^{k-l}$, $1 \leq l \leq k$ and $\mathbf{a}^{k-l,l} = [a^{k-l+1}\cdots a^k]$ i.e. $\mathbf{a}^k = [\mathbf{a}^{k-l}\mathbf{a}^{k-l,l}]$ (see Section 3.2 for details about this notation). We define the middle operator $\overline{M}$ on $\mathcal{E}_{\mathbf{a}^k}[w]$ which "puts" only $\mathbf{a}^{k-l,l}$ to the $\mathbb{Z}$–slot in the following way:

$$\begin{aligned}\overline{M}_{\mathbf{B}^l}^{\mathbf{a}^{k-l,l}} &: \mathcal{E}_{\mathbf{a}^k}[w] \longrightarrow \mathcal{E}_{\mathbf{a}^{k-l}\mathbf{B}^l}[w-l] \\ \overline{M}_{\mathbf{B}^l}^{\mathbf{a}^{k-l,l}} f_{\mathbf{a}^k} &= (n+w-2k)\mathbb{Z}_{\mathbf{B}^l}^{\mathbf{b}^l} f_{\mathbf{a}^{k-l}\mathbf{b}^l} - l\mathbb{X}_{B^1\dot{\mathbf{B}}^l}{}^{\dot{\mathbf{b}}^l}\nabla^{b^1} f_{\mathbf{a}^{k-l}\mathbf{b}^l}\,.\end{aligned}$$

Conformal invariance follows from properties of projecting parts of M, see Theorem 2.1.4. Clearly $\overline{M}$ can be generalised to various versions on V. We will use $\overline{M}$ on $\mathcal{E}_{\mathbf{a}^k}[k+1]$ later in Section 3.2.

2.2 *DSplit* and the gBGG splitting operator

Every irreducible bundle V is of the form $V \subseteq E_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w]$ where "$\subseteq$" is equality in the tensor case. That is, we constructed the operator $DSplit_b^t(m)$ in Section 2.1 for all irreducibles. The main aim of this section is to determine parameters t, m and b (depending on $r_1, \ldots, r_{n'}$, w) such that $DSplit_b^t(m)$ is a splitting operator on $E_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w]$, suitable for our implementation of the curved translation principle outlined in 1.3.6. According to Theorem 2.1.7, we have to ensure the parameters satisfy $s(t, m) > 0$.

Let us remind the gBGG splitting operator defined in 1.3.7. This is a splitting operator for V in the standard pattern with the same projecting part as the splitting from [20]. The latter is characterised by the condition "$\partial\partial^* + \partial^*\partial = 0$" (in the notation of [20]) but we call any splitting operator with the same projecting part the gBGG splitting operator. This operator is unique in the flat case, see Appendix A.

If V is in a standard pattern, we will find t, m and b such that $DSplit_b^t(m)$ is the gBGG splitting operator. On the way to this we will show on which position in the pattern (see 1.3.3) V appears. We will do the latter also for singular and non–standard patterns. We do not have any distinguished splitting operator for these two cases. However, $DSplit_b^t(m)$ is well–defined in all cases and during the discussion on the standard positions, we also suggest appropriate parameters t, m and b for singular and non–standard ones.

Since we will need the symbolism of Dynkin diagrams during this section, henceforth we assume the complex setting. But most of the results hold also in the real case. We shall comment upon differences between the real and complex case briefly in 2.2.3.

Throughout this chapter we shall use the notation from 1.1.3 and 1.3.3. We will assume $w \in \mathbb{AW}$, see (1.61). Some of these weights correspond to bundles with no nontrivial operator in the flat case and we do not need any splitting for these bundles.

Quantities for the Weyl's construction

Below we will need various quantities determined by the representation of the form

$$\mathbb{E}_{(\pm)}(l; s_1, \ldots, s_r)_0[w] = \mathbb{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w].$$

Beside $s, s^j, r, r_i \in \frac{1}{2}\mathbb{N}_0$, $s_i, \tilde{s}^j \in \mathbb{N}_0$, see (1.1) and Table 1.3, and the conformal weight w, we will use also

$$\begin{aligned} t_i &= w - s + i \\ o_i &= w - s - \tilde{r}^{i+1} + i + 1 \\ o_i^* &= w - s + \tilde{r}^{i+1} + (n - i - 1) \\ o &= 2(w - s) + n \\ w^* &= -w + 2s - n \end{aligned} \tag{2.91}$$

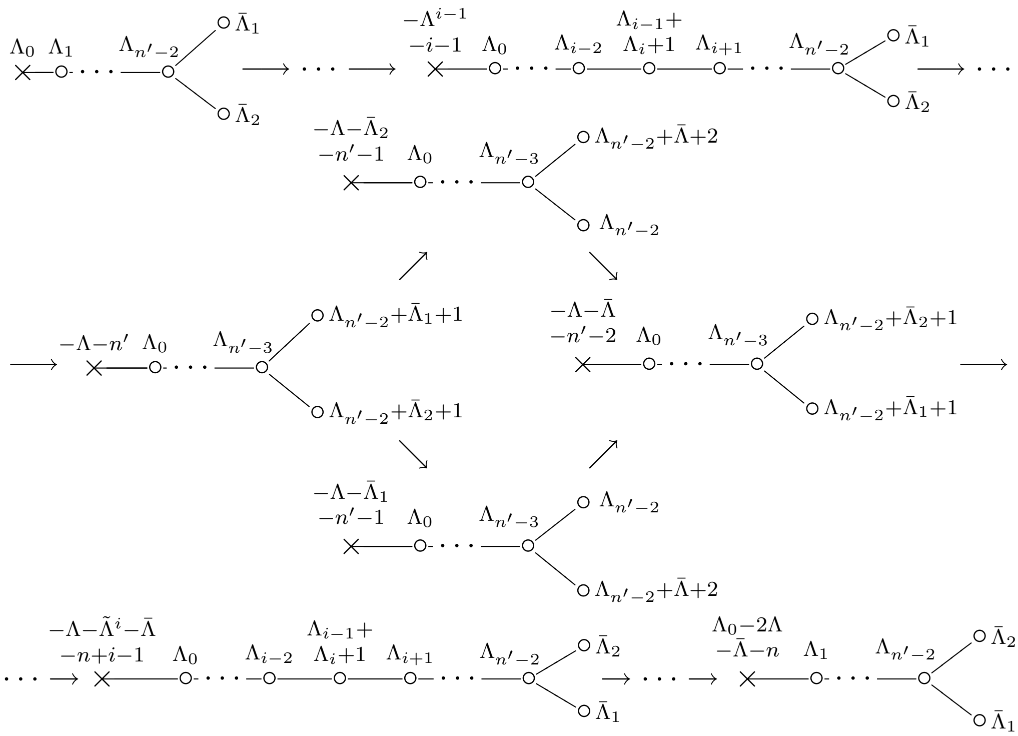
for $0 \leq i \leq n'$.

2.2.1. Pattern and identification of $V \subseteq E_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w]$ therein. Our aim is to determine the position of the bundle V in the pattern. As this position will depend only on $r_1, \cdots, r_{n'}$, w and possibly the sign, we can consider this as the position of $E_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w]$. We will describe how this position depends on w with remaining parameters fixed.

The pattern corresponds to the Hasse graph structure on the subset $W^{\mathfrak{p}} \subseteq W$ of the Weyl group, see Table 1.1. We obtain the pattern for the weight Λ by passing the Hasse graph structure for $w \in W^{\mathfrak{p}}$ to the set of weights $\{w.\Lambda \mid w \in W^{\mathfrak{p}}\}$, see 1.3.3 for more details. Looking at $w \in W^{\mathfrak{p}}$ expressed as composition of simple reflections in Table 1.1, we can easily compute $w.\Lambda$ because simple reflections act in a simple way (see [2]) on weights given by Dynkin diagrams with coefficients. The result is displayed in Table 2.1.

The pattern with Dynkin diagrams

where $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor - 1$ corresponds either to the ith or $(n-i)$th degree



Remark: This is the pattern from 1.3.3 for $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$. If $\bar{\Lambda}_1 \leq \bar{\Lambda}_2$, we have to interchange the positions in the middle.

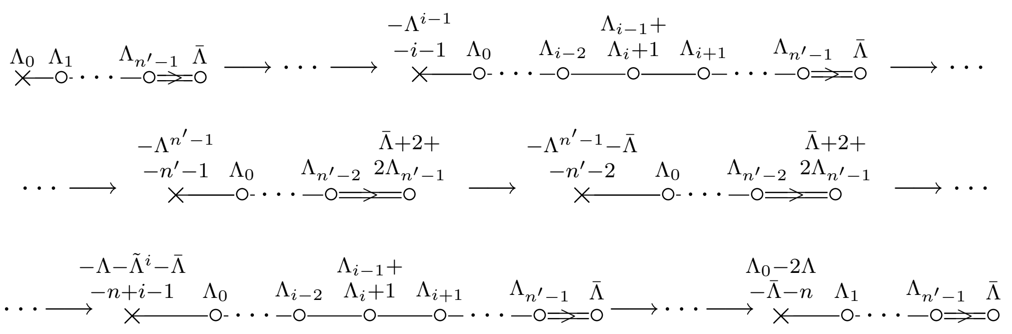


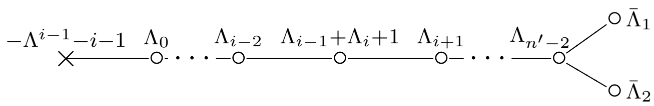
Table 2.1: The pattern with Dynkin diagrams.

For n even, we distinguish the positions n'_X and n'_Y in the middle according to the coefficient over the cross. (Beside this, they differ only in the sign. The coefficient over the cross indicates which of them is of the higher/lower homogenity. The latter is given by the action of the grading element (1.3).) The greater coefficient corresponds to n'_Y and the smaller one to n'_X. Hence Table 2.1 corresponds to the pattern 1.3.3 for $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$. In this case, n'_Y corresponds to the sign $+$ in the notation of Young diagrams, see Table 1.3.

We shall use Table 2.1 to find V in the pattern. Using the relation between notations for representation via Dynkin an Young diagrams, see Table 1.3, the coefficients r_k determine most of the coefficients Λ_k and it remains to compare $w - s - r$ with the coefficient over the cross. We will show this computation in details only for the first half of the pattern (including n'_Y and n'_X for n even), the computation in the remaining cases is analogous. Results for the whole pattern are in Table 2.3 for n even and Table 2.4 for n odd. We will use the notation from 1.1.3 and also (1.1).

Even dimensional case. In the case of the zero degree, $\Lambda_0 \geq 0$ corresponds to a regular and $\Lambda_0 = -1$ to a singular position. Since $w-s-r = \Lambda_0 = o_0-1$, we get $w - s \geq r$ for the regular position 0 and $w - s = r - 1$ for the singular position $0, 1$.

In the case of degree i, $1 \leq i \leq n' - 2$, let us compare $\mathbb{E}_+\{r_1, \dots, r_{n'}\}_0[w]$ with the representation



see the ith degree in the pattern. Recall we use the notation (1.1) here. This means that $w - s - r = -\Lambda^{i-1} - i - 1$ and using the relation $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$ (given by the sign), the last display means

$$r_1 = \Lambda_0, \ldots, r_{i-1} = \Lambda_{i-2},\ r_i = \Lambda_{i-1} + \Lambda_i + 1,\ r_{i+1} = \Lambda_{i+1}, \ldots, r_{n'-2} = \Lambda_{n'-2}$$
$$r_{n'-1} + 2r_{n'} = \bar{\Lambda}_1,\ r_{n'-1} = \bar{\Lambda}_2.$$

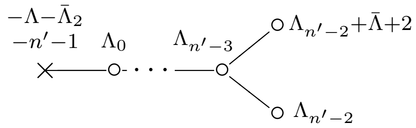
This implies $w - s - r = -r^i + (\Lambda_i + 1) - i - 1$ or equivalently $\Lambda_i = w - s - \tilde{r}^{i+1} + i = o_i - 1$. We have $0 \leq \Lambda_i \leq r_i - 1$ in the regular case and $\Lambda_i \in \{-1, r_i\}$ in the singular ones. This yields the interval

$$\tilde{r}^{i+1} - i \leq w - s \leq \tilde{r}^i - i - 1$$

for regular cases on the ith position and two possibilities $w - s = \tilde{r}^i - i$ and $w - s = \tilde{r}^{i+1} - i - 1$ for singular cases on positions $i-1, i$ and $i, i+1$, respectively.

The $(n'-1)$th degree is similar. We get $r_{n'-1} = \Lambda_{n'-2} + \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1$ and $\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o_{n'-1} - 1$ and $\max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o^*_{n'-1} - 1$. This yields the interval for the regular position $n'-1$ and the choice of w for the singular position $n'-2, n'-1$ as in the Table 2.3. The singular position $n'-1, n'$ is discussed below.

Finally, let us try to identify the representation $\mathbb{E}_{(+)}\{r_1, \ldots, r_{n'}\}_0[w]$ with one of the representation corresponding to the degree n'. This is necessarily



due to the sign. Here $\bar{\Lambda} = \bar{\Lambda}_1 + \bar{\Lambda}_2$. The display means that $w - s - r = -\Lambda - \bar{\Lambda}_2 - n' - 1$ and

$$r_1 = \Lambda_0, \ldots, r_{n'-1} = \Lambda_{n'-2},\ 2r_{n'} = \bar{\Lambda}_1 + \bar{\Lambda}_2 + 2$$

which implies $w - s - r = -r^{n'-1} - (2r_{n'} - \bar{\Lambda}_1 - 2) - n' - 1$ or equivalently $\bar{\Lambda}_1 = w - s + r_{n'} + n' - 1 = o^*_{n'-1} - 1$. Therefore $\bar{\Lambda}_2 = -o_{n'-1} - 1$ because

$o^*_{n'-1} - o_{n'-1} = 2r_{n'}$. Regular cases satisfy $0 \le \bar{\Lambda}_1 \le 2r_{n'} - 2$ and singular ones $\bar{\Lambda}_1 \in \{-1, 2r_{n'} - 1\}$. Thus we have the interval

$$-r_{n'} - n' + 1 \le w - s \le r_{n'} - n' - 1$$

for regular positions and two possibilities $w - s \in \{r_{n'} - n', -r_{n'} - n'\}$ for singular positions $n'-1, n'_Y$ and $n', n'_X + 1$. Let us note these two possibilities coincide for $r_{n'} = 0$. The latter is the singular case with $\bar{\Lambda}_1 = \bar{\Lambda}_2 = -1$ called "middle". It remains to distinguish two regular positions according to the homogenity. If our representations is of the higher homogenity i.e. on the position n'_Y, then $\bar{\Lambda}_1 \ge \bar{\Lambda}_2$ which yields the interval $w - s \in \langle -n'; r_{n'} - n' - 1 \rangle$. The position n'_X corresponds analogously to the interval $\langle -r_{n'} - n' + 1; -n' \rangle$.

Odd dimensional case. We will be less detailed as the situation is similar to the even dimensional case. Note $w \in \mathbb{Z}$ for regular and singular positions and $w \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ for nonstandard ones.

The zero degree is completely analogous to the even dimensional case for regular and singular cases. The nonstandard ones correspond to $-\frac{1}{2} \le \Lambda_0$.

Let us consider degrees $1 \le i \le n' - 1$. Similar as above, we obtain $\Lambda_i = w - s - \tilde{r}^{i+1} + i = o_i - 1$ and $r_i = \Lambda_{i-1} + \Lambda_i + 1$ from Table 2.1. Hence we have the interval $-\frac{1}{2} \le \Lambda_i \le r_i - \frac{1}{2}$ (where the upper bound corresponds to $\Lambda_{i-1} = -\frac{1}{2}$) for the regular and nonstandard position i. This yields $w - s \in \langle \tilde{r}^{i+1} - i - \frac{1}{2}; \tilde{r}^i - i - \frac{1}{2} \rangle$. The singular positions $i - 1, i$ and $i, i + 1$ correspond to $\Lambda_{i-1} = -1$ and $\Lambda_i = -1$, i.e. $w - s = \tilde{r}^i - i$ and $w - s = \tilde{r}^{i+1} - i - 1$, respectively.

In the case of degree n', we obtain the data $\bar{\Lambda} = 2(w - s + n') = o - 1$ and $2r_{n'} = 2\Lambda_{n'-1} + \bar{\Lambda} + 2$ in a similar way as above. Concerning regular and nonstandard positions, we have the range $0 \le \bar{\Lambda} \le 2r_{n'} - 1$ where the upper bound corresponds to $\Lambda_{n'-1} = -\frac{1}{2}$. Thus we obtain the interval $w - s \in \langle -n'; r_{n'} - n' - \frac{1}{2} \rangle$ for the regular and nonstandard position n'. The singular cases $n'-1, n'$ and $n', n'+1$ correspond to $\Lambda_{n'-1} = -1$ and $\bar{\Lambda} = -1$, respectively. This means $w - s = r_{n'} - n'$ and $w - s = -n' - \frac{1}{2}$, respectively.

2.2.2. gBGG splitting operator. Having the detailed description of the pattern in Table 2.1, our aim now is to suggest an appropriate splitting operator of the form $DSplit^t_b(m)$ for the bundle $V \subseteq E_{(+)}\{r_1, \ldots, r_{n'}\}_0[w]$ ("$\subseteq$"is the Cartan component) in the pattern in such a way that we obtain the gBGG splitting operator in standard cases.

Standard positions are of the form $V^{w.\Lambda}$ for a $\mathfrak{g}$–dominant weight Λ and $w \in W^{\mathfrak{p}}$, see 1.1.1 and 1.3.1 for the notation. The gBGG splitting operator $\mathcal{V}^{w.\Lambda} \cong \mathcal{E}^i \boxtimes \mathcal{V}^{w\Lambda} \longrightarrow \mathcal{E}^i \boxtimes \mathcal{V}^{\Lambda}$ yields the target bundle $E^i \boxtimes V^{\Lambda}$ for $DSplit^t_b(m)$. Recall $\mathbb{V}^{w.\Lambda} \hookrightarrow \mathbb{E}^i \boxtimes \mathbb{V}^{\Lambda}$ is unique because $w\Lambda$ is on the orbit of the highest weight Λ. (The latter means $\mathbb{V}^{w\Lambda} \hookrightarrow \mathbb{V}^{\Lambda}$ is unique.)

We described projecting parts of $DSplit^t_b(m)$ via TFP–components in 2.1.7. Here we develop a suitable description for the (irreducible) projecting parts $V^{w.\Lambda} \hookrightarrow E^i \boxtimes V^{\Lambda}$. To be able to deal with the bundle V^{Λ} (which is not a TFP–bundle), we will, roughly speaking, interpret strings of $\mathbb{X}$, $\mathbb{Y}$ etc., which describe TFP–components, as Cartan products (instead of tensor product, see 1.2.6). To make a precise definition, let us start with the observation that V^{Λ}, as a $\mathfrak{g}$–module, is the Cartan part of the tractor bundle W as follows:

$$\mathbb{V}^{\Lambda} = \overset{\Lambda_0}{\circ}\!-\!\overset{\Lambda_1}{\circ}\cdots-\!\overset{\Lambda_{n'-2}}{\circ}\!\!<\!\!\begin{array}{l}\circ\,\bar{\Lambda}_1\\ \circ\,\bar{\Lambda}_2\end{array} \subseteq \mathbb{W} := \bigotimes^{\Lambda_0} \mathbb{T}^0 \cdots \bigotimes^{\Lambda_{n'-2}} \mathbb{T}^{n'-2} \bigotimes^{\Lambda'_{n'-1}} \mathbb{T}^{n'-1} \bigotimes^{\Lambda'_{n'}} \mathbb{T}^{n'}, \quad n \text{ even}$$

$$\mathbb{V}^{\Lambda} = \overset{\Lambda_0}{\circ}\!-\!\overset{\Lambda_1}{\circ}\cdots-\!\overset{\Lambda_{n'-1}}{\circ}\!\Rightarrow\!\overset{\bar{\Lambda}}{\circ} \subseteq \mathbb{W} := \bigotimes^{\Lambda_0} \mathbb{T}^0 \cdots \bigotimes^{\Lambda_{n'-1}} \mathbb{T}^{n'-1} \bigotimes^{\frac{1}{2}\Lambda_{n'}} \mathbb{T}^{n'}, \qquad n \text{ odd}$$

see (1.44) for the definition of $\mathbb{T}^k$, where $\Lambda'_{n'-1} = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$ and $\Lambda'_{n'} = \frac{1}{2}|\bar{\Lambda}_1 - \bar{\Lambda}_2|$. Clearly $\Lambda'_{n'}$ and $\frac{1}{2}\bar{\Lambda}$ are integers for tensor representations. In spinor cases, we set

$$\bigotimes^{i} \mathbb{T}^{n'} := \bigotimes^{\lfloor i \rfloor} \mathbb{T}^{n'} \otimes \mathbb{E}_{\Lambda}$$

The pattern with ITFP–components where $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor - 1$ corresponds either to the ith or $(n-i)$th degree
$(\mathbb{Y}^0)^{\Lambda_0} \cdots (\mathbb{Y}^{n'-2})^{\Lambda_{n'-2}} (\mathbb{Y}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{Y}^{n'}_+)^{\Lambda'_{n'}} \longrightarrow$ $\cdots \longrightarrow \mathrm{Id}_{E^i} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^i)^{\Lambda_{i-1}} (\mathbb{Y}^i)^{\Lambda_i} \cdots (\mathbb{Y}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{Y}^{n'}_+)^{\Lambda'_{n'}} \longrightarrow \cdots$ $\mathrm{Id}_{E^{n'}_+} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^{n'-1})^{\Lambda_{n'-2}} (\mathbb{Z}^{n'}_+)^{\Lambda'_{n'-1}} (\mathbb{Y}^{n'}_+)^{\Lambda'_{n'}}$ $\nearrow \qquad\qquad \searrow$ $\longrightarrow \mathrm{id}_{E^{n'-1}} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^{n'-1})^{\Lambda_{n'-2}} (\mathbb{Y}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{Y}^{n'}_+)^{\Lambda'_{n'}}$ $\mathrm{id}_{E^{n'+1}} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^{n'-1})^{\Lambda_{n'-2}} (\mathbb{X}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{X}^{n'}_-)^{\Lambda'_{n'}} \longrightarrow$ $\searrow \qquad\qquad \nearrow$ $\mathrm{id}_{E^{n'}_-} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^{n'-1})^{\Lambda_{n'-2}} (\mathbb{Z}^{n'}_-)^{\Lambda'_{n'-1}} (\mathbb{X}^{n'}_-)^{\Lambda'_{n'}}$ $\cdots \longrightarrow \mathrm{id}_{E^{n-i}} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^i)^{\Lambda_{i-1}} (\mathbb{X}^i)^{\Lambda_i} \cdots (\mathbb{X}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{X}^{n'}_-)^{\Lambda'_{n'}} \longrightarrow \cdots$ $\longrightarrow \mathrm{id}_{E^n} \boxtimes (\mathbb{X}^0)^{\Lambda_0} \cdots (\mathbb{X}^{n'-2})^{\Lambda_{n'-2}} (\mathbb{X}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{X}^{n'}_-)^{\Lambda'_{n'}}$ Remark: This corresponds to the pattern in Table 2.1 if $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$. If we switch all the signs above, we will obtain the pattern for $\bar{\Lambda}_1 \leq \bar{\Lambda}_2$. Recall, $\Lambda'_{n'-1} = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$ and $\Lambda'_{n'} = \frac{1}{2}\lvert\bar{\Lambda}_1 - \bar{\Lambda}_2\rvert$.
$(\mathbb{Y}^0)^{\Lambda_0} \cdots (\mathbb{Y}^{n'-1})^{\Lambda_{n'-1}} (\mathbb{Y}^{n'})^{\frac{1}{2}\bar{\Lambda}} \longrightarrow \cdots$ $\cdots \longrightarrow \mathrm{id}_{E^i} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^i)^{\Lambda_{i-1}} (\mathbb{Y}^i)^{\Lambda_i} \cdots (\mathbb{Y}^{n'-1})^{\Lambda_{n'-1}} (\mathbb{Y}^{n'})^{\frac{1}{2}\bar{\Lambda}} \longrightarrow \cdots$ $\longrightarrow \mathrm{id}_{E^{n'}} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^{n'})^{\Lambda_{n'-1}} (\mathbb{Y}^{n'})^{\frac{1}{2}\bar{\Lambda}} \longrightarrow \mathrm{id}_{E^{n'+1}} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^{n'})^{\Lambda_{n'-1}} (\mathbb{X}^{n'})^{\frac{1}{2}\bar{\Lambda}} \longrightarrow$ $\cdots \longrightarrow \mathrm{id}_{E^{n-i}} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^i)^{\Lambda_{i-1}} (\mathbb{X}^i)^{\Lambda_i} \cdots (\mathbb{X}^{n'-1})^{\Lambda_{n'-1}} (\mathbb{X}^{n'})^{\frac{1}{2}\bar{\Lambda}} \longrightarrow \cdots$ $\cdots \longrightarrow \mathrm{id}_{E^n} \boxtimes (\mathbb{X}^0)^{\Lambda_0} \cdots (\mathbb{X}^{n'-1})^{\Lambda_{n'-1}} (\mathbb{X}^{n'})^{\frac{1}{2}\bar{\Lambda}}$

Table 2.2: The pattern with ITFP–components.

Pattern and $E_{(\pm)}\{r_1,\ldots,r_{n'}\}_0[w]$, n even, $w \in \mathbb{Z}$						
Weight	**Pattern**		**Splitting**			
$w - s \in$	Position	Type	t	m	b	w'
$\langle r; \infty)$	0	R	t_0	0	0	0
$\{r-1\}$	$0, 1$	S	t_1	0	0	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\tilde{r}^i - i\}$	$i-1, i$	S	t_i	$r-t$	0	$-i$
$\langle \tilde{r}^{i+1} - i; \tilde{r}^i - i - 1\rangle$	i	R	t_i	$r-t-1$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\langle r_{n'} - n' + 1; \tilde{r}^{n'-1} - n'\rangle$	$n'-1$	R	$t_{n'-1}$	$r-t-1$	0	0
$\{r_{n'} - n'\}$ for $r_{n'} > 0$ $r_{n'} \geq 1$ $r_{n'} = \frac{1}{2}$	$n'-1, n'$	S	 $t_{n'}-1$ 0	 $r-t$ $\lfloor r \rfloor$	0	$-n'+1$
$\langle -n'; r_{n'} - n' - 1\rangle$	n'_Y	R	$t_{n'}$	$r-t-1$	0	0
$\{-n'\}$ for $r'_n = 0$	middle	S				
$\langle -r_{n'} - n' + 1; -n'\rangle$	n'_X	R	0	$r-b-1$	$-t_{n'}$	0
$\{-r_{n'} - n'\}$ for $r_{n'} > 0$ $r_{n'} = \frac{1}{2}$ $r_{n'} \geq 1$	$n', n'+1$	S	0	 $\lfloor r \rfloor$ $r-b$	 0 $-t_{n'}-1$	$-n'-1$
$\langle -r^{n'-1} - n'; -r_{n'} - n' - 1\rangle$	$n'+1$	R	0	$r-b-1$	$-t_{n'+1}$	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\langle -\tilde{r}^i - n + i + 1; -\tilde{r}^{i+1} - n + i\rangle$	$n-i$	R	0	$r-b-1$	$-t_{n-i}$	$-n+2i$
$\{-\tilde{r}^i - n + i\}$	$n-i,$ $n-i+1$	S	0	$r-b$	$-t_{n-i}$	$-(n-i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{-r-n+1\}$	$n-1, n$	S	0	0	$-t_{n-1}$	$-(n-1)$
$(-\infty; -r-n\rangle$	n	R	0	0	$-t_n$	$-n$

Table 2.3: Parameters for gBGG splitting operators (see 2.2.3)

for $i \in \frac{1}{2}\mathbb{N}_0 \setminus \mathbb{N}_0$, where $\mathbb{E}_\Lambda =: \bigotimes^{1/2} \mathbb{T}^{n'}$ is the tractor spinor representation defined in 1.2.4. Let us note the TFP–bundle $W := \mathcal{G} \times_P \mathbb{W}$ is uniquely determined by V^Λ in this way and V^Λ is the Cartan part of W (up to the sign), both considered as $\mathfrak{g}$–representations. Henceforth we shall use the simpler notation

$$(\mathbb{T}^k)^i := \bigotimes^i \mathbb{T}^k.$$

Now we define the set $ITFPC(V)$ of *irreducible tractor form product components* of a $\mathfrak{g}$–irreducible tractor bundle V as $ITFPC(V) := \{pr \in TFPC(W) \mid pr^*|_V \neq 0\}$ where W is a TFP–bundle and $V \subseteq W$ is the inclusion of the Cartan part. Then we define the set of *ITFP–components* of the bundle $E^i \boxtimes V[w]$, $0 \leq i \leq n$ as $ITFPC(E^i \boxtimes V[w]) := \{\mathrm{id}_{E^i} \boxtimes pr \mid pr \in ITFPC(V)\}$.

TFP–components of W can be expressed as juxtapositions of $\mathbb{X}^i$'s, $\mathbb{Y}^i$'s, $\mathbb{Z}^i$'s, $\mathbb{W}^i$'s and at most one X or Y. We shall use these juxtapositions, interpreted as the Cartan product, as a notation for ITFP–components of V. For n even, we can use also $\mathbb{X}^{n'}_\pm$, $\mathbb{Y}^{n'}_\pm$ etc., see (1.54). (The non-triviality of $pr^*|V$ will be obvious in cases we will need.) We shall also use the abbreviations

$$\begin{aligned}
(pr)^i &:= \overbrace{pr \boxtimes \cdots \boxtimes pr}^{i} \in ITFPC\big(\overset{i}{\boxtimes}\, T^k\big), \quad pr \in ITFPC(T^k),\ i \in \mathbb{N}_0 \\
(\mathbb{X}^{n'}_\pm)^i &:= \underbrace{\mathbb{X}^{n'}_\pm \boxtimes \cdots \boxtimes \mathbb{X}^{n'}_\pm}_{\lfloor i \rfloor} \boxtimes X_\pm, \quad n \text{ even},\ i \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}
\end{aligned} \tag{2.92}$$

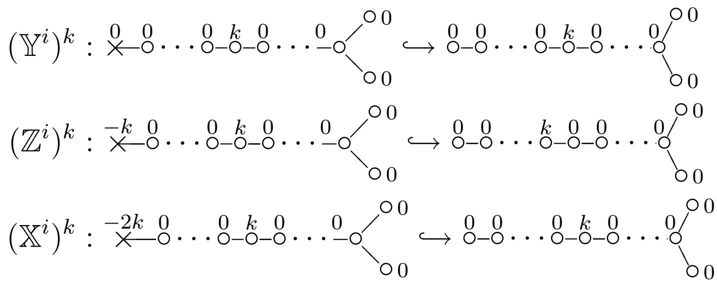
and similarly for $\mathbb{Y}^{n'}_\pm$.

We will use the developed notation to describe the pattern in Table 2.1 in terms of $ITFPC$'s. Assume $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$ or odd dimension. Then the regular position 0 in the pattern is the highest weight $\mathfrak{g}_0$–component of $\mathbb{V}^\Lambda$. Comparing the weights, this obviously corresponds to the ITFP–component of V^Λ of the form $\mathbb{Y}_{(+)} \cdots \mathbb{Y}_{(+)}$ or $\mathbb{Y}_{(+)} \cdots \mathbb{Y}_{(+)} Y_{(+)}$ where the sign applies in the even dimensional case for Y and for $\mathbb{Y}$'s of the tensor valence $n' = \frac{n}{2}$. Using the abbreviations (2.92), this is exactly the position 0 in Table 2.2.

It is not difficult to translate the whole Table 2.1 (for regular patterns) into the symbolism of $ITFPC$'s. This describes not only the bundles $V^{w.\Lambda}$, $w \in W^{\mathfrak{p}}$ in Table 2.1 but also the inclusion $V^{w.\Lambda} = \mathbb{E}^i \boxtimes \mathbb{V}^{w\Lambda} \hookrightarrow \mathbb{E}^i \otimes \mathbb{V}^{\Lambda}$. On the representation level, we have

$$\mathbb{Y}^i : \mathbb{E}^i[i+1] \hookrightarrow \mathbb{T}^i, \ \mathbb{Z}^i : \mathbb{E}^i[i] \hookrightarrow \mathbb{T}^{i-1}, \ \mathbb{X}^i : \mathbb{E}^i[i-1] \hookrightarrow \mathbb{T}^i$$
$$Y : \mathbb{E}_\lambda[1] \hookrightarrow \mathbb{E}_\Lambda, \ X : \mathbb{E}_\lambda \hookrightarrow \mathbb{E}_\Lambda$$

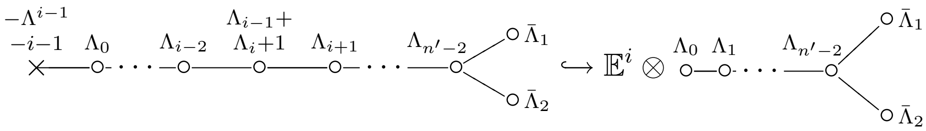
using (1.44) and (1.45) for tensor cases and (1.33) for spinors. From this, we easily see the form of $(\mathbb{Y}^i)^k$, $(\mathbb{Z}^i)^k$ and $(\mathbb{X}^i)^k$. Using Dynkin diagrams, these are the inclusions



(2.93)

for $1 \leq i \leq n' - 2$, n even and similarly for remaining cases, see Table 1.2. Here k is the coefficient over the ith node in all diagrams with the exception of the right hand side of $(\mathbb{Z}^i)^k$. Using (2.93), we easily obtain more complicated $ITFPC$'s, e.g.

$$\mathrm{id}_{\mathbb{E}^i} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^i)^{\Lambda_{i-1}} (\mathbb{Y}^i)^{\Lambda_i} \cdots (\mathbb{Y}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{Y}^{n'}_{+})^{\Lambda'_{n'}} :$$



for $1 \leq i \leq n' - 2$ and n even. This shows the correspondence of the ith positions, $1 \leq i \leq n' - 2$ in Tables 2.1 and 2.2. One can similarly check this correspondence for all positions. Summarising, Table 2.2 describes the regular pattern in terms of ITFP–projecting parts.

Now we are ready to choose an appropriate splitting operator $DSplit^t_b(m)$ for $E_{(+)}\{r_1, \ldots, r_{n'}\}_0[w]$ in the pattern. In regular cases, Table 2.2 shows exactly t, m and b for the gBGG splitting operator. In all cases, we have to verify that the chosen parameters satisfy $s(t,m) > 0$, see Theorem 2.77. We will follow the discussion in 2.2.1. We will also use (2.91) and the notation form Table (1.3).

Even dimensional case. We show the detailed computation only for the first half of the pattern (with the exception of n'_X), the rest is analogous. Results for the whole pattern are in Table 2.3. Recall $\Lambda'_{n'-1} = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$ and $\Lambda'_{n'} = \frac{1}{2}|\bar{\Lambda}_1 - \bar{\Lambda}_2|$. Also remind $w \in \mathbb{Z}$ due to (1.61), page 67.

The position 0 in Table 2.2 shows $b = m = 0$ and $t = \Lambda^{n'-2} + \Lambda'_{n'-1} + \Lambda'_{n'} = w - s$. Here the first equality follows from the form of position 0 in Table 2.2 and the second one from $\Lambda_0 = w - s - r$, see 2.2.1. Note $t \geq 0$ because $w - s \geq r$ according to Table 2.3. The scalar $s(t,m)$ is equal either to $w - s - t + s_{\lfloor t \rfloor} + 1$ or $w - s - t + \frac{n}{2}$, see (2.81). Hence $s(t,0) = s(w-s,0) > 0$ in both cases.

In the regular case i, $1 \leq i \leq n' - 2$, we have $\tilde{r}^{i+1} - i \leq w - s \leq \tilde{r}^i - i - 1$ and the corresponding ITFP–component is of the form

$$\mathrm{id}_{\mathbb{E}^i} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^i)^{\Lambda_{i-1}} (\mathbb{Y}^i)^{\Lambda_i} \cdots (\mathbb{Y}^{n'-2})^{\bar{\Lambda}_{n'-2}} (\mathbb{Y}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{Y}^{n'}_+)^{\Lambda'_{n'}}. \quad (2.94)$$

This means that $t = \tilde{\Lambda}^i + \Lambda'_{n'-1} + \Lambda'_{n'}$ and $m = r - t - 1$. Obviously $\Lambda'_{n'-1} = r_{n'-1}$ and $\Lambda'_{n'} = r_{n'}$. Using the relation between Λ's and r's from 2.2.1, and since $\Lambda_i = w - s - \tilde{r}^{i+1} + i = o_i - 1$, we obtain

$$t = \tilde{r}^{i+1} + \Lambda_i = w - s + i = t_i.$$

Now let us consider the scalar $s(t,m)$. If $t \geq \frac{1}{2}$ the either $s(t,m) = w -$

$s - t + s_{\lfloor t \rfloor} + 1$ or $s(t,m) = w - s - t + \frac{n}{2}$ and $s_{\lfloor t \rfloor} = i$ from (2.94). Thus $s(w-s+i,m) \geq 0$. If $t = 0 \wedge m \geq 1$ then $s(0,m) = n + w - \lfloor s \rfloor - s_{\bar{m}} + \bar{m} - 1$ where, in our case, $\bar{m} = 2$ and $w - s = -i$. The last two equalities follow from, respectively, (2.94) and the last display. Also note $t = 0$ requires $r_{n'} = 0$ i.e. $s = \lfloor s \rfloor$. Hence $s(t,m) = s(0, r-1) = n - i - s_{\bar{m}} + 1 > 0$.

Let us briefly comment the regular position $n' - 1$. We put $t := \Lambda'_{n'-1} + \Lambda'_{n'}$ according to Table 2.2. That is,

$$t = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + \frac{1}{2}|\bar{\Lambda}_1 - \bar{\Lambda}_2| = \frac{1}{2}\left(\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + \max\{\bar{\Lambda}_1, \bar{\Lambda}_2\}\right).$$

We have shown in 2.2.1 that $\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o_{n'-1} - 1$ and $\max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o^*_{n'-1} - 1$. Now it follows from (2.91) and the last display that $t = w - s + n' - 1 = t_{n'-1}$. A similar discussion as in the previous paragraph shows that $s(t,m) > 0$.

Now let us focus on the singular positions $i-1, i$ for $1 \leq i \leq n'-1$. This means $w - s = \tilde{r}^i - i$ according to Table 2.3. Although we cannot use Table 2.2, we can put directly

$$t := \tilde{r}^i = w - s + i = t_i, \quad m = r - t.$$

As above, this means $s_t \geq i$, but now $\bar{m} = 1$. Thus in the case $t \geq \frac{1}{2}$, we obtain $s(t,m) > 0$ analogously as in the regular case i, $1 \leq i \leq n' - 2$. If $t = 0$ then $s(t,m) = s(0,r) = n - i - s_{\bar{m}} > 0$ because $i \leq n' - 1$. The resulting weight is $w' = (w - s) - t = -i$.

The regular case on the position n'_Y yields similar results. The range for the weight is $-n' \leq w - s \leq r_{n'} - n' - 1$ and $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$ due to the sign of $E_{(+)}\{r_1, \ldots, r_{n'}\}_0[w]$. This is the "upper" case in the middle in Table 2.2 i.e. the irreducible projecting part

$$\mathrm{id}_{E^{n'}_+} \boxtimes (\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^{n'-1})^{\Lambda_{n'-2}} (\mathbb{Z}^{n'}_+)^{\bar{\Lambda}_2} (\mathbb{Y}^{n'}_+)^{\frac{1}{2}(\bar{\Lambda}_1 - \bar{\Lambda}_2)}. \tag{2.95}$$

This means $t = \frac{1}{2}(\bar\Lambda_1 - \bar\Lambda_2)$ and $m = r - t - 1$. Since $r_{n'} = \frac{1}{2}(\bar\Lambda_1 + \bar\Lambda_2) + 1$ and $\bar\Lambda_1 = w - s + r_{n'} + n' - 1$, see 2.2.1, we obtain

$$t = \bar\Lambda_1 - r_{n'} + 1 = w - s + n' = t_{n'}.$$

From this we get $t < r_{n'}$ because $t = r_{n'}$ would mean $w - s = r_{n'} - n'$ and the interval for $w - s$ in Table 2.3 would be empty. (This is actually obvious for any position corresponding to the degree n'.) Hence if $t \geq 1$ then $s(t, m) = w - s - t + s_{\lfloor t \rfloor} + 1$ and, in our case, $s(w - s + n', m) = 1 > 0$ because $s_{\lfloor t \rfloor} = n'$. If $t = \frac{1}{2}$ then $t + m < r$ hence $s(\frac{1}{2}, m) = w - s + \frac{1}{2} + n' \geq \frac{1}{2} > 0$. Finally, if $t = 0 \wedge m \geq 1$ then $\bar m = 2$ and $w = s - n'$ (in particular, $s \in \mathbb{Z}$) hence $s(0, r - 1) = n' - s_{\bar m} + 1 > 0$.

Concerning singular positions in the first half of the pattern, it remains to discuss $n' - 1, n'$ i.e. $w - s = r_{n'} - n'$ for $r_{n'} > 0$. (There are no operators in the pattern on the position "middle" so we do not need any splitting in this case.) Now the usual choice for singular cases $t := t_{n'}$ and $m = r - t$ yields $t = r_{n'}$ whence $s(r_{n'}, m) = 0$. We put

$$m = r - t \quad \text{where} \quad t = \begin{cases} r_{n'} - 1 = w - s + n' - 1 = t_{n'} - 1 & r_{n'} \geq 1 \\ 0 & r_{n'} = \frac{1}{2}. \end{cases}$$

(Note $r - t \in \mathbb{N}_0$). If $t \geq \frac{1}{2}$ we obtain $s(t_{n'} - 1, m) > 0$ similarly as in the regular case. If $t = 0$ and $m = 1$ then $s(t, m) = n + w - \lfloor s \rfloor - s_{\bar m} + \bar m - 1$ where $\bar m = 1$, $\lfloor s \rfloor = s - \frac{1}{2}$ and $w - s = \frac{1}{2} - n'$. Hence $s(0, r) = n + (w - s) + \frac{1}{2} - s_1 = n' - s_1 + 1 > 0$.

Remark. Let us remind the importance of the top operator $\tilde{\tilde{T}}$, see Remark 2.1.5. Without this, the stronger condition $w - s - t + \frac{n}{2} > 0$ would be required for all regular cases n'_Y. (This is not satisfied.) Now the stronger condition affects only the singular case, where we have more freedom in the choice of a splitting.

Odd dimensional case. We shall show the detailed computation only for the first half of the pattern, the rest is analogous. Remind $w \in \mathbb{Z}$ for regular and singular positions and $w \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ for nonstandard ones, see 1.3.3. The discussion below is analogous to the even dimensional case where, roughly speaking, the coefficient $\bar{\Lambda}$ over the last node now plays the role of $|\bar{\Lambda}_1 - \bar{\Lambda}_2|$.

In the case of the regular and nonstandard positions 0, we put $t := t_0$ and $t := t_0 + \frac{1}{2}$, respectively, and $m = 0$. Here, recall, $t_0 = w - s$. Then $s(w-s, 0) > 0$ and $s(w - s + \frac{1}{2}, 0) > 0$ for $t \geq \frac{1}{2}$ which follows immediately from (2.81).

Let us consider the regular and nonstandard positions i, $1 \leq i \leq n' - 1$. We put

$$t := \tilde{r}^{i+1} + \Lambda_i = w - s + i = t_i \quad \text{and} \quad m := r - t + 1$$

in the regular cases and

$$t := \tilde{r}^{i+1} + \Lambda_i + \frac{1}{2} = w - s + i + \frac{1}{2} = t_i + \frac{1}{2} \quad \text{and} \quad m := r - t$$

in the nonstandard ones. Therefore $0 \leq t \leq \tilde{r}^i$ (recall $r_i = \Lambda_{i-1} + \Lambda_i + 1$ says $\Lambda_i \leq r_i - \frac{1}{2}$) which means $s_t \leq i$. Now if $t \geq \frac{1}{2}$ then $s(t, m) > 0$ because $w - s - t \in \{-i - \frac{1}{2}, -i\}$ (which follows from last two displays) and $i \leq n' - 1$. Now suppose $t = 0 \wedge m \geq 1$. Then $\bar{m} \in \{1, 2\}$ hence $s(t, m) = s(0, m) = n + w - \lfloor s \rfloor - s_{\bar{m}} + \bar{m} - 1 \geq n + w - s - s_1 > 0$ where the last inequality follows from $w - s \in \{-i - \frac{1}{2}, -i\}$. The resulting weight is 0 for the regular and $-i - \frac{1}{2}$ for the nonstandard positions.

It remains to discuss regular and nonstandard positions n'. They correspond to the interval $w - s \in \langle -n'; r_{n'} - n' - \frac{1}{2} \rangle$ according to Table 2.4. Moreover we have shown in 2.2.1 that $\bar{\Lambda} = 2(w - s + n') = o - 1$ and $2r_{n'} = 2\Lambda_{n'-1} + \bar{\Lambda} + 2$. We shall discuss both cases separately.

In the case of regular position n', we put $t := \frac{1}{2}\bar{\Lambda} = w - s + n' = t_{n'}$ and $m := r - t - 1$. Then obviously $t \leq r_{n'}$ hence if $t \geq \frac{1}{2}$ then $s(t, m) = w - s - t + \frac{n}{2}$

according to (2.81). In our case, $w - s - t = -n'$ thus $s(t_{n'}, m) = \frac{1}{2} > 0$. The parameters in the case $t = 0 \wedge m \geq 1$ satisfy $w - s = -n'$ (in particular $s \in \mathbb{N}$) and $\bar{m} = 2$. Thus $s(0, r-1) = n - n' - s_2 + 1 > 0$.

For the nonstandard position n', we have to modify a bit the previous choice. We put

$$m = r - t \quad \text{where} \quad t = \begin{cases} w - s + n' - \frac{1}{2} = t_{n'} - \frac{1}{2} & w - s + n' \geq \frac{1}{2} \\ 0 & w - s + n' = 0. \end{cases}$$

It follows from the interval for $w - s$ that $t \leq r_{n'}$. Hence if $t \geq \frac{1}{2}$ then $s(t, m) = w - s - t + \frac{n}{2} = 1 > 0$ because $w - s - t = -n' + \frac{1}{2}$ according to the last display. Finally, let us consider the case $t = 0 \wedge m \geq 1$. Then $s(t, m) = s(0, r) = n + w - \lfloor s \rfloor - s_1 + 1 - 1 \geq n + w - s - s_1 \geq 1$ because $w - s \in \{-n', \frac{1}{2} - n'\}$ according to the last display.

2.2.3. Pattern, Young symmetries and parameters for the splitting: summary. All results from this section are summarised in Tables 2.3 (for n even) and 2.4 (for n odd). The input for these tables is the bundle $E_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w]$ which yields the parameters w, s, r, t_j, r^j and $\tilde{r}^j$ according to 1.1.3 and and the parameters in (2.91). The tables are organised as follows.

The first three columns identify the position in the pattern. Regular positions (type R) are degrees of the cohomology $i \in \{0, \ldots, n\}$ and two components of the degree n' in even dimensions are distinguished according to the form of the projecting part (see Table 2.2) or, equivalently, according to the conformal weight w. Singular positions (type S) and nonstandard positions (type NS) are defined in 1.3.3.

The next three columns show numbers of applications of the top, middle and bottom operators, i.e. the parameters of $DSplit_b^t(m)$, see 2.1.7. This yields the gBGG splitting operator in regular cases. The last column shows

Pattern and $E\{r_1,\ldots,r_{n'}\}_0[w]$, n odd, $w \in \begin{cases} \mathbb{Z} & \text{type R} \\ \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z} & \text{type NS} \end{cases}$						
Weight	**Pattern**		**Splitting**			
$w-s\in$	Position	Type	t	m	b	w'
$\langle r-\frac{1}{2};\infty)$	0	R NS	t_0 $t_0+\frac{1}{2}$	0	0	0 $-\frac{1}{2}$
$\{r-1\}$	$0,1$	S				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{\tilde{r}^i-i\}$	$i-1,i$	S				
$\langle\tilde{r}^{i+1}-i-\frac{1}{2};\tilde{r}^i-i-\frac{1}{2}\rangle$	i	R NS	t_i $t_i+\frac{1}{2}$	$r-t-1$ $r-t$	0	0 $-i-\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{r_{n'}-n'\}$	$n'-1,n'$	S				
$\langle-\frac{n}{2}+1;r_{n'}-n'-\frac{1}{2}\rangle$	n'	R NS	$t_{n'}$ $t_{n'}-\frac{1}{2}$	$r-t-1$ $r-t$	0	0 $-n'+\frac{1}{2}$
$\{-n'\}=\{\frac{1}{2}-\frac{n}{2}\}$	n'	R NS	0	$r-1$ $\lfloor r\rfloor$	0	0 $-n'+\frac{1}{2}$
$\{-\frac{n}{2}\}$	$n',n'+1$	S				
$\{-n'-1\}=\{-\frac{1}{2}-\frac{n}{2}\}$	$n'+1$	R NS	0	$r-1$ $\lfloor r\rfloor$	0	-1 $-n'-\frac{1}{2}$
$\langle-r_{n'}-n'+\frac{1}{2};-\frac{n}{2}-1\rangle$	$n'+1$	R NS	0	$r-b-1$ $r-b$	$-t_{n'}$ $-t_{n'}-\frac{1}{2}$	-1 $-n'-\frac{1}{2}$
$\{-r_{n'}-n'\}$	$n'+1,n'+2$	S				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\langle-\tilde{r}^i-n+i-\frac{1}{2};$ $-\tilde{r}^{i+1}-n+i-\frac{1}{2}\rangle$	$n-i$	R NS	0	$r-b-1$ $r-b$	$-t_{n-i}$ $-t_{n-i}+\frac{1}{2}$	$-n+2i$ $-(n-i)+\frac{1}{2}$
$\{-\tilde{r}^i-n+i\}$	$n-i,n-i+1$	S				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{-r-n+1\}$	$n-1,n$	S				
$(-\infty;-r-n+\frac{1}{2}\rangle$	n	R NS	0	0	$-t_n$ $-t_n+\frac{1}{2}$	$-n$ $-n+\frac{1}{2}$

Table 2.4: Parameters for gBGG splitting operators (see 2.2.3)

the conformal weight after application of $DSplit_b^t(m)$. (This has to be zero for regular cases in the first half of the pattern.) The last four columns are sometimes omitted. These cases do not admit any nontrivial operator thus we do not need any splitting for them.

Let us note we used the sign + in 2.2.1 and 2.2.2 for even dimensional cases. But this is not an essential point. The opposite sign corresponds to the same position in the pattern in Table 2.1 with interchanged $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$.

We assumed the complex setting up to now. But in the case of the odd dimensional real pattern and an even dimensional real pattern with two positions n'_Y and n'_X in the middle, we can use all the results from Tables 2.3 and 2.4. In particular, $DSplit_b^t(m)$ is a splitting operator for the parameters b, m and t from the Tables. If the complex and real patterns are different then the weight of the pattern Λ satisfies $\mathbb{V}^\Lambda = \mathbb{E}\{r_1, \cdots, r_{n'-1}, 0\}_0[w]$ and n even, see page 67. That is, $\bar{\Lambda}_1 = \bar{\Lambda}_2$. Then a bundle $E\{\ldots, r_{n'}\}_0[w']$ on the position n'_X or n'_Y in this complex pattern satisfies $w - s = -n'$. (This follows from the corresponding discussion in 2.2.1 after a short computation.) Then the parameters from Table 2.3 corresponding to the positions n'_X and n'_Y coincide and can be applied to the real pattern with one position n' in the middle.

2.2.4 Example. We shall demonstrate the structure of the pattern on the space $\mathcal{E}(k,l)_0[w]$. This can have two irreducible components for $k = \frac{n}{2}$ but we do not need to distinguish them: they appear on the same position (although in different patterns) and also the numbers of top, middle and bottom operators are the same for both of them. (If the real and complex cases are different, the two complex position n'_X and n'_Y coincide.) The parameters provided by the Young diagram (k,l) are $r_k = r_l = 1$ and $r_i = 0$, $l \neq i \neq k$ in the case $l < k$, or $r_k = 2$ and $r_i = 0$, $i \neq k$ in the case $k = l$. Further

Identification of $\mathcal{E}_{(\pm)}(k,l)_0[w]$, $n' \geq k \geq l \geq 1$ in the pattern					
Regular positions: $w \in \mathbb{Z}$, n odd or even					
Weight	**Position**	**Splitting**			
		t	m	b	w'
$w \geq k+l+2$	0	$-w_{kl}$	0	0	0
$w = k+1$	$l_{(Y)}$	1	0	0	0
$w = l$	$k_{(Y)}$	0	1	0	0
$w = 2k+l-n$	$(n-k)_{(X)}$	0	1	0	$2k-n$
$w = k+2l-n-1$	$(n-l)_{(X)}$	0	0	1	$2l-n$
$w \leq k+l-n-2$	n	0	0	$w_{kl}-n$	$-n$
where $i_{(Y)} := \begin{cases} i, & i < \frac{n}{2} \\ n'_Y, & i = \frac{n}{2} \end{cases}$, $i_{(X)} := \begin{cases} i, & i < \frac{n}{2} \\ n'_X, & i = \frac{n}{2} \end{cases}$, $w_{kl} = k+l-w$					

Singular positions $i-1, i$ **:** $w, w' \in \mathbb{Z}$, n even **Nonstandard positions** i **:** $w, w' \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, n odd					
Weight	**Position**	**Splitting**			
	i	t	m	b	w'
$w \in \langle k+l+\frac{3}{2}, \infty\rangle$, n odd	0	$\lceil -w_{kl} \rceil$	0	0	$-\frac{1}{2}$
$w \in \langle k+\frac{3}{2}, k+l+1\rangle$ for $(l, \lceil w \rceil) \neq (n', n'+2)$	$\lfloor w_{kl} \rfloor + 2$	2	0	0	$-i$ or $-i-\frac{1}{2}$
$(l, \lceil w \rceil) = (n', n'+2)$	n'	1	1	0	$-n'+1$ or $-n'+\frac{1}{2}$
$w \in \langle l+\frac{1}{2}, k+\frac{1}{2}\rangle$ for $(l, \lceil w \rceil) \neq (n', n'+1)$	$\lfloor w_{kl} \rfloor + 1$	1	1	0	$-i$ or $-i-\frac{1}{2}$
$(l, w) = (n', n'+\frac{1}{2})$, n odd	n'	0	2	0	$-n'+1$ or $-n'+\frac{1}{2}$
$w \in \langle k+l-n'+\frac{1}{2}, l-\frac{1}{2}\rangle$	$\lfloor w_{kl} \rfloor$	0	2	2	$-i$ or $-i-\frac{1}{2}$
where $w_{kl} = k+l-w$ and the second half of the pattern is omitted					

Table 2.5: Tables 2.3 and 2.4 for $\mathcal{E}_{(\pm)}(k,l)_0[w]$ (see Example 2.2.4)

$s = k + l$ and $r = 2$. Therefore

$$\tilde{r}^1 = \cdots = \tilde{r}^l = 2,\ \tilde{r}^{l+1} = \cdots = \tilde{r}^k = 1,\ \tilde{r}^{k+1} = \cdots = \tilde{r}^{n'} = 0.$$

Since the regular position $i > 0$ (or also $n - i$) requires $\tilde{r}^{i+1} < \tilde{r}^i$, we see immediately the only possible regular positions are $i \in \{0, l, k, n-k, n-l, n\}$. In the case of singular and non–standard positions, we need to consider only the first half of the pattern as these positions in the second half admit no nontrivial operator in the flat case. The results are summarised in Table 2.5. The parameters therein have the same meaning as in Tables 2.3 and 2.4.

Chapter 3

Applications

We will present two applications of the technology developed until now. The first one is a universal and algorithmic construction of curved analogues of the operators from the pattern, see 1.3.5, page 70. This will require most of the calculus developed in Section 2.1. The second application concerns the conformal Killing equation on forms. This demonstrate a range of possible further applications, pursued in more detail in one particular case.

3.1 Invariant operators on irreducible spaces

Following 1.3.6, we use the operators $DSplit$, d, $\Box$ and their formal adjoints to construct tractor formulae of curved analogues of all strongly invariant flat operators between irreducible bundles which are known to exist. The main point will be to establish the non-triviality of the operators defined by these formulae. Also, our aim is to avoid any additional projections (symmetrizations, taking trace-free parts etc.) on the tractor level. This makes the process a bit more complicated but on the other hand it is useful if one needs to transform tractor formulae to tensor ones. Similarly as in Section

2.2, we shall henceforth assume the complex setting. We will comment upon the real case briefly in 3.1.5.

We shall start with an example indicating our construction covers standard operators not treated in [14] and also that *DSplit* and the splitting from [20] are different in the curved case.

3.1.1 Example. **The operator $S_{n'}$ for n odd.** We shall describe the curved translation explicitly on the space $\mathcal{E}(n', n')_0[n'+1]$ for dimensions $n = 2n'+1$. This bundle appears on the standard pattern on the position n' and we obtain the corresponding operator

$$S_{n'} : \mathcal{E}(n', n')_0[n' + 1] \longrightarrow \mathcal{E}(n', n' + 1)_0[n' - 1] \cong \mathcal{E}(n', n')_0[n' - 2]$$

from d as follows. (The target space of $S_{n'}$ follows from Table 2.1.) Following Table 2.4, we apply $T = DSplit^1(0)$ first. Using the formula (2.55), the result is

$$T_{A^0 \mathbf{A}}{}^{\mathbf{a}} : \mathcal{E}(n', n')_0[n' + 1] \longrightarrow \mathcal{E}_{[A^0 \mathbf{A}^{n'}]\mathbf{b}^{n'}}[0]$$

$$\begin{aligned} T_{A^0 \mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} = {} & 6\mathbb{Y}_{A^0 \mathbf{A}}{}^{\mathbf{a}} f_{\mathbf{ab}} + 2\mathbb{Z}_{A^0 \mathbf{A}}{}^{a^0 \mathbf{a}} \big[3\nabla_{a^0} f_{\mathbf{ab}} - n' \boldsymbol{g}_{a^0 b^1} \nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}}\big] + 2n' \mathbb{W}_{A^0 A^1 \dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}} \nabla^p f_{p\dot{\mathbf{a}}\mathbf{b}} \\ & + \mathbb{X}_{A^0 \mathbf{A}}{}^{\mathbf{a}} \Big\{ -6\big[\Delta + (1 - n')P\big] f_{\mathbf{ab}} + 2n' \big[3\nabla_{a^1} \nabla^p f_{p\dot{\mathbf{a}}\mathbf{b}} + 2\nabla_{b^1} \nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}}\big] \\ & \quad + 6n' \big[3P_{a^1}{}^p f_{p\dot{\mathbf{a}}\mathbf{b}} + 2\nabla_{b^1} \nabla^p f_{\mathbf{a}p\dot{\mathbf{b}}}\big] - 3(n')^2 \boldsymbol{g}_{a^1 b^1} \big[\nabla^p \nabla^q + 3P^{pq}\big] f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}} \Big\} \end{aligned}$$

for $f_{\mathbf{ab}} \in \mathcal{E}(n', n')_0[n' + 1]$, $\mathbf{a} = \mathbf{a}^{n'}$, $\mathbf{b} = \mathbf{b}^{n'}$ where we skew over $[b^1 \dot{\mathbf{b}}]$ on the right hand side. Now we can apply the exterior derivative d. After a tedious computation we obtain the result

$$\begin{aligned}
\nabla_{b^0} T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ab}} = {} & \mathbb{Z}_{A^0}{}^{a^0}{}^{\mathbf{a}}_{\mathbf{A}} \boldsymbol{g}_{a^0 b^0} \Big[3n'(n'-1) C_{a^1 a^2}{}^{pq} f_{pq\ddot{\mathbf{a}}\mathbf{b}} + 6(n')^2 C_{a^1}{}^{p}{}_{b^1}{}^{q} f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}} \Big] \\
& + \mathbb{Z}_{A^0}{}^{a^0}{}^{\mathbf{a}}_{\mathbf{A}} \boldsymbol{g}_{a^0 b^0} \boldsymbol{g}_{a^1 b^1} \varphi_{\dot{\mathbf{a}}\dot{\mathbf{b}}} + \mathbb{W}_{A^0 A^1}{}^{\dot{\mathbf{a}}}_{\mathbf{A}} \boldsymbol{g}_{a^1 b^1} \tilde{\varphi}_{\dot{\mathbf{a}}\dot{\mathbf{b}}} \\
+ \mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} \Big\{ & -6(n'+1) \Big[\nabla_{b_0} \big(\nabla^p \nabla_{[p} f_{\mathbf{a}]\mathbf{b}} + P^p{}_{[p} f_{\mathbf{a}]\mathbf{b}} \big) - P_{b^0}{}^{p} \nabla_{[p} f_{\mathbf{a}]\mathbf{b}} \Big] \\
& + 6n' \Big[\nabla_{b^0} P_{a^1}{}^{p} f_{p\dot{\mathbf{a}}\mathbf{b}} + P_{b^0 a^1} \nabla^p f_{p\dot{\mathbf{a}}\mathbf{b}} + \nabla_{b^0} P_{b^1}{}^{p} f_{\mathbf{a}p\dot{\mathbf{b}}} \Big] \\
& + 3n'(n'-1) \nabla_{b^0} C_{a^1 a^2}{}^{pq} f_{pq\ddot{\mathbf{a}}\mathbf{b}} + 6(n')^2 \nabla_{b^0} C_{a^1}{}^{p}{}_{b^1}{}^{q} f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}} \\
& + 2(n')^2 C_{b^0 b^1 a^1}{}^{p} \nabla^q f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}} + \boldsymbol{g}_{b^0 a^1} \psi_{\dot{\mathbf{a}}\mathbf{b}} \Big\}
\end{aligned}$$

where we skew over $[b^0\mathbf{b}]$, and where $\varphi, \tilde{\varphi} \in \mathcal{E}_{\dot{\mathbf{a}}^{n'}\dot{\mathbf{b}}^{n'}}[n'-3]$ and $\psi \in \mathcal{E}_{\dot{\mathbf{a}}^{n'}\mathbf{b}^{n'}}[n'-3]$. The target space of $S_{n'}$ is the Cartan component of the bottom slot, see Table 2.2. Contrary to the gBGG splitting operator from [20], the projection to this component is not invariant. This follows from the form of the $\mathbb{Z}$–slot and (1.47). That is, the trace part of the $\mathbb{Z}$–slot affects the conformal invariance of projection to the target space of $S_{n'}$. However, the double trace part of the $\mathbb{Z}$–slot and also the $\mathbb{W}$–slot do not, see (1.47). Therefore we do not need to know the sections φ and $\tilde{\varphi}$. (Note they are only curvature terms.)

We have shown $DSplit^1$ differs from the splitting from [20]. We use formal adjoints from Example 2.1.17 to solve this problem. This is the operator

$$T^{*A^0\mathbf{A}}{}_{\mathbf{a}} = \tilde{T}^{*A^0\mathbf{A}}{}_{\mathbf{a}} : \mathcal{E}_{[A^0\mathbf{A}^k]b^l}[w'] \longrightarrow \mathcal{E}(k,l)_0[w'+k-1]$$

given by the formula (2.88), page (148) where $k \geq l$. To satisfy the latter condition, we apply the volume form to $\nabla_{d^0} T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ad}}$ first. This yields $\boldsymbol{\epsilon}_{\mathbf{b}}{}^{d^0\mathbf{d}} \nabla_{d^0} T_{A^0}{}^{\mathbf{a}}_{\mathbf{A}} f_{\mathbf{ad}} \in \mathcal{E}_{[A^0\mathbf{A}^{n'}]\mathbf{b}^{n'}}[-1]$ where $\mathbf{d} = \mathbf{d}^{n'}$. Therefore $w' = -1$ and $k = l = n'$ in the formula (2.88) for T^*. From this it follows that $c_1, c_2, d, \tilde{d} \neq 0$ in Example 2.1.17. Therefore the result

$$S_{n'} := T^{*C^0\mathbf{C}}{}_{\mathbf{a}} \boldsymbol{\epsilon}_{\mathbf{b}}{}^{d^0\mathbf{d}} \nabla_{d^0} T_{C^0}{}^{\mathbf{c}}_{\mathbf{C}} f_{\mathbf{cd}} \in \mathcal{E}(n',n')_0[n'-2]$$

is nonvanishing in the flat case, cf. (2.88). Since this is always invariant, the last display yields a tractor formula for $S_{n'}$.

Before we proceed to the formulae in the general case, let us comment upon the (non)irreducibility of $V = E_{(\pm)}\{r_1, \cdots, r_{n'}\}_0[w]$. Actually, all the formulae we will consider below act between spaces $\Phi : \mathcal{V} \longrightarrow \mathcal{V}'$ where $\mathcal{V}'$ is of the similar form as V. (That is, expressed via Young symmetries.) Denoting the Cartan components of both sides by $\mathcal{V}^{\boxtimes} \subseteq V$ and $\mathcal{V}'^{\boxtimes} \subseteq V'$, the operator (formula) Φ yields the operator

$$\mathcal{V}^{\boxtimes} \hookrightarrow \mathcal{V} \xrightarrow{\Phi} \mathcal{V}' \twoheadrightarrow \mathcal{V}'^{\boxtimes}$$

from the pattern, cf. (1.11) and (1.12). But we do not need the inclusion and projection above for the construction of Φ. So henceforth we consider $\mathcal{V} = E_{(\pm)}\{r_1, \cdots, r_{n'}\}_0[w]$ as the source space. The position (i.e. the position of $V^{\boxtimes}$) in the pattern is described in Tables 2.3 and 2.4

3.1.2. Identification of operators via Young symmetries. The operators from the pattern are uniquely determined by the source and target spaces in the flat case. That is, the notation

$$i \longrightarrow i+1 \qquad \text{or} \qquad i, i+1 \longrightarrow n-i-1, n-i$$

determines the operator uniquely (for a given pattern). In this sense, all operators are identified in patterns in Tables 2.1 and 2.2. However, since we prefer to consider the source space in the form $\mathcal{V} = \mathcal{E}_{(\pm)}\{r_1, \cdots, r_{n'}\}_0[w]$, we would like to know the target space in a similar form. Also, we will determine the order from parameters of $\mathcal{V}$. ([24] shows how to do this from Table 2.1. The order is given by the difference of the action of the grading element (see (1.3)) on the source and target representations. It turns out this difference is always equal to one of coefficients over the nodes, increased by

Short operators on $\mathcal{E}_{(\pm)}\{r_1,\dots,r_{n'}\}_0[w],\ w\in\mathbb{Z}$		
Operator	**Order**	**Signs**
$S_i^{(Y)}: i \longrightarrow i+1_{(Y)}$ $0\le i\le n'-1$	$o_i = \begin{cases} \Lambda_i+1 & i\neq \frac{n}{2}-1 \\ \min\{\bar\Lambda_1,\bar\Lambda_2\}+1 & i=\frac{n}{2}-1 \end{cases}$	same
$S_0 : \mathcal{E}_{(\pm)}\{r_1,r_2,\dots,r_{n'}\}_0[w] \longrightarrow \mathcal{E}_{(\pm)}\{r_1+o_0,r_2,\dots,r_{n'}\}_0[w]$ $S_i^{(Y)} : \mathcal{E}_{(\pm)}\{r_1,\dots,r_i,r_{i+1},\dots,r_{n'}\}_0[w] \longrightarrow$ $\longrightarrow \mathcal{E}_{(\pm)}\{r_1,\dots,r_i-o_i,r_{i+1}+o_i,\dots,r_{n'}\}_0[w],\ i>0$ Formula: $f_{(\pm)} \mapsto \left[DSplit_{t-o_i+1}(m+o_i-1)^*\circ d\circ DSplit^t(m)\,f\right]_{(\pm)}$		
$S_{n'-1}^X : n'-1 \longrightarrow n'_X$	$o^*_{n'-1} = \max\{\bar\Lambda_1,\bar\Lambda_2\}+1$	different
$S_{n'-1}^X : \mathcal{E}_{(\pm)}\{r_1,\dots,r_{n'}\}_0[w] \longrightarrow$ $\longrightarrow \mathcal{E}_{\mp}\{r_1,\dots,r_{n'-1}-o_{n'-1},r_{n'}+o_{n'-1}\}_0[w-2r_{n'}]$ Formula: $f_{(\pm)} \mapsto \left[(S_{n'}^Y)^* f\right]_{\mp}$		
$S_{n'} : n' \longrightarrow n'+1$ n odd	$o=\bar\Lambda+1$	same
$S_{n'} : \mathcal{E}\{r_1,\dots,r_{n'}\}_0[w] \longrightarrow \mathcal{E}\{r_1,\dots,r_{n'}\}_0[w-o]$ Formula: $f\mapsto DSplit^t(m)^*\circ\tilde{\boldsymbol\epsilon}\circ d\circ DSplit^t(m)\,f$ where the isomorphism $\tilde{\boldsymbol\epsilon} : \mathcal{E}^{n'+1}\longrightarrow\mathcal{E}^{n'}$ is induced by the volume form $\boldsymbol\epsilon$		
$S_{n'}^Y : n'_Y \longrightarrow n'+1$	$o^*_{n'-1} = \max\{\bar\Lambda_1,\bar\Lambda_2\}+1$	different
$S_{n'}^Y : \mathcal{E}_{\pm}\{r_1,\dots,r_{n'}\}_0[w] \longrightarrow$ $\longrightarrow \mathcal{E}_{(\mp)}\{r_1,\dots,r_{n'-1}-o_{n'-1},r_{n'}+o_{n'-1}\}_0[w-2r_{n'}]$ Formula: $f_{\pm} \mapsto \left[DSplit^{t-o_{n'-1}-1}(m+o_{n'-1}+1)^*\circ d^*\circ DSplit^t(m)\,f\right]_{(\mp)}$		
$S_{n-i-1}^{(X)} : n-i-1_{(X)} \longrightarrow n-i$ $0\le i\le n'-1$	$o^*_i = \begin{cases} \Lambda_i+1 & i\neq \frac{n}{2}-1 \\ \min\{\bar\Lambda_1,\bar\Lambda_2\}+1 & i=\frac{n}{2}-1 \end{cases}$	same
$S_{n-1} : \mathcal{E}_{(\pm)}\{r_1,r_2,\dots,r_{n'}\}_0[w] \longrightarrow \mathcal{E}_{(\pm)}\{r_1-o^*_0,r_2,\dots,r_{n'}\}_0[w-2o^*_0]$ $S_{n-i-1}^{(X)} : \mathcal{E}_{(\pm)}\{r_1,\dots,r_i,r_{i+1},\dots,r_{n'}\}_0[w] \longrightarrow$ $\longrightarrow \mathcal{E}_{(\pm)}\{r_1,\dots,r_i+o^*_i,r_{i+1}-o^*_i,\dots,r_{n'}\}_0[w-2o^*_i],\ i>0$ Formula: $f_{(\pm)} \mapsto \left[(S_i^{(Y)})^* f\right]_{(\pm)}$		

Table 3.1: Formulae for short operators (see 3.1.5)

Long operators on $\mathcal{E}_{(\pm)}\{r_1,\ldots,r_{n'}\}_0[w]$ $w \in \begin{cases} \mathbb{Z} & n \text{ even} \\ \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z} & n \text{ odd} \end{cases}$		
Operator	**Order**	**Signs**
$L_0 : 0 \longrightarrow n$, n odd $L_i : i \longrightarrow n-i$, $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$ $L_i : i-1, i \longrightarrow n-i, n-i+1$, $1 \leq i \leq n'$	$o = 2(w-s)+n$	different
$\mathcal{E}_{(\pm)}\{r_1,\ldots,r_{n'}\}_0[w] \longrightarrow \mathcal{E}_{(\mp)}\{r_1,\ldots,r_{n'}\}_0[-w+2s-n]$ Formulae: $L_i(f_{(\pm)}) = \left[DSplit^t(m')^* \Box_{2(n'-i)} DSplit^t(m')f\right]_{(\mp)},\ i \leq n'-1$ $L_{n'}(f_{(\pm)}) = \left[DSplit^t(m')^* \Box DSplit^t(m')f\right]_{(\mp)},\ w-s > \frac{1-n}{2}$ $L_{n'}(f_{(\pm)}) = \left[DSplit^t(m')^* \not{D} DSplit^t(m')f\right]_{(\mp)},\ w-s = \frac{1-n}{2}$ • positions are nonstandard for n odd and regular or singular for n even • $m' = m+1$ on regular positions and $m' = m$ in remaining cases		

Table 3.2: Formulae for long operators (see 3.1.5)

one.) We will discuss only the first half of the pattern in detail, the second one is analogous. All results are summarised in Tables 3.1 and 3.2. Recall the quantities (2.91); we will use them often in the Tables and throughout this section. In the discussion below, $\mathcal{V} = \mathcal{E}_{(\pm)}\{r_1, \cdots, r_{n'}\}_0[w]$, will be always the source space.

Short operators. These are operators S_i and $S^Y_{n'-1}$, $S^X_{n'-1}$ and $S^Y_{n'}$, $S^X_{n'}$ in the middle diamond for n even. Using the pattern in Table 2.1, the order of operators in the first half of the pattern is $\Lambda_i + 1$ for S_i, $\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1$ for $S^Y_{n'-1}$ and $\max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1$ for $S^X_{n'-1}$. The orders in the second half follow via the duality.

(a) $i = 0$: We can use the similar arguments as in the case (b) below, $\Lambda_0 = o_0 - 1$ plays the role of Λ_i therein.

(b) $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor - 1$: The target space of this operator is on the position $i + 1$. Comparing the projecting parts in patterns in Table 2.2 on positions i and $i + 1$, we see two differences: the tensor part is changed from $\mathrm{id}_{\mathbb{E}^i}$ to $\mathrm{id}_{\mathbb{E}^{i+1}}$ and $(\mathbb{Y}^i)^{\Lambda_i}$ is replaced by $(\mathbb{Z}^{i+1})^{\Lambda_i}$. This does not change the conformal weight which follows from

$$\begin{gathered} \mathbb{Y}^i\colon \mathbb{E}^i[i+1] \longrightarrow \mathbb{T}^i, \quad \mathbb{Z}^{i+1}\colon \mathbb{E}^{i+1}[i+1] \longrightarrow \mathbb{T}^i, \quad \mathbb{X}^i : \mathbb{E}^i[i-1] \longrightarrow \mathbb{T}^i \\ \mathbb{W}^{i-1}\colon \mathbb{E}^{i-1}[i-1] \longrightarrow \mathbb{T}^i, \quad Y : E_\lambda[1] \longrightarrow E_\Lambda, \quad X : E_\lambda \longrightarrow E_\Lambda. \end{gathered} \tag{3.1}$$

Considering both these changes and since $\Lambda_i = o_i - 1$ (see 2.2.1), we obtain the result

$$S_i : \mathcal{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w] \longrightarrow \mathcal{E}_{(\pm)}\{r_1, \ldots, r_i - o_i, r_{i+1} + o_i, \ldots, r_{n'}\}_0[w]$$

of the order o_i where both signs in the even dimensional case are the same. (The latter note follows from Table 2.2.)

(c) $S^Y_{n'-1}$ for n even: We can use the same construction as in (b) if we replace Λ_i by $\Lambda'_{n'-1} = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$, see Table 2.2. Using $\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o_{n'-1} - 1$ from 2.2.1, the result in Table 3.1 follows.

(d) $S^X_{n'-1}$ for n even: Comparing the positions $n'-1$ and n'_X in the pattern in Table 2.2, we observe that beside the tensor part, there are two differences: $(\mathbb{Y}^{n'-1})^{\Lambda'_{n'-1}}$ is replaced by $(\mathbb{Z}^{n'}_-)^{\Lambda'_{n'-1}}$ and $(\mathbb{Y}^{n'}_+)^{\Lambda'_{n'}}$ by $(\mathbb{X}^{n'}_-)^{\Lambda'_{n'}}$. According to (3.1), only the latter changes the conformal weight and the difference is $2\Lambda'_{n'} = |\bar{\Lambda}_1 - \bar{\Lambda}_2| = \max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} - \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$, see 2.2.2. We have shown in 2.2.1 that $\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o_{n'-1} - 1$ and $\max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o^*_{n'-1} - 1$. Therefore $2\Lambda'_{n'} = o^*_{n'-1} - o_{n'-1} = 2r_{n'}$ according to (2.91). The order is $\max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1 = o^*_{n'-1}$. The signs are different, see Table 2.2.

(e) $S^Y_{n'}$ for n even: We can treat $S^Y_{n'}$ in a similar way as $S^X_{n'-1}$from (d). (Actually, $S^Y_{n'}$ is the formal adjoint of $S^X_{n'-1}$.) The order is $\max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1 = o^*_{n'-1}$ according to 2.2.1 where $\bar{\Lambda}_1 \geq \bar{\Lambda}_2$. It follows from Table 2.2 and (3.1) that the tractor part lowers the conformal weight by $2\Lambda'_{n'-1} + 2\Lambda'_{n'} = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + \max\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = o^*_{n'-1} - o_{n'-1} - 2 = 2r_{n'} - 2$. Using $E^{n'+1} \simeq E^{n'-1}[-2]$, the resulting change of the weight is $2r_{n'}$. The change of the parameters $r_{n'-1}$ and $r_{n'}$ is determined by $\Lambda'_{n'-1} + 1 = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1 = -o_{n'-1}$. ("+1" here is due to the tensor part.)

(f) $S^X_{n'}$ for n even: Looking at Table 2.1, we see the sign $+$ appears on the middle position of the lower homogenity n'_X for $\bar{\Lambda}_1 \leq \bar{\Lambda}_2$. Hence we obtain the order $\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1 = \bar{\Lambda}_1 + 1 = o^*_{n'-1}$. Also $r_{n'-1}$ and $r_{n'}$ are changed, according to Table 2.2, by $\Lambda'_{n'-1} + 1 = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} + 1$. The conformal weight is lowered by $2\min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = 2o^*_{n'-1} - 2$ due to the tractor part and by 2 due to the isomorphism $E^{n'+1} \cong E^{n'-1}[-2]$.

(g) $S_{n'}$ for n odd: The order is $\bar{\Lambda} + 1 = o$ according to 2.2.1. Using Table 2.2 together with (3.1) and the isomorphism $\mathcal{E}^{n'+1} \simeq \mathcal{E}^{n'}[-1]$, the conformal weight is lowered by $2(\frac{1}{2}\bar{\Lambda}) + 1 = o$. (Parameters $r_1, \ldots, r_{n'}$ of $\mathcal{V}$ are not changed.)

Long operators. We observe from Table 2.1 that the difference between the source and target spaces is in the coefficient over the cross and that the coefficients over the "legs" are interchanged for n even. Thus the signs are different. The difference between coefficients over the cross is $2\tilde{\Lambda}^i - 2i + \bar{\Lambda} + n$. The analysis of the pattern from 2.2.1 for the degree ≥ 1 shows $w - s - r = -\Lambda^{i-1} - i - 1$ and since clearly $2r = 2\Lambda + \bar{\Lambda} + 2$, we get the difference between the coefficients over the cross (i.e. the order of the operator) is $2\tilde{\Lambda}^i - 2i + \bar{\Lambda} + n = 2(w - s) + n = o$. The data for the zero degree in 2.2.1 are slightly different but the result is the same. (Note o is also the difference between conformal weights of the source and target spaces.) Hence the result in Table 3.2 follows.

Remark. Let us emphasise that we have talked about orders - not formal orders – above. Table 2.1 follows representation theory hence yields actual orders of the operators.

Now we shall discuss explicit formulae for the curved analogues. The source space space will be always $\mathcal{V} = \mathcal{E}_{(\pm)}\{r_1, \cdots, r_{n'}\}_0[w]$ and the first step in the construction of the operator will be $DSplit_b^t(m)$ for t, m and b from Tables 2.3 and 2.4. Then we apply d, $\Box_{2k}$ or $\not{D}$ see 1.3.6 and try to obtain the resulting operator in one of projecting parts of $d \circ DSplit_b^t(m)$, $\Box_{2k} \circ DSplit_b^t(m)$ or $\not{D} \circ DSplit_b^t(m)$. This may not be possible in the curved case, cf. Example 3.1.1. The unique nontrivial TFP–projecting part $pr_b^t(m)$ of $DSplit_b^t(m)$ is described in Theorem 2.1.7, page 136. It is always $pr^t(m)$ or $pr_b(m)$ because $b = 0$ or $t = 0$ in Tables 2.3 and 2.4.

3.1.3. Formulae for short operators. Let us consider a short operator $S : \mathcal{V} \longrightarrow \mathcal{V}'$ from the pattern where both V and V' are given via Young symmetries as in Table 3.1. Following Tables 2.3 and 2.4, we have the splitting operators

$$DSplit_b^t(m) : \mathcal{V} \longrightarrow \mathcal{E}^i \otimes \mathcal{E}_{\mathfrak{A}} \quad \text{and} \quad DSplit_{b'}^{t'}(m') : \mathcal{V}' \longrightarrow \mathcal{E}^{i+1} \otimes \mathcal{E}_{\mathfrak{A}}$$

where $\mathfrak{A}$ is given by (2.80) and $E_{\mathfrak{A}}$ is a TFP–bundle. The splitting operators $DSplit_b^t(m)$ and $DSplit_{b'}^{t'}(m')$ correspond to the positions of $\mathcal{V}$ and $\mathcal{V}'$ in the pattern and their (unique nontrivial) TFP–projecting parts are $pr_b^t(m)$ and $pr_{b'}^{t'}(m')$, respectively. It follows from the gBGG theory [20] that

$$S = pr_{b'}^{t'}(m')^*[d \circ DSplit_b^t(m)]$$

on the Cartan component $\mathcal{V}^{\boxtimes}$ of $\mathcal{V}$ in the flat case. Hence S is non-vanishing on $\mathcal{V}^{\boxtimes}$. Also let us emphasise that by this construction we obtain formulae for short operators with the formal order equal the actual one. This follows from Table 2.2 and the difference between homogenities of the source and target positions. This difference shows that the formal order is equal to a coefficient of the highest weight Λ increased by one. That is, the formal order is equal to the actual order. (Cf. Remark 3.1.2.)

The projection $(pr_{b'}^{t'}(m'))^*$ is provided by $pr_{b'}^{t'}(m')^{\perp} = pr_{t'}^{b'}(m')$ in the sense of (1.56). It may not be invariant in the curved case so will use the formal adjoint $DSplit_{t'}^{b'}(m')^*$ instead. This is always invariant but we have to discuss carefully the non-triviality now. First, $DSplit_{t'}^{b'}(m')^*$ is a multiple of identity on $pr_{b'}^{t'}(m')(\mathcal{V}') \subseteq \mathcal{E}^{i+1} \otimes \mathcal{E}_{\mathfrak{A}}$, see Proposition 2.1.8, page 143. Thus we have to show this multiple is nonzero. Second, beside elements of $pr_{b'}^{t'}(m')(\mathcal{V}') \subseteq \mathcal{E}^{i+1} \otimes \mathcal{E}_{\mathfrak{A}}$, $DSplit_{t'}^{b'}(m')^*$ depends only on $pr(\tilde{f}) \in \mathcal{E}^{i+1} \otimes \mathcal{E}_{\mathfrak{A}}$ where $pr \in TFPC(E_{\mathfrak{A}})$ such that $h(pr) > h(pr_{b'}^{t'}(m'))$. (Here $\tilde{f}$ is a section of the source space of pr.) Both these observations follow from properties of formal adjoints. Summarising, we need to show that

1. $DSplit_{t'}^{b'}(m')^* \circ pr_{b'}^{t'}(m')$ is a nonzero multiple of identity and
2. if $pr \in TFPC(E_{\mathfrak{A}})$ such that $h(pr) > h(pr_{b'}^{t'}(m'))$ then $pr^*[d \circ DSplit_b^t(m)]$ vanishes in the flat case. (3.2)

We will verify these properties only for the operators in the first half of the pattern (with one exception, see below). This is sufficient as their formal adjoints provide formulae for the remaining ones.

(a) We shall start with the short operator $S_i : i \longrightarrow i+1$ where $0 \le i \le n'-1$ with the convention that if n is even then $S_{n'-1}$ denotes the operator $S^Y_{n'-1}$ and n' denotes the position n'_Y in this paragraph. This will simplify the notation. Recall $S^Y_{n'-1}$ is the operator of the lower order from the couple $S^Y_{n'-1}$ and $S^X_{n'-1}$.

Obviously, the parameters of *DSplit* on positions i and $i+1$ satisfy $b = b' = 0$. Moreover, comparing positions i and $i+1$ in Table 2.2 we see that $t' = t - \Lambda_i$ and $m' = m + \Lambda_i$ where $\Lambda_{n'-1} := \Lambda'_{n'-1}$ for n even. Hence we obtain the formula

$$S_i := DSplit_{t'}(m')^* \circ d \circ DSplit^t(m)$$

where $t' = t - o_i + 1$ and $m' = m + o_i - 1$ because $\Lambda_i = o_i - 1$ according to 2.2.1. It remains to verify (3.2).

1. Proposition 2.1.8 shows a way how to establish the first property: it is sufficient to check that $DSplit_{t'}(m')$ is a splitting operator on $(\mathcal{V}')^*[-n]$. Recall

$$\text{if } DSplit^* : \mathcal{E}^{i+1} \otimes \mathcal{E}_{\mathfrak{A}} \longrightarrow \mathcal{V}' \text{ then } DSplit : (\mathcal{V}')^*[-n] \longrightarrow (\mathcal{E}^{i+1} \otimes \mathcal{E}_{\mathfrak{A}})^*[-n]$$

with omitted parameters t' and m', see details in 2.1.8. Using the form of $\mathcal{V}'$ in Table 3.1 and (1.14), we get

$$(\mathcal{V}')^*[-n] = \mathcal{E}_{(\pm)}\{r_1, \ldots, r_i - o_i, r_{i+1} + o_i, \ldots, r_{n'}\}_0[w']$$

where $w' = -w + 2s' - n$ and $s' = s + o_i$ denotes the number of tensor indices of $\mathcal{V}'$, increase by $\frac{1}{2}$ if $r_{n'} \notin \mathbb{N}$. (We do not need to discuss the sign.) According to Theorem 2.1.7, we need to check $s(0, m') > 0$ for $DSplit_{t'}(m')$ applied to $(\mathcal{V}')^*[-n]$. Looking at the form of $s(0, m')$ in (2.81), we need to know, beside s' and w', also $\bar{m}'$ and $s'_{\bar{m}'}$, i.e. parameters corresponding to $\bar{m}$ and $s_{\bar{m}}$ from (2.81) but related to the last display. Obviously the Young diagrams for $\mathcal{V}$ and $\mathcal{V}'$ have the same number of columns (namely $\lfloor r \rfloor$) thus $\bar{m}' = \lfloor r \rfloor - m' + 1$. Using $m' = m + o_i - 1$, we obtain

$$\bar{m}' = \lfloor r \rfloor - (m + o_i - 1) + 1 = \bar{m} - o_i + 1 = \lfloor t + 2 \rfloor - o_i + 1. \tag{3.3}$$

Properties of $s'_{\bar{m}'}$ follow from Table 2.2. The positions i and $i+1$ in this Table, considered as TFP–components, are

$$\cdot\cdot (\mathbb{Z}^i)^{\Lambda_{i-1}} \mathrm{id}_{E^i} (\mathbb{Y}^i)^{\Lambda_i} (\mathbb{Y}^{i+1})^{\Lambda_{i+1}} \cdot\cdot \longrightarrow \cdot\cdot (\mathbb{Z}^i)^{\Lambda_{i-1}} \mathrm{id}_{E^{i+1}} (\mathbb{Z}^{i+1})^{\Lambda_i} (\mathbb{Y}^{i+1})^{\Lambda_{i+1}} \cdot\cdot$$

where we displayed only how these positions differ. Now recall, by definition, $\bar{m}'$ indicates the longest column of the Young diagram – on the left in the previous display – to which we apply the middle operator in $DSplit_{t'}(m')$, and $s'_{\bar{m}'}$ is the length of this column. Hence either $s'_{\bar{m}'} = i + 1$ (for $\Lambda_i > 0$) or $s'_{\bar{m}'} \leq i$ (for $\Lambda_i = 0$) and we conclude $s'_{\bar{m}'} \leq i + 1$. Using this and (3.3), we obtain from (2.81) the result

$$\begin{aligned} s(0, m') &= n + w' - \lfloor s' \rfloor - s'_{\bar{m}'} + \bar{m}' - 1 \\ &= n + w' - \lfloor s' \rfloor - s'_{\bar{m}'} + (\lfloor t \rfloor - o_i + 3) - 1 \\ &= -w + s - s'_{\bar{m}'} + t + 2 = i - s'_{\bar{m}'} + 2 \geq 1 \end{aligned}$$

where we have also used $-\lfloor s' \rfloor + \lfloor t \rfloor = -s' + t,\ s' = s + o_i$ and $w' = -w + 2s' - n$ in the third equality and $t = w - s + i$ in the last equality.

2. Assume $pr \in TFPC(E_{\mathfrak{A}})$ such that $h(pr) > h(pr^t(m))$ and $E := pr^*[d \circ DSplit^t(m)] \neq 0$ in the flat case. Then the formula E defines an operator from the pattern with the source space on the position i, or the identity. Let us suppose E is not identity first and try to vary the dimension n. This does not change t, m and the position of V hence E is given by the same formula for all dimensions. The order of the long operator (on any

position) is $2(w-s)+n$ which depends on n. Therefore E is a short operator. But this is not possible as $h(pr) > h(pr^t(m))$ implies that order of E is lower then the order of S_i. (Recall if $i = n' - 1$ then the second short operator has higher (or equal) order than S_i.) More precisely, we can use this argument only for the formal order of E, see 1.2.6. But the actual order of E is the same or even lower then the formal one. (We have mentioned above both orders agree for our formula of S_i.)

Finally, suppose E is a multiple of identity. This requires $h(pr) = h(pr^t(m)) + 1$ because d is of the first order. Recall the form of the TFP–projecting part of $DSplit^t(m)f$, $f \in \mathcal{V}$. It is

$$pr^t(m)f = \mathbb{Y}_{[A_1^0\mathbf{A}_1]}{}^{\mathbf{a}_1} \cdots \mathbb{Y}_{[A_t^0\mathbf{A}_t]}{}^{\mathbf{a}_t} \mathbb{Z}_{\mathbf{A}_{t+2}}^{\mathbf{a}_{t+2}} \cdots \mathbb{Z}_{\mathbf{A}_r}^{\mathbf{a}_r} f_{\mathbf{a}_1\cdots\mathbf{a}_t\mathbf{a}_{t+1}\mathbf{a}_{t+2}\cdots\mathbf{a}_r}$$

for $t \in \mathbb{N}_0$ with the free form index $\mathbf{a}_{t+1}$. (The case $t \notin \mathbb{N}_0$ is analogous.) The only way how can d increase the homogenity is to apply the derivative to $\mathbb{Z}_{\mathbf{A}_j}^{\mathbf{a}_j}$, $t+2 \leq j \leq r$, and consider the $\mathbb{Y}$–term on the right hand side of

$$\nabla_p \mathbb{Z}_{\mathbf{A}_j}^{\mathbf{a}_j} = -(k+1)\delta_p^{a_j^1} \mathbb{Y}_{A_j^1\dot{\mathbf{A}}_j}{}^{\dot{\mathbf{a}}_j} - (k+1)P_p{}^{a_j^1} \mathbb{X}_{A_j^1\dot{\mathbf{A}}_j}{}^{\dot{\mathbf{a}}_j},$$

see (1.49). But $\mathbf{a}_{t+1} = \mathbf{a}_{t+1}^{s_{t+1}}$ is the "longest" form index in the Young subdiagram $(s_{t+1}, \cdots, s_{\lfloor r \rfloor})$ hence the result requires skewing over $s_{t+1}+1$ indices in this subdiagram which vanishes.

(b) $S_{n'}^Y : n'_Y \longrightarrow n'+1$, n even. We shall follow (b) where we replace d by d^*. It follows from Table 2.2 that in the flat case, $S_{n'}^Y$ is given by the projection $pr_{t'}(m')^*[d^* DSplit^t(m)]$ where $m' = m - \Lambda'_{n'-1}$ and $t' = t + \Lambda'_{n'-1}$. Hence in the curved case we obtain the formula

$$S_{n'}^Y := DSplit^{t'}(m')^* \circ d \circ DSplit^t(m).$$

Since $\Lambda'_{n'-1} = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\} = -o_{n'-1} - 1$ according to 2.2.1 (note this minimum is just $\bar{\Lambda}_2$ in 2.2.1), we see $m' = m + o_{n'-1} + 1$ and $t' = t - o_{n'-1} - 1$. Now we verify (3.2).

1. Following (a) and using similar notation as therein, we are going to show $DSplit^{t'}(m')$ is a splitting operator on

$$(\mathcal{V}')^*[-n] = \mathcal{E}_{(\pm)}\{r_1, \dots, r_{m'-1} - o_{n'-1}, r_{n'} + o_{n'-1}\}_0[w'] \qquad (3.4)$$

where $w' = -(w - 2r_{n'}) + 2s' - n$ and $s' = s + o_{n'-1}$ corresponds to the number of indices in the previous display similarly as in (a). (Cf. Table 3.1 for the term $w - 2r_{n'}$.) Clearly $w' = w$ using $o_{n'-1} = w - s - r_{n'} + n'$. Further, using $s' + t' = s + t - 1$ and $t = t_{n'} = w - s + n'$, we obtain

$$w' - s' - t' = w - s - t + 1 = -n' + 1 \quad \text{and} \quad t' = t - o_{n'-1} - 1 = r_{n'} - 1.$$

We need to show $s(t', m') > 0$ for the space (3.4). According to (2.81), there are three possibilities. First, if $t' \geq 1$ then $s(t', m') = w' - s' - t' + s'_{\lfloor t' \rfloor} + 1$ where primes indicate parameters related to (3.4). It follows from Table 2.2, in particular, because $t' = \Lambda'_{n'-1} + \Lambda'_{n'}$ therein, that $s'_{\lfloor t' \rfloor} \in \{n' - 1, n'\}$. Thus $s(t', m') > 0$ using the last display. Second, if $t' = \frac{1}{2}$ then $s(t', m') = w' - s' - t' + \frac{n}{2} + 1 > 0$ using also the previous display. Third, if $t' = 0$ and $m' \geq 1$ then $s \in \mathbb{Z}$ hence $s(0, m') = n + w' - s' - s'_{\bar{m}'} + \bar{m}' - 1 > 0$ because $w' - s' = -n' + 1$ using the last display.

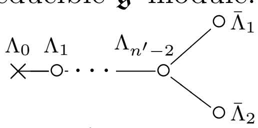
2. The same consideration as in (a) verifies this property. (It is even easier now as there are no long operators on the position n'_Y.)

(c) $S_{n'}$ for n odd. We put $S_{n'} := DSplit^t(m)^* \circ \tilde{\boldsymbol{\epsilon}} \circ d \circ DSplit^t(m)$ where $\tilde{\boldsymbol{\epsilon}}$ is induced by the isomorphism $\tilde{\boldsymbol{\epsilon}} : \mathcal{E}^{n'+1} \longrightarrow \mathcal{E}^{n'}[-1]$ determined by the volume form $\boldsymbol{\epsilon} \in \mathcal{E}^n[n]$. To show non-triviality, we have to verify (3.2). To establishes the part 1., note the target space of $S_{n'}$ has the conformal weight $w - o = -w + 2s - n$ (see Table 3.1) hence we need to show $DSplit^t(m)$ is a splitting operator on

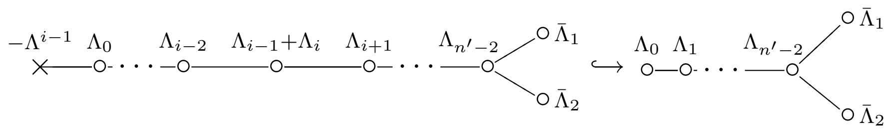
$$(\mathcal{V}')^*[-n] = \mathcal{E}\{r_1, \dots, r_{n'}\}_0[w']$$

where $w' = -(-w+2s-n)+2s-n = w$. But $DSplit^t(m)$ is the D–splitting operator on $\mathcal{V} \cong (\mathcal{V}')^*[-n]$ according to Table 2.4. (In the other words, $S_{n'}$ is formally self–adjoint.) The part 2. of (3.2) follows using the same arguments as in (a).

(d) All the remaining operators can be expressed as formal adjoint of some of operators treated above. In particular, $S^X_{n'-1} = (S^Y_{n'})^*$, $S^X_{n'} = (S^Y_{n'-1})^*$ and $S_{n-i-1} = (S_i)^*$ for $i \le \lfloor \frac{n}{2} \rfloor - 1$. Let us note the formal adjoints appear on the same pattern for n odd; if n is even they appear on the same pattern or on the pattern with interchanged coefficients over the legs.

3.1.4. Formulae for long operators. We shall start with the following observation, summarised in the Lemma below. We defined TFP–projective parts of the form $pr^t(m)$ and $pr_b(m)$ in Theorem 2.1.7. Following 2.2.2, we can consider these also as irreducible projecting parts. Let us consider a tractor bundle V^Λ such that $\mathbb{V}^\Lambda$ is an irreducible $\mathfrak{g}$–module. Suppose Λ is of the same form as in Table 2.1 i.e. $\mathbb{V}^\Lambda =$  and similarly for n odd. Then in even dimensions, we define $pr^t(m)_+ \in ITFPC(V^\Lambda)$ as

$$pr^t(m)_+ := \overbrace{(\mathbb{Z}^1)^{\Lambda_0} \cdots (\mathbb{Z}^i)^{\Lambda_{i-1}}}^{m=\Lambda^{i-1}} \overbrace{(\mathbb{Y}^i)^{\Lambda_i} \cdots (\mathbb{Y}^{n'-1})^{\Lambda'_{n'-1}} (\mathbb{Y}^{n'}_+)^{\Lambda'_{n'}}}^{t=\tilde{\Lambda}^i+\bar{\Lambda}} :$$



if $\bar{\Lambda}_1 \ge \bar{\Lambda}_2$ and we define similarly $pr^t(m)_-$ if $\bar{\Lambda}_2 \ge \bar{\Lambda}_1$. Recall $\Lambda'_{n'-1} = \min\{\bar{\Lambda}_1, \bar{\Lambda}_2\}$ and $\Lambda'_{n'} = \frac{1}{2}|\bar{\Lambda}_1 - \bar{\Lambda}_2|$ in the last display. Analogously, we define $pr_b(m)_\pm$ as $pr^b(m)_\pm$ with all $\mathbb{Y}$'s replaced by $\mathbb{X}$'s. Similarly, we have $pr^t(m)$ and $pr_b(m)$ for n odd. As usually, the notation $pr^t(m)_{(\pm)}$ and $pr_b(m)_{(\pm)}$ indicates that the appropriate sign applies for n even if $|\bar{\Lambda}_1 - \bar{\Lambda}_2| > 0$. It follows from the pattern in Table 2.2 that $pr^t(m)_{(\pm)}$ and $pr_b(m)_{(\pm)}$ form the orbit of the highest weight Λ. Therefore every $pr^t(m)_{(\pm)}$ and $pr_b(m)_{(\pm)}$ is unique as an irreducible $\mathfrak{g}_0$–component of V^Λ. More generally, we have the following.

Lemma. *Let V be an irreducible tractor bundle. Then $pr^t(m)_{(\pm)}, pr_b(m)_{(\pm)} \in ITFPC(V[w])$ are unique as irreducible $\mathfrak{g}_0$–components of $V[w]$ for any scalar w.* □

The construction of formulae for long operators is similar as in 3.1.3 but we will use the strongly invariant operators

$$\Box_{2k} : \mathcal{E}[k - n/2] \longrightarrow \mathcal{E}[-k - n/2]$$
$$\not{D} : \mathcal{E}_\lambda[1 - n/2] \longrightarrow \mathcal{E}_\lambda[-n/2]$$

instead of the exterior derivative d, see 1.3.6. Recall $k < n'$ for n even. These operators are elliptic for the Riemannian signature hence their null space is finite dimensional in this case. Note they are formally self–adjoint.

Let us consider a long operator L_i from the pattern. The source space $\mathcal{V}$ is on the regular (for n even) or nonstandard (for n odd) position i or a singular position $i-1, i$ (for n even) and we have the splitting $DSplit^t(m)$ with t, m from Tables 2.3 and 2.4. To apply $\Box_{2k}$ or $\not{D}$, we have to get rid of free tensor indices in $DSplit^t(m)f$, $f \in \mathcal{V}$ first. We put $m' := m+1$ in regular cases and $m' := m$ in remaining ones. The target space of $DSplit^t(m')$ in regular cases has the conformal weight 0 hence $M \circ DSplit^t(m) = DSplit^t(m')$ is D–splitting operator, cf. with (2.14). (Note $w = 0$ and $k = i \leq n' - 1$ therein.)

Let us suppose $i \leq n' - 1$ first. Note $t + m' = r$ according to our choice of m' in regular cases and according to Tables 2.3 and 2.4 in remaining ones. ($m := r - t$ is the choice of m in the Tables.) Then $DSplit^t(m')f \in \mathcal{E}_{\mathfrak{A}}[w']$ where $\mathfrak{A}$ is given by (2.80), in particular there are no free spinor indices. The conformal weight is $w' = -i$ for n even and $w' = -i - \frac{1}{2}$ for n odd. Putting $k := n' - i$, we can apply $\Box_{2k}$. The restriction $k < n'$ for n even excludes the operator L_0 from this construction, cf. 1.3.5. Then in the Riemannian case, there is $pr \in TFPC(E_{\mathfrak{A}})$ such that $E := pr^* \Box_{2k} DSplit^t(m')f$ is non-vanishing in the flat case. ($\Box_{2k}$ has finite dimensional kernel hence cannot vanish identically.) Hence E yields a formula for a nontrivial differential operator in the flat case and we can assume this operator is invariant (by an appropriate choice of pr). But then E is nontrivial for any signature because the formula $\Box_{2k} DSplit^t(m')$ does not depend on the signature and E cannot vanish identically.

Let us consider the operator E. We will show it has to be the long one (in the flat case). First, let us consider $pr^* DSplit^t(m')f \in \mathcal{E}_{\mathfrak{a}}[\tilde{w}]$, $\tilde{w} \in \mathbb{R}$ where $pr \in TFPC(E_{\mathfrak{A}})$ is arbitrary. (Here $\mathfrak{a}$ is an appropriate system of indices.) If $pr = pr^t(m')$ then $\tilde{w} = w$. Therefore if $h(pr) = hh(E_{\mathfrak{A}})$ then also $\tilde{w} = w$, see (3.1). (Recall there are only $\mathbb{Y}$'s and $\mathbb{Z}$'s in $pr^t(m')$.) Using (3.1) once more, a moment of thinking reveals $\tilde{w} \leq w$ for arbitrary pr. But since $\Box_{2k}$ lowers the weight by 2, we conclude the target space of E has the conformal weight $\leq w - 2k$. Clearly symmetrizations do not change the weight and taking the trace lowers the weight even more. Summarising, the conformal weight of target space of $pr^* DSplit^t(m')$ has the conformal weight $\leq w - 2k$. Thus E is neither a short operator nor the identity. Therefore, E is the long operator L_i.

Now we show that $pr = pr_t(m')$ using the three following observations. First, there has to be an irreducible $\mathfrak{g}_0$–component $\tilde{pr}$ of $E^{\boxtimes}_{\mathfrak{A}} \subseteq E_{\mathfrak{A}}$ providing L_i, where $E^{\boxtimes}_{\mathfrak{A}}$ is the Cartan component of $E_{\mathfrak{A}}$. (That is, $\mathbb{E}^{\boxtimes}_{\mathfrak{A}} \subseteq \mathbb{E}_{\mathfrak{A}}$ as $\mathfrak{g}$–modules.) The reason is $pr^t(m')f \in \mathcal{E}^{\boxtimes}_{\mathfrak{A}}[w']$ and $\Box_{2k}$ commutes with the projection $\mathcal{E}_{\mathfrak{A}} \twoheadrightarrow \mathcal{E}^{\boxtimes}_{\mathfrak{A}}$. Second, $pr := pr_t(m') \in ITFPC(E^{\boxtimes}_{\mathfrak{A}})$ is a possible candidate because then $pr^* \Box_{2k} DSplit^t(m') : \mathcal{V} \longrightarrow \mathcal{V}[-2(w-s)-n]$ is the "right" target space according to Table 3.1. To verify the conformal weight $-2(w-s)-n$ note $\Box_{2k}$ lowers the weight by $2k = 2(n'-i)$ and the difference between conformal weights of target spaces of $pr^t(m')^*$ and $pr_t(m')^*$ is $2t$, cf (3.1). Here $t = t_i = w-s+i$ for n even and $t = t_i + \frac{1}{2}$ for n odd. Hence we get the difference $2k+2t = 2(w-s)+n$ between conformal weights of the source and target spaces. Finally, $pr_t(m')$ is the only possible candidate because, using Lemma 3.1.4, $pr_t(m')$ is unique as an irreducible $\mathfrak{g}_0$–component (i.e. an ITFP–component) of $\mathcal{E}_{\mathfrak{A}}^{\boxtimes}[w' - 2k]$. (Note that considering $pr_t(m')$ as an ITFPC–component, we actually obtain $pr_t(m')_{(\pm)} \in ITFPC(E_{\mathfrak{A}})$ for an appropriate sign.)

The previous paragraph concerned the flat case. In general, we put

$$L_i := DSplit^t(m')^* \Box_{2k} DSplit^t(m') : \\ \mathcal{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w] \longrightarrow \mathcal{E}_{(\mp)}\{r_1, \ldots, r_{n'}\}_0[-w + 2s - n].$$

The signs on both sides for n even are different according to Table 2.1 hence this is exactly the formula from Table 3.2.

Now we should establish an analogue of (3.2). We note only that 2. follows immediately from the discussion above and 1. is analogous to the case of the (formally self–adjoint) operator $S_{n'}$ for n odd.

It remains to discuss $L_{n'}$ between singular positions $n'-1, n' \longrightarrow n', n'+1$ for n even or nonstandard positions $n' \longrightarrow n'+1$ for n odd. The first step is again the splitting $DSplit^t(m)$. (Note $m = m'$ in this case.) Then, if there is no spinor index, we use $\Box_2 = \Box$, and if there is (one) spinor index, i.e. $w - s = \frac{1-n}{2}$ (see Tables 2.3 and 2.4), we apply the Dirac operator $\not{D}$. Then the same reasons as above establish the result in Table 3.2.

Finally, let us note the long operators we have constructed are formally self–adjoint due to duality between $DSplit$ and $DSplit^*$ and self–duality of $\Box_{2k}$ and $\not{D}$.

Remark. In the flat case and for n even, we obtain also formulae for the operator L_0 because we can replace $\Box_n$ by $Y^{A_1} \cdots Y^{A_{k-1}} \Box D_{A_1} \cdots D_{A_{k-1}}$ which is invariant and nontrivial [35]. The rest follows the consideration above, the result is the formula

$$L_n = pr_t(m)^* Y^{A_1} \cdots Y^{A_{k-1}} \Box D_{A_1} \cdots D_{A_{k-1}} Split^t(m).$$

3.1.5. Conformally invariant operators: summary. All results from this section are summarised in Tables 3.1 and 3.2. The source space of all operators is of the form $\mathcal{E}_{(\pm)}\{r_1, \ldots, r_{n'}\}_0[w]$. This yields the parameters s, r, o_j, o_j^*, o etc. according to 1.1.3 and (2.91), and t and m according to Tables 2.3 and 2.4. The operators correspond (in the flat case) to the pattern from 2.1 which yields the coefficients $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ (for n even), $\bar{\Lambda}$ (for n odd) and Λ_j.

Short (long) operators are denoted by S (L) with a subscript indicating the position, see details in 1.3.3. Moreover, we use the notation

$$S_i^{(Y)} = \begin{cases} S_{n'-1}^Y & i = n'-1,\ n \text{ even} \\ S_i & \text{otherwise}, \end{cases} \qquad S_j^{(X)} = \begin{cases} S_{n'}^X & j = n',\ n \text{ even} \\ S_j & \text{otherwise}. \end{cases} \tag{3.5}$$

to simplify the presentation of results in Table 3.1. The first line in each part in both Tables describes positions of both spaces in the pattern, order of the operator and whether the operator preserves or changes the sign. The latter concerns only the even dimensional case, signs have no meaning in odd dimensions. The second line shows Young symmetries of both spaces and a tractor formula for the operator. Of course, the weight w therein must satisfy the condition for the corresponding position from Tables 2.3 and 2.4.

We assumed the complex setting up to now, cf. 2.2.3. The real case is different if $n = 2n'$, $n' - p$ is odd (here (p, q) is the signature of the metric) and $r_{n'} > 0$ on the position $n' - 1$, see 1.3.3. But then the complex operators

$S^X_{n'-1}$ and $S^Y_{n'-1}$ (and similarly $S^X_{n'}$ and $S^Y_{n'}$) differ only in the sign. Hence, the real operators $S_{n'-1}$ and $S_{n'}$ correspond to sums of (complex operators) $S^X_{n'-1} \oplus S^Y_{n'-1}$ and $S^X_{n'} \oplus S^Y_{n'}$, respectively.

3.1.6 Example. Following 2.2.4, we shall demonstrate results from this section on the space $\mathcal{E}_{(\pm)}(k,l)_0[w]$. That is, we provide explicit formulae for curved analogues of all flat invariant operators on $\mathcal{E}_{(\pm)}(k,l)_0[w]$ with the exception of L_0 for n even. We express them as compositions of operators B, M, T and their formal adjoints developed in Section 2. Let us remind we computed explicit formulae for all these operators in examples therein. We shall also present usual (i.e. tensor) formulae for operators up to the order 2.

(a) *Short operators.* Recall a simple recurrent procedure for (tensor) formulae of all standard operators with the exception of $S_{n'}$ for n odd is developed in [14]. This does not use the tractor calculus. Our result shows how these operators fit to the general picture.

The formulae for all standard operators in terms of B, M, T and ∇ are displayed in Tables 3.3 and 3.4 (cf. Table 3.1). Note we use the notation $S^{(X,Y)}_{n'-1}$ for $S^Y_{n'-1}$ and $S^X_{n'-1}$ if the latter two operators differ only in the sign. We use $S^{(X,Y)}_{n'}$ in a similar way. This happens if $\Lambda'_{n'} = 0$ in Table 2.1.

We met formulae in terms of ∇, $\boldsymbol{g}$ and the curvature of many of the operators throughout Examples in this thesis. In the review below, Proj always denotes projection to the target space. (In the Tables 3.3 and 3.4, this projection is not stated as this is achieved by formal adjoints B^*, M^* and T^*).

From the formulae (2.55) and (2.57) for the top operator, we obtained

Conformally invariant operators on $\mathcal{E}_{(\pm)}(k,l)_0[w]$, $n' \geq k \geq l \geq 1$ **I. Short operators in the 1st half of the pattern: $w \in \mathbb{Z}$**		
Conditions	**Formula**	**Order**
$w \geq k+l+2$	$S_0 : \mathcal{E}_{(\pm)}(k,l)_0[w] \longrightarrow \mathcal{E}_{(\pm)}(k,l,\underbrace{1,\cdots,1}_{o=w-k-l})_0[w]$ $S_0 f_{(\pm)\mathbf{ab}} = \big(B^{*D^0\mathbf{D}}{}_{\mathbf{a}} B^{*E^0\mathbf{E}}{}_{\mathbf{b}} M^{*C_o}{}_{c_o} \cdots M^{*C_2}{}_{c_2}$ $\nabla_{c_1} D_{C_2} \cdots D_{C_o} T_{E^0\mathbf{E}}{}^{\mathbf{e}} T_{D^0\mathbf{D}}{}^{\mathbf{d}} f_{(\pm)\mathbf{de}}\big)_{(\pm)}$	o
$l < k$	$S_l^{(Y)} : \mathcal{E}_{(\pm)}(k,l)_0[k+1] \longrightarrow \mathcal{E}_{(\pm)}(k,l+1)_0[k+1]$ $S_l^{(Y)} f_{(\pm)\mathbf{ab}} = \big(B^{*C^0\mathbf{C}}{}_{\mathbf{a}} \nabla_{[b^0} T_{\lvert C^0\mathbf{C}}{}^{\mathbf{c}} f_{(\pm)\mathbf{c}\lvert\mathbf{b}]}\big)_{(\pm)}$ where $S_l^{(Y)} = \begin{cases} S_l & l \neq \frac{n}{2}-1 \\ S_{n'-1} & l = \frac{n}{2}-1 \end{cases}$	1
$(l,k)=(n'-1,n')$ n even	$S^X_{n'-1} : \mathcal{E}_{\pm}(n',n'-1)_0[n'+1] \longrightarrow \mathcal{E}_{\mp}(n',n')_0[n'-1]$ $S^X_{n'-1} f_{\pm\mathbf{ab}} = \big(T^{*C^0\mathbf{C}}{}_{\mathbf{a}} \nabla_{[b^0} T_{\lvert C^0\mathbf{C}}{}^{\mathbf{c}} f_{\pm\mathbf{c}\lvert\mathbf{b}]}\big)_{\mp}$ $S^X_{n'-1} = (S^Y_{n'})^*$ where $S^Y_{n'} : \mathcal{E}_{\pm}(n',n')_0[n'+1] \longrightarrow \mathcal{E}_{\mp}(n',n'-1)_0[n'-3]$	3
$l \neq \frac{n}{2}$	$S_k : \mathcal{E}_{(\pm)}(k,l)_0[l] \longrightarrow \mathcal{E}_{(\pm)}(k+1,l)_0[k+1]$ $S_k f_{(\pm)\mathbf{ab}} = \big(M^{*\mathbf{C}}{}_{\mathbf{b}} \nabla_{[a^0} M_{\lvert\mathbf{C}\rvert}{}^{\mathbf{c}} f_{(\pm)\mathbf{a}]\mathbf{c}}\big)_{(\pm)}$	1
$l = k \leq n'-1$	$S_k^{(X,Y)} : \mathcal{E}(k,k)_0[k+1] \longrightarrow \mathcal{E}_{(\pm)}(k+1,k+1)_0[k+1]$ $S_k^{(X,Y)} f_{\mathbf{ab}} = \big(M^{*C^0\mathbf{C}}{}_{a^0\,\mathbf{a}} \nabla_{[b^0} T_{\lvert C^0\mathbf{C}}{}^{\mathbf{c}} f_{\mathbf{c}\lvert\mathbf{b}]}\big)_{(\pm)}$ where $S_k^{(X,Y)} = \begin{cases} S_k & k \neq \frac{n}{2}-1 \\ S^Y_{n'-1} \text{ or } S^X_{n'-1} & k = \frac{n}{2}-1 \end{cases}$	2
$l = k = n'$ n odd	$S_{n'} : \mathcal{E}(n',n')_0[n'+1] \longrightarrow \mathcal{E}(n',n')_0[n'-2]$ $S_{n'} f_{\mathbf{ab}} = T^{*C^0\mathbf{C}}{}_{\mathbf{a}} \boldsymbol{\epsilon}_{\mathbf{b}}{}^{d^0\mathbf{d}} \nabla_{d^0} T_{C^0\mathbf{C}}{}^{\mathbf{c}} f_{\mathbf{cd}}$ $S_{n'}$ is self–adjoint i.e. $(S_{n'})^* = S_{n'}$	3

Table 3.3: Table 3.1 for $\mathcal{E}_{(\pm)}(k,l)_0[w]$, Part I (see Example 3.1.6)

Conformally invariant operators on $\mathcal{E}_{(\pm)}(k,l)_0[w]$, $n' \geq k \geq l \geq 1$

II. Short operators in the 2nd half of the pattern: $w \in \mathbb{Z}$

Conditions	Formula	Order
$k=l=n'$ n even	$S^Y_{n'} : \mathcal{E}_{\pm}(n',n')_0[n'+1] \longrightarrow \mathcal{E}_{\mp}(n',n'-1)_0[n'-3]$ $S^Y_{n'-1} f_{\pm\mathbf{ab}} = \left(T^{*C^0\mathbf{C}}{}_{\mathbf{a}} \nabla^{b^1} T_{C^0}{}^{\mathbf{c}}{}_{\mathbf{C}} f_{\pm\mathbf{cb}}\right)_{\mp}$	3
$l=k$	$S^{(X,Y)}_{n-k} : \mathcal{E}_{(\pm)}(k,k)_0[3k-n] \longrightarrow \mathcal{E}(k-1,k-1)_0[3k-n-4]$ $S^{(X,Y)}_{n-k} f_{(\pm)\mathbf{ab}} = T^{*C^1\dot{\mathbf{C}}}{}_{\dot{\mathbf{b}}} \nabla^{a^1} M^{\mathbf{c}}_{\mathbf{C}} f_{(\pm)\mathbf{ac}}$ where $S^{(X,Y)}_{n-k} = \begin{cases} S_{n-k} & k \neq \frac{n}{2} \\ S^Y_{n'} \text{ or } S^X_{n'} & k = \frac{n}{2} \end{cases}$	2
	$S_{n-k} = (S_{k-1})^*$ where $S_{k-1} : \mathcal{E}(k-1,k-1)_0[k] \longrightarrow \mathcal{E}_{(\pm)}(k,k)_0[k]$	
$l<k$	$S_{n-k} : \mathcal{E}_{(\pm)}(k,l)_0[2k+l-n] \longrightarrow \mathcal{E}(k-1,l)_0[3k+l-n-2]$ $S_{n-k} f_{(\pm)\mathbf{ab}} = M^{\mathbf{C}}_{\mathbf{b}} \nabla^{a^1} M^{\mathbf{c}}_{\mathbf{C}} f_{(\pm)\mathbf{ac}}$	1
	$S_{n-k} = (S_{k-1})^*$ where $S_{k-1} : \mathcal{E}(k-1,l)_0[l] \longrightarrow \mathcal{E}_{(\pm)}(k,l)_0[l]$	
	$S^{(X)}_{n-l} : \mathcal{E}_{(\pm)}(k,l)_0[2k+l-n-1] \longrightarrow \mathcal{E}_{(\pm)}(k,l-1)_0[k+2l-n-3]$ $S^{(X)}_{n-l} f_{(\pm)\mathbf{ab}} = \left(T^{*C^0\mathbf{C}}{}_{\mathbf{a}} \nabla^{b^1} B_{C^0}{}^{\mathbf{c}}{}_{\mathbf{C}} f_{(\pm)\mathbf{cb}}\right)_{(\pm)}$ where $S^{(X)}_{n-l} = \begin{cases} S_{n-l} & l \neq \frac{n}{2} \\ S_{n'} & l = \frac{n}{2} \end{cases}$	1
	$S^{(X)}_{n-l} = (S^{(Y)}_{l-1})^*$ where $S^{(Y)}_{l-1} : \mathcal{E}(k,l-1)_0[k+1] \longrightarrow \mathcal{E}_{(\pm)}(k,l)_0[k+1]$	

Table 3.4: Table 3.1 for $\mathcal{E}_{(\pm)}(k,l)_0[w]$, Part II (see Example 3.1.6)

Conformally invariant operators on $\mathcal{E}_{(\pm)}(k,l)_0[w]$, $n' \geq k \geq l \geq 1$

III. Long operators $\mathcal{E}_{(\pm)}(k,l)_0[w] \longrightarrow \mathcal{E}_{(\mp)}(k,l)_0[w-o]$ where

- $w \in \mathbb{Z}$ for n even and $w \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ for n odd
- the order is $o := 2(w-k-l)+n$

Conditions	Formula
$w \geq k+\frac{3}{2}$ for $(l, \lceil w \rceil) \neq (n', n'+2)$	$f_{(\pm)\mathbf{ab}} \mapsto \left(T^{*C^0}{}^{\mathbf{C}}_{\mathbf{a}} T^{*D^0}{}^{\mathbf{D}}_{\mathbf{b}} \Box_{2(n'-i)} T_{D^0}{}^{\mathbf{d}}_{\mathbf{D}} T_{C^0}{}^{\mathbf{c}}_{\mathbf{C}} f_{(\pm)\mathbf{cd}}\right)_{(\mp)}$ where $i = 2+k+l-\lceil w \rceil$ and either n is odd or $w \leq k+l+1$
$(l, \lceil w \rceil) = (n', n'+2)$	$f_{(\pm)\mathbf{ab}} \mapsto \left(T^{*C^0}{}^{\mathbf{C}}_{\mathbf{a}} M^{*}{}^{\mathbf{D}}_{\mathbf{b}} \Box M^{\mathbf{d}}_{\mathbf{D}} T_{C^0}{}^{\mathbf{c}}_{\mathbf{C}} f_{(\pm)\mathbf{cd}}\right)_{(\mp)}$
$w \in \langle l+\frac{1}{2}; k+1 \rangle$ for $(l, \lceil w \rceil) \neq (n', n'+1)$	$f_{(\pm)\mathbf{ab}} \mapsto \left(T^{*C^0}{}^{\mathbf{C}}_{\mathbf{a}} M^{*}{}^{\mathbf{D}}_{\mathbf{b}} \Box_{2(n'-i)} M^{\mathbf{d}}_{\mathbf{D}} T_{C^0}{}^{\mathbf{c}}_{\mathbf{C}} f_{(\pm)\mathbf{cd}}\right)_{(\mp)}$ where $i = 1+k+l-\lceil w \rceil$
$(l, w) = (n', n'+\frac{1}{2})$, n odd	$f_{(\pm)\mathbf{ab}} \mapsto \left(M^{*}{}^{\mathbf{D}}_{\mathbf{b}} M^{*}{}^{\mathbf{C}}_{\mathbf{a}} \Box M^{\mathbf{c}}_{\mathbf{C}} M^{\mathbf{d}}_{\mathbf{D}} f_{(\pm)\mathbf{cd}}\right)_{(\mp)}$
$w \in \langle k+l+\frac{1}{2}-n'; l \rangle$	$f_{(\pm)\mathbf{ab}} \mapsto \left(M^{*}{}^{\mathbf{D}}_{\mathbf{b}} M^{*}{}^{\mathbf{C}}_{\mathbf{a}} \Box_{2(n'-i)} M^{\mathbf{c}}_{\mathbf{C}} M^{\mathbf{d}}_{\mathbf{D}} f_{(\pm)\mathbf{cd}}\right)_{(\mp)}$ where $i = k+l-\lceil w \rceil$

Table 3.5: Table 3.2 for $\mathcal{E}_{(\pm)}(k,l)_0[w]$, (see Example 3.1.6)

the operators

$$S^{(X)}_{n-l} : \mathcal{E}_{(\pm)}(k, l-1)_0[k+2l-n-1] \longrightarrow \mathcal{E}_{(\pm)}(k, l-1)_0[k+2l-n-3]$$
$$S^{(X)}_{n-l}(f_{\mathbf{ab}}) = \nabla^{b^1} f_{\mathbf{ab}}$$
$$S_{n-k} : \mathcal{E}_{(\pm)}(k, l)_0[2k+l-n] \longrightarrow \mathcal{E}(k-1, l)_0[2k+l-n-2],\ l < k$$
$$S_{n-k}(f_{\mathbf{ab}}) = \mathrm{Proj}\nabla^{a^1} f_{\mathbf{ab}}$$
$$S^{(X,Y)}_{n-k} : \mathcal{E}_{(\pm)}(k, k)_0[3k-n] \longrightarrow \mathcal{E}(k-1, k-1)_0[3k-n-4]$$
$$S^{(X,Y)}_{n-k}(f_{\mathbf{ab}}) = (\nabla^{(a^1}\nabla^{b^1)} + P^{a^1b^1})f_{\mathbf{ab}}$$
$$S_k : \mathcal{E}_{(\pm)}(k, l)_0[l] \longrightarrow \mathcal{E}_{(\pm)}(k+1, l)_0[l],\ l \neq \frac{n}{2}$$
$$S_k(f_{\mathbf{ab}}) = \mathrm{Proj}\nabla_{[a^1} f_{\mathbf{a}]\mathbf{b}}.$$

The following operators

$$S^{(Y)}_{l} : \mathcal{E}_{(\pm)}(k, l)_0[k+1] \longrightarrow \mathcal{E}_{(\pm)}(k, l+1)_0[k+1],\ l < k$$
$$S^{(Y)}_{l}(f_{\mathbf{ab}}) = \mathrm{Proj}\ \nabla_{[b^0} f_{|\mathbf{a}|\mathbf{b}]}$$
$$S^{(X,Y)}_{k} : \mathcal{E}(k, k)_0[k+1] \longrightarrow \mathcal{E}_{(\pm)}(k+1, k+1)_0[k+1],\ k < n'$$
$$S^{(X,Y)}_{k}(f_{\mathbf{ab}}) = \mathrm{Proj}(\nabla_{a^0}\nabla_{b^0} + P_{a^0b^0})f_{\mathbf{ab}}$$

where we skew over $[a^0\mathbf{a}]$ and $[b^0\mathbf{b}]$ in the latter before the projection Proj, can be obtained most easily as formal adjoints of $S^{(Y)}_{n-l}$ and $S^{(X,Y)}_{n-k}$ displayed above, respectively. It remains to discuss the operators S_0 for $w \geq k+l+2$. We shall mention only the orders 1 and 2:

$$S_0 : \mathcal{E}(k, l)_0[k+l+2] \longrightarrow \mathcal{E}_{(\pm)}(k, l, 1)_0[k+l+2]$$
$$S_0(f_{\mathbf{ab}}) = \mathrm{Proj}\nabla_{(c} f_{a^1|\dot{\mathbf{a}}|b^1)\dot{\mathbf{b}}}$$
$$S_0 : \mathcal{E}(k, l)_0[k+l+3] \longrightarrow \mathcal{E}_{(\pm)}(k, l, 1, 1)_0[k+l+3]$$
$$S_0(f_{\mathbf{ab}}) = \mathrm{Proj}\left[\nabla_{(d}\nabla_c f_{a^1|\dot{\mathbf{a}}|b^1)\dot{\mathbf{b}}} + P_{(dc} f_{a^1|\dot{\mathbf{a}}|b^1)\dot{\mathbf{b}}}\right].$$

These formulae as well as higher order cases can be obtained e.g. from [14].

Beside S_0, we have three other operators of order three: $S^X_{n'-1}$ and $S^Y_{n'}$ (mutually self adjoint) for n even and S'_n (a self adjoint operator) for n odd. [14] provides formulae for the first two but not for $S_{n'}$. This case has been treated in more details in Example 3.1.1. Using this, the tractor formula from Table 3.3 can be easily developed into a (long!) formula in terms of ∇, $\boldsymbol{g}$ and R only.

(b) *Long operators.* The order of these operators is $2(w-k-l)+n$ hence the lowest nonzero order, 2, corresponds to the operator

$$\mathcal{E}(k,l)_0[k+l+1-\frac{n}{2}] \longrightarrow \mathcal{E}(k,l)_0[k+l-1-\frac{n}{2}],\ l \neq \frac{n}{2}.$$

The restriction $l \neq \frac{n}{2}$ follows from the observation that in even dimensions, the displayed operator is a long operator between regular positions. But $\mathcal{E}_{\pm}(\frac{n}{2},\frac{n}{2})_0[\frac{n}{2}+1]$ appears on a position in the middle of the pattern where the long operators do not exist. (If $l = k = \frac{n}{2}$, the lowest positive order of a long operator will be 4.) We obtained a formula for the displayed operator in Example 2.1.7. This can differ from the formula in Table 3.5 in curvature terms (consider e.g. $C_{a^1}{}^{p}{}_{b^1}{}^{q} f_{p\dot{\mathbf{a}}q\dot{\mathbf{b}}}$). For higher orders, formula expressed only in ∇, $\boldsymbol{g}$ and R are getting too complicated. Table 3.5 shows their manageable version.

3.2 Conformal Killing equation on k–forms

We use the calculus developed until now to construct prolongations of the conformal Killing equation on forms. The latter is an overdetermined system of partial differential equations. Then we apply these results to the solution space. We obtain various relations between solutions.

To simplify subsequent computation we use the convention that indices labelled with sequential superscripts which are at the same level (i.e. all contravariant or all covariant) will indicate a completely skew set of indices. Formally we set $a^1 \cdots a^k = [a^1 \cdots a^k] = \mathbf{a}^k$.

Beside the conformal metric $\boldsymbol{g}_{ab}$, we will also use $\boldsymbol{g}_{\mathbf{a}^k\mathbf{b}^k}$ (and similarly $\boldsymbol{g}_{\dot{\mathbf{a}}^k\dot{\mathbf{b}}^k}$) for $\boldsymbol{g}_{a^1b^1} \cdots \boldsymbol{g}_{a^kb^k}$ where all a–indices and all b–indices are skewed over.

By definition, we require $k \geq l$ in the notation $\mathcal{E}(k,l)$. Here we will use also the opposite order for spaces $\mathcal{E}(1,k) := \mathcal{E}(k,1)$ and $\mathcal{E}(2,k) := \mathcal{E}(k,2)$ and similarly for the trace–free parts. The order of form indices in the notation for sections will be $f_{\mathbf{c}\mathbf{a}^k} \in \mathcal{E}(1,k)$ and $\tilde{f}_{\mathbf{c}^2\mathbf{a}^k} \in \mathcal{E}(2,k)$. We will later need the following identities

$$f_{a^1p\dot{\mathbf{a}}^k} = \frac{1}{k} f_{p\mathbf{a}^k} \quad \text{and} \quad \tilde{f}_{a^1qp\dot{\mathbf{a}}^k} = \frac{1}{k} \tilde{f}_{pq\mathbf{a}^k} \tag{3.6}$$

for $f_{\mathbf{c}\mathbf{a}^k} \in \mathcal{E}(1,k)[w]$ and $\tilde{f}_{\mathbf{c}^2\mathbf{a}^k} \in \mathcal{E}(2,k)[w]$. Similarly as (1.7), this follows from the skewing $[p\mathbf{a}^k]$ which vanishes in both cases. Using the second of these we recover, for example, the well known identities

$$R_{[a}{}^{b}{}_{c]}{}^{d} = \frac{1}{2} R_{ac}{}^{bd} \quad \text{and} \quad C_{[a}{}^{b}{}_{c]}{}^{d} = \frac{1}{2} C_{ac}{}^{bd}\ .$$

Using (3.6) and (1.18), a short computation reveals the transformations

$$\begin{aligned} \hat{\nabla}_{a^0} f_{\mathbf{c}\mathbf{a}^k} &= \nabla_{a^0} f_{\mathbf{c}\mathbf{a}^k} + (w-1)\Upsilon_{a^0} f_{\mathbf{c}\mathbf{a}^k} + \boldsymbol{g}_{ca^0} \Upsilon^p f_{p\mathbf{a}^k} \\ \hat{\nabla}^c f_{\mathbf{c}\mathbf{a}^k} &= \nabla^c f_{\mathbf{c}\mathbf{a}^k} + (n+w-k-1)\Upsilon^c f_{\mathbf{c}\mathbf{a}^k} \\ \hat{\nabla}^{c^1} \tilde{f}_{\mathbf{c}^2\mathbf{a}^k} &= \nabla^{c^1} \tilde{f}_{\mathbf{c}^2\mathbf{a}^k} + (n+w-k-3)\Upsilon^{c^1} \tilde{f}_{\mathbf{c}^2\mathbf{a}^k} \end{aligned} \tag{3.7}$$

for $f_{\mathbf{c}\mathbf{a}^k} \in \mathcal{E}(1,k)_0[w]$ and $\tilde{f}_{\mathbf{c}^2\mathbf{a}^k} \in \mathcal{E}(2,k)_0[w]$.

3.2.1. The conformal Killing equation on forms. The space $\mathcal{E}_{\mathbf{c}\mathbf{a}^k} = \mathcal{E}_c \otimes \mathcal{E}_{a^1\cdots a^k}$ is completely reducible for $1 \leq k \leq n$ and we have the $O(g)$-decomposition $\mathcal{E}_{\mathbf{c}\mathbf{a}^k}[w] \cong \mathcal{E}_{[\mathbf{c}\mathbf{a}^k]}[w] \oplus \mathcal{E}_{\{\mathbf{c}\mathbf{a}^k\}_0}[w] \oplus \mathcal{E}_{\mathbf{a}^{k-1}}[w-2]$ where the bundle $\mathcal{E}_{\{\mathbf{c}\mathbf{a}^k\}_0}[w]$ consists of rank $k+1$ trace-free tensors $T_{\mathbf{c}\mathbf{a}^k}$ (of conformal weight w) that are skew on the indices $a^1 \cdots a^k$ and have the property that $T_{[ca^1\cdots a^k]} = 0$.

(Note that the three spaces on the right-hand side are $SO(g)$-irreducible if $k \notin \{n/2, n/2 \pm 1\}$). On the space $\mathcal{E}_{\mathbf{ca}^k}[w]$ there is a projection $\mathcal{P}_{\{\mathbf{ca}^k\}_0}$ to the component $\mathcal{E}_{\{\mathbf{ca}^k\}_0}[w]$ and we will use the notation

$$T_{\mathbf{ca}^k} \overset{\{\mathbf{ca}^k\}_0}{=} S_{\mathbf{ca}^k} \quad \text{or} \quad T_{\mathbf{ca}^k} =_{\{\mathbf{ca}^k\}_0} S_{\mathbf{ca}^k}$$

to mean that $\mathcal{P}_{\{\mathbf{ca}^k\}_0}(T) = \mathcal{P}_{\{\mathbf{ca}^k\}_0}(S)$. We will also use the projection $\mathcal{P}_{\{\mathbf{ca}^k\}}$ to $\mathcal{E}(1,k)[w] =: \mathcal{E}_{\{\mathbf{ca}^k\}}[w]$.

Each metric from the conformal class determines a corresponding Levi-Civita connection ∇ and for $1 \leq k \leq n-1$ and $\sigma_{\mathbf{a}^k} \in \mathcal{E}^k[k+1]$, we may form $\nabla_c \sigma_{\mathbf{a}^k}$. This is not conformally invariant. However it is straightforward to verify that its projection $\mathcal{P}_{\{\mathbf{ca}^k\}_0}(\nabla\sigma)$ is conformally invariant. That is, this is independent of the choice of metric (and corresponding Levi-Civita connection) from the conformal class. Thus the equation

$$\nabla_{\{c}\sigma_{\mathbf{a}^k\}_0} = 0, \qquad 1 \leq k \leq n-1 \tag{3.8}$$

called the (form) *conformal Killing equation*, is conformally invariant.

Suppose $\tilde{\nabla}$ is a connection on another vector bundle (or space of sections thereof) $\mathcal{E}_\bullet$. For this connection coupled with the Levi-Civita connection let us also write $\tilde{\nabla}$. Since it is a first order equation (3.8) is strongly invariant (cf. [30, 21]). That is, if now $\sigma_{\mathbf{a}^k} \in \mathcal{E}_{\mathbf{a}^k\bullet}[k+1] = \mathcal{E}_{\mathbf{a}^k}[k+1] \otimes \mathcal{E}_\bullet$ then $\tilde{\nabla}_{\{c}\sigma_{\mathbf{a}^k\}_0} = 0$ is also conformally invariant. We will also call any such equation a conformal Killing equation (or sometimes for emphasis a *coupled conformal Killing equation*).

The volume form $\boldsymbol{\epsilon}$ determines the Hodge operator $\tilde{\boldsymbol{\epsilon}}$ on density valued forms, see 1.1.3. We will denote this operator by $\star$ here because this is the usual notation (see e.g. [11]) and because we do not need to use $\tilde{\boldsymbol{\epsilon}}$ with attached indices. That is, we have a mapping

$$\star : \mathcal{E}^k[w] \longrightarrow \mathcal{E}^{n-k}[n-2k+w] \quad k \in \{0, 1, \cdots, n\} .$$

In particular we have

$$\star : \mathcal{E}^k[k+1] \longrightarrow \mathcal{E}^{n-k}[n-k+1],$$

and from elementary classical SO(n)-representation theory it follows easily that $\sigma \in \mathcal{E}^k[k+1]$ solves (3.8) for k-forms if and only if $\star\sigma$ solves the version of (3.8) for $(n-k)$-forms. Thus on oriented manifolds it is only strictly necessary to study this equation for (weighted) k-forms with $k \leq n/2$. Also it follows that on even dimensional oriented manifolds a form in $\mathcal{E}^{n/2}[n/2+1]$ is a solution of (3.8) if and only if its self-dual and anti-self dual parts are separately solutions. Nevertheless, since the redundancy does us no harm, we shall ignore these observations and in the following simply treat the equation on k-forms for $1 \leq k \leq n-1$.

3.2.2. Invariant prolongation for conformal Killing forms. In this section, and in much of the subsequent work, we will write $f_{\mathbf{a}}$ (rather than $f_{\mathbf{a}^k}$) to denote a section in $\mathcal{E}_{\mathbf{a}^k}[k+1]$. That is, the superscript of the form index $\mathbf{a}$ will be omitted but can be taken to be k (or otherwise if clear from the context).

Before we start with the construction of the prolongation, we will introduce some notation for certain algebraic actions of the curvature on tensors. Let us write $\sharp$ (which we will term *hash*) for the natural action of sections A of End(TM) on tensors. For example, on a covariant 2-tensor T_{ab}, we have

$$A\sharp T_{ab} = -A^c{}_a T_{cb} - A^c{}_b T_{ac}.$$

If A is skew for a metric g, then at each point, A is $\mathfrak{so}(g)$-valued. The hash action thus commutes with the raising and lowering of indices and preserves the SO(g)-decomposition of tensors. For example the Riemann tensor may be viewed as an End(TM)-valued 2-form R_{ab} and in this notation we have

$$[\nabla_a, \nabla_b]T = R_{ab}\sharp T\ ,$$

for an arbitrary tensor T. Similarly we have $C_{ab}\sharp T$ for the Weyl curvature. As a section of the tensor square of the g-skew bundle endomorphisms of TM, the Weyl curvature also has a double hash action that we denote $C\sharp\sharp T$.

We need some more involved actions of the Weyl tensor on $\mathcal{E}_{\mathbf{a}^k}[w]$ for $k \geq 2$. These are given by

$$\begin{aligned}(C\blacklozenge f)_{c\dot{\mathbf{a}}} :=& \frac{k-2}{k}\left(C_{ca^2}{}^{pq}f_{pq\ddot{\mathbf{a}}} + C_{a^3a^2}{}^{pq}f_{pqc\cdot\ddot{\mathbf{a}}\cdot}\right) \in \mathcal{E}_{c\dot{\mathbf{a}}^k}[w-2] \\ (C\lozenge f)_{\mathbf{ca}} :=& C_{c^1c^2a^1}{}^{p}f_{p\dot{\mathbf{a}}} + C_{a^1a^2c^1}{}^{p}f_{pc^2\ddot{\mathbf{a}}} + \frac{k}{n-k}\boldsymbol{g}_{c^1a^1}(C\blacklozenge f)_{c^2\dot{\mathbf{a}}} \\ & \in \mathcal{E}_{\mathbf{c}^2\mathbf{a}^k}[w]\end{aligned} \tag{3.9}$$

where $\mathbf{c} = \mathbf{c}^2$ and $f_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[w]$. Note that $C\lozenge f$ vanishes for $k = n-1$ since $\mathcal{E}(2, n-1)_0$ is trivial. (We have used the Lemma 3.2.3 (ii) here, see below). For the sake of complete clarity we have given these explicit formulae but note that, up to a multiple, the first of these is simply $C\sharp f \in \mathcal{E}_{\mathbf{c}^2\mathbf{a}^k}$ followed by projection to $\mathcal{E}(1, k-1)[w-2]$ (the projection involves a trace), while the second is $C\sharp f$ followed by projection to $\mathcal{E}(2, k)_0[w]$. This is clear except for the final projection in each case which we now verify.

3.2.3 Lemma. *Let us suppose $k \geq 2$. Then*

(i) $(C\blacklozenge f)_{c\dot{\mathbf{a}}} = C_{\{ca^2}{}^{pq}f_{|pq|\ddot{\mathbf{a}}\}} \in \mathcal{E}(1, k-1)_0[w-2]$

(ii) $(C\lozenge f)_{\mathbf{ca}} \in \mathcal{E}(2, k)_0[w]$

Proof. (i) It follows from (3.9) and the Bianchi identity that $(C\blacklozenge f)_{c\dot{\mathbf{a}}}$ is trace-free. Moreover

$$C_{\{ca^2}{}^{pq}f_{|pq|\ddot{\mathbf{a}}\}} = C_{ca^2}{}^{pq}f_{pq\ddot{\mathbf{a}}} - C_{[ca^2}{}^{pq}f_{|pq|\ddot{\mathbf{a}}]} = (C\blacklozenge f)_{c\dot{\mathbf{a}}}. \tag{3.10}$$

where the first equality is just the definition of the projection $\{..\}$ and the second follows from re-expressing of the skew symmetrisation $[c\dot{\mathbf{a}}]$ in the last display.

(ii) According to the definition of $\mathcal{E}(2,k)_0$, we are required to show that $(C\lozenge f)_{c^1[c^2\mathbf{a}]} = (C\lozenge f)_{[\mathbf{c}\dot{\mathbf{a}}]a^{k+1}} = 0$ (note $(C\lozenge f)_{[\mathbf{ca}]} = 0$ is obvious from (3.9)) and also that $C\lozenge f$ is trace-free. Both skew symmetrisation's $[c^2\mathbf{a}]$ and $[\mathbf{c}\dot{\mathbf{a}}]$ kill the last term of $C\lozenge f$ in (3.9), because $(C\blacklozenge f)_{[\mathbf{c}\dot{\mathbf{a}}]} = 0$ according to the Lemma (i). Applying the symmetrisation $[c^2\mathbf{a}]$ to the first two terms in (3.9) and using the Bianchi identity yields

$$C_{c^1[c^2a^1}{}^p f_{|p|\dot{\mathbf{a}}]} + \frac{1}{2}C_{[a^1a^2|c^1}{}^p f_{p|c^2\ddot{\mathbf{a}}]},$$

where the indices c^1c^2 are *not* skewed over. This is zero because $C_{c^1[c^2a^1]}{}^p = -\frac{1}{2}C_{c^2a^1c^1}{}^p$. The second skew symmetrisation $[\mathbf{c}\dot{\mathbf{a}}]$ is similar.

It remains to prove $\boldsymbol{g}^{c^1a^1}(C\lozenge f)_{\mathbf{ca}} = 0$. Tracing the last term in (3.9) yields

$$\frac{k}{n-k}\boldsymbol{g}^{c^1a^1}\boldsymbol{g}_{c^1a^1}(C\blacklozenge f)_{c^2\dot{\mathbf{a}}} = \frac{1}{2}(C\blacklozenge f)_{c^2\dot{\mathbf{a}}}$$

after a short computation. Further computations reveal

$$\boldsymbol{g}^{c^1a^1}C_{c^1c^2a^1}{}^p f_{p\dot{\mathbf{a}}} = -\frac{k-1}{2k}C_{c^2a^2}{}^{pq} f_{pq\ddot{\mathbf{a}}}$$

and

$$\boldsymbol{g}^{c^1a^1}C_{a^1a^2c^1}{}^p f_{pc^2\ddot{\mathbf{a}}} = -\frac{k-2}{2k}C_{a^3a^2}{}^{pq} f_{pqc^2\dddot{\mathbf{a}}} + \frac{1}{2k}C_{c^2a^2}{}^{pq} f_{pq\ddot{\mathbf{a}}}.$$

Summing the last three displays, the Lemma part (ii) follows from (3.9) for $C\blacklozenge f$. □

Introducing new variables, the equation (3.8) may be re-expressed in the form

$$\nabla_c\sigma_{\mathbf{a}} = \mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}\ ,$$

where $\mu_{a^0\mathbf{a}} \in \mathcal{E}_{a^0\mathbf{a}^k}[k+1]$ and $\nu_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}^k}[k-1]$. These capture some of the 1-jet information: we have $\mu_{a^0\mathbf{a}} = \nabla_{a^0}\sigma_{\mathbf{a}}$, and $\nu_{\dot{\mathbf{a}}} = \frac{k}{n-k+1}\nabla^p\sigma_{p\dot{\mathbf{a}}}$. We need a further set of variables to complete (3.8) to a first order closed system. There is some choice here, but, for the purposes of studying conformal invariance, it turns out that $\rho_{\mathbf{a}} := -\frac{1}{k}\nabla_{a^1}\nu_{\dot{\mathbf{a}}} + \frac{1}{nk}\nabla^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0} - P_{a^1}{}^p\sigma_{p\dot{\mathbf{a}}}$ is a judicious choice. We then have the following result.

3.2.4 Proposition. *Solutions of the conformal Killing equation (3.8), for $1 \leq k \leq n-1$, are in 1-1 correspondence with solutions of the following system on $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$, $\mu_{a^0\mathbf{a}} \in \mathcal{E}_{a^0\mathbf{a}^k}[k+1]$, $\nu_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}^k}[k-1]$ and $\rho_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k-1]$:*

$$\begin{aligned}
&\nabla_c\sigma_{\mathbf{a}} = \mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}\ ;\\
&\nabla_c\mu_{a^0\mathbf{a}} = (k+1)\left[\boldsymbol{g}_{ca^0}\rho_{\mathbf{a}} - P_{ca^0}\sigma_{\mathbf{a}} - \frac{1}{2}C_{a^0a^1c}{}^p\sigma_{p\dot{\mathbf{a}}}\right]\ ;\\
&\nabla_c\nu_{\dot{\mathbf{a}}} = -k\left[\rho_{c\dot{\mathbf{a}}} + P_c{}^p\sigma_{p\dot{\mathbf{a}}}\right] + \frac{k(k-1)}{2(n-k)}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}\ ;\\
&\nabla_c\rho_{\mathbf{a}} = P_{ca^1}\nu_{\dot{\mathbf{a}}} - P_c{}^p\mu_{p\mathbf{a}} + \frac{1}{2}A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} - A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}}\\
&\qquad\qquad + \frac{1}{2}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} - \frac{k}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \qquad \text{for } k \geq 2;\\
&\nabla_c\rho_{a^1} = P_{ca^1}\nu - P_c{}^p\mu_{pa^1} + A_{a^1pc}\sigma^p \qquad\qquad \text{for } k = 1.
\end{aligned} \tag{3.11}$$

The mapping from solutions $\sigma_{\mathbf{a}}$ of (3.8) to solutions $(\sigma_{\mathbf{a}}, \mu_{a^0\mathbf{a}}, \nu_{\dot{\mathbf{a}}}, \rho_{\mathbf{a}})$ of the system above is

$$\begin{aligned}
\sigma_{\mathbf{a}} \mapsto \Big(&\sigma_{\mathbf{a}},\ \nabla_{a^0}\sigma_{\mathbf{a}},\ \frac{k}{n-k+1}\nabla^p\sigma_{p\dot{\mathbf{a}}},\\
&\frac{1}{nk}\nabla^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0} - \frac{1}{n-k+1}\nabla_{a^1}\nabla^p\sigma_{p\dot{\mathbf{a}}} - P_{a^1}^{\ p}\sigma_{p\dot{\mathbf{a}}}\Big)
\end{aligned} \tag{3.12}$$

Proof. As mentioned above the first equation $\nabla_c\sigma_{\mathbf{a}} = \mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$ is simply a restatement of the conformal Killing equation (3.8) afforded by introducing the new variables $\mu_{a^0\mathbf{a}} \in \mathcal{E}_{[a^0\mathbf{a}]}[k+1]$ and $\nu_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}}[k-1]$. (At this point and until further notice below we take the rank of σ to be in the range $1 \leq k \leq n-1$.)

This equation also gives $\mu_{a^0\mathbf{a}}$ and $\nu_{\dot{\mathbf{a}}}$ in terms of derivatives of $\sigma_{\mathbf{a}}$. Thus the Proposition is clear except that we should verify that if $\sigma_{\mathbf{a}}$ solves (3.8) then we have the second, third and fourth equations of (3.11).

To establish the second equation, let us observe $(k+2)\nabla_{[c}\nabla_{a^0}\sigma_{\mathbf{a}]} = \nabla_c\nabla_{a^0}\sigma_{\mathbf{a}} - (k+1)\nabla_{a^1}\nabla_{[a^0}\sigma_{c\dot{\mathbf{a}}]}$, and that the left-hand-side vanishes due to the Bianchi identity. The first term on the right hand side is just $\nabla_c\mu_{a^0\mathbf{a}}$ thus

$$\begin{aligned}\nabla_c\mu_{a^0\mathbf{a}} &= (k+1)\nabla_{a^1}\mu_{a^0c\dot{\mathbf{a}}} = (k+1)\nabla_{a^1}\left(\nabla_{a^0}\sigma_{c\dot{\mathbf{a}}} - \boldsymbol{g}_{a^0[c}\nu_{\dot{\mathbf{a}}]}\right)\\ &= (k+1)\left(\frac{1}{2}R_{a^1a^0c}{}^p\sigma_{p\dot{\mathbf{a}}} - \frac{1}{k}\boldsymbol{g}_{ca^0}\nabla_{a^1}\nu_{\dot{\mathbf{a}}}\right)\end{aligned}$$

where the second equality follows from the first equation in (3.11) and the third equality from the Bianchi identity. Now the equation for $\nabla_c\mu_{a^0\mathbf{a}}$ in (3.11) follows from the last display using (1.16) and from the relation $\rho_{\mathbf{a}} = -\frac{1}{k}\nabla_{a^1}\nu_{\dot{\mathbf{a}}} - P_{a^1}{}^p\sigma_{p\dot{\mathbf{a}}}$, which we have for solutions.

The second equation in (3.11) concerns $\nabla_c\nu_{\dot{\mathbf{a}}} = \frac{k}{n-k+1}\nabla_c\nabla^p\sigma_{p\dot{\mathbf{a}}}$. Commuting the covariant derivatives we get $\nabla_c\nabla^p = R_c{}^p\sharp + \nabla^p\nabla_c$ where, recall, $\sharp$ captures the action of the Riemann curvature tensor R. Therefore

$$\begin{aligned}(n-k+1)\nabla_c\nu_{\dot{\mathbf{a}}} &= k\left[R_c{}^p{}_p{}^q\sigma_{q\dot{\mathbf{a}}} + (k-1)R_c{}^p{}_{a^2}{}^q\sigma_{pq\ddot{\mathbf{a}}} + \nabla^p\left(\mu_{cp\dot{\mathbf{a}}} + \boldsymbol{g}_{c[p}\nu_{\dot{\mathbf{a}}]}\right)\right]\\ &= k\Big[-\mathrm{Ric}_c{}^p\,\sigma_{p\dot{\mathbf{a}}} + \frac{1}{2}(k-1)R_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} - \nabla^p\mu_{pc\dot{\mathbf{a}}} + \frac{1}{k}\nabla_c\nu_{\dot{\mathbf{a}}}\Big]\end{aligned}$$

where we have used $\nabla^p\nu_{p\ddot{\mathbf{a}}} = \frac{k}{n-k+1}\nabla^p\nabla^q\sigma_{qp\ddot{\mathbf{a}}} = 0$. Note that the last term here is a multiple of the left-hand-side. We consider the other terms on the right-hand-side. Recall that (1.16) gives $\mathrm{Ric}_{ab} = (n-2)\rho_{ab} + \P\boldsymbol{g}_{ab}$. Using (1.16) also for the second term on the right-hand-side, and the equation for $\nabla_c\mu_{a^0\mathbf{a}}$ in (3.11) for the third, a computation yields

$$\begin{aligned}-\mathrm{Ric}_c{}^p\,\sigma_{p\dot{\mathbf{a}}} &= -(n-2)P_c{}^p\sigma_{p\dot{\mathbf{a}}} - P\sigma_{c\dot{\mathbf{a}}}\\ \frac{1}{2}(k-1)R_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} &= \frac{1}{2}(k-1)C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} + 2(k-1)\delta^p{}_{[c}P_{a^2]}{}^q\sigma_{pq\ddot{\mathbf{a}}}\\ &= \frac{1}{2}(k-1)C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} - (k-1)\left(P_{a^2}{}^p\sigma_{pc\ddot{\mathbf{a}}} - P_c{}^p\sigma_{p\dot{\mathbf{a}}}\right)\\ -\nabla^p\mu_{pc\dot{\mathbf{a}}} &= -(n-k)\rho_{c\dot{\mathbf{a}}} + P\sigma_{c\dot{\mathbf{a}}} - kP_{[c}{}^p\sigma_{|p|\dot{\mathbf{a}}]} - \frac{1}{2}(k-1)C_{[a^2c}{}^{qp}\sigma_{|pq|\ddot{\mathbf{a}}]}.\end{aligned}$$

Hence the last but one display says that $\frac{n-k}{k}\nabla_c\nu_{\dot{\mathbf{a}}}$ is equal to the sum of the right hand sides of the last display. Now using the relation $-kP_{[c}{}^p\sigma_{|p|\dot{\mathbf{a}}]} = -P_c{}^p\sigma_{p\dot{\mathbf{a}}}+(k-1)P_{a^2}{}^p\sigma_{pc\ddot{\mathbf{a}}}$ and (3.10) we obtain immediately the third equation in (3.11).

The last equation requires more computation. Let us first make an observation about its skew-symmetric part $\nabla_{[c}\rho_{\mathbf{a}]}$. Using the definition of ρ and the Bianchi identity, we have $\nabla_{[c}\rho_{\mathbf{a}]} = -\nabla_{[c}P_{a^1}{}^p\sigma_{|p|\dot{\mathbf{a}}]}$. Using the Leibniz rule and the first equation in (3.11) for the right hand side, we obtain

$$\nabla_{[c}\rho_{\mathbf{a}]} = -\frac{1}{2}A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} - P_{[c}{}^p\mu_{|p|\mathbf{a}]}, \tag{3.13}$$

since the term $P_{a^1}{}^p\boldsymbol{g}_{c[p}\nu_{\dot{\mathbf{a}}]}$ vanishes after the skew symmetrisation $[c\mathbf{a}]$. Now to compute the full section $\nabla_c\rho_{\mathbf{a}}$, we shall start with the equation for $\nabla_c\nu_{\dot{\mathbf{a}}}$ from (3.11). We apply ∇_{a^1} to both sides of this equation and skew over all a–indices. Commuting the covariant derivatives on the left-hand-side, we obtain $\nabla_{a^1}\nabla_c = \nabla_c\nabla_{a^1} + R_{a^1c}\sharp$. The first term on the right hand side is $-k\nabla_{a^1}\rho_{c\dot{\mathbf{a}}} = (k+1)\nabla_{[c}\rho_{\mathbf{a}]} - \nabla_c\rho_{\mathbf{a}}$. Through these observations, and using (3.13), we obtain

$$\begin{aligned}\nabla_c\nabla_{a^1}\nu_{\dot{\mathbf{a}}} + (k-1)R_{a^1ca^2}{}^p\nu_{p\ddot{\mathbf{a}}} &= -(k+1)\big(\frac{1}{2}A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} + P_{[c}{}^p\mu_{|p|\mathbf{a}]}\big)\\ &\quad -\nabla_c\rho_{\mathbf{a}} - k\nabla_{a^1}P_c{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}.\end{aligned}$$

Many terms can be simplified and we shall start with the the first term on the left-hand-side. We have

$$\nabla_c\nabla_{a^1}\nu_{\dot{\mathbf{a}}} = -k\big(\nabla_c\rho_{\mathbf{a}} + \nabla_cP_{a^1}^{\ p}\sigma_{|p|\dot{\mathbf{a}}}\big)$$

which follows from the equation for $\nabla_c\nu_{\dot{\mathbf{a}}}$ in (3.11). Combining the last two displays we obtain

$$\begin{aligned}-(k-1)\nabla_c\rho_{\mathbf{a}} &= 2k\nabla_{[c}P_{a^1]}{}^p\sigma_{p\dot{\mathbf{a}}} - (k+1)\big(\frac{1}{2}A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} + P_{[c}{}^p\mu_{|p|\mathbf{a}]}\big)\\ &\quad -\frac{1}{2}(k-1)R_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}.\end{aligned}$$

where we have also used $R_{a^1ca^2}{}^p = \frac{1}{2}R_{a^1a^2c}{}^p$. Note that for the case of (the rank of σ being) $k = 1$ both sides of the equality above vanish and we get no information. Now we simplify terms on the right hand side: the first term using the Leibniz rule and the equation for $\nabla_c\sigma_{\mathbf{a}}$, the next two terms re-expressing the skew symmetrisation $[c\mathbf{a}]$ and the first curvature term using the decomposition (1.16). This yields

$$
\begin{aligned}
2k\nabla_{[c}P_{a^1]}{}^p\sigma_{p\dot{\mathbf{a}}} &= kA^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} + 2kP_{[a^1}{}^p\mu_{c]p\dot{\mathbf{a}}} + 2kP_{[a^1}{}^p\boldsymbol{g}_{c][p}\nu_{\dot{\mathbf{a}}]} \\
&= kA^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} + kP_{a^1}{}^p\mu_{cp\dot{\mathbf{a}}} - kP_c{}^p\mu_{a^1p\dot{\mathbf{a}}} + (k-1)\boldsymbol{g}_{ca^1}P_{a^2}{}^p\nu_{p\ddot{\mathbf{a}}} \\
-\frac{1}{2}(k+1)A^p{}_{[ca^1}\sigma_{|p|\dot{\mathbf{a}}]} &= -A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} + \frac{1}{2}(k-1)A^p{}_{a^2a^1}\sigma_{pc\ddot{\mathbf{a}}} \\
-(k+1)P_{[c}{}^p\mu_{|p|\mathbf{a}]} &= -P_c{}^p\mu_{p\mathbf{a}} + kP_{a^1}{}^p\mu_{pc\dot{\mathbf{a}}} \\
-\frac{1}{2}(k-1)R_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} &= -\frac{1}{2}(k-1)\left[C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}} + 2\boldsymbol{g}_{ca^1}P_{a^2}{}^p\nu_{p\ddot{\mathbf{a}}} + 2P_{ca^1}\nu_{\dot{\mathbf{a}}}\right].
\end{aligned}
$$

Substituting these in the previous display, the Proposition for $k \geq 2$ follows. The case $k = 1$ can be checked directly by tracing $\frac{1}{2}R_{c^0c^1}\sharp\mu_{a^0a^1} = \nabla_{c^0}\nabla_{c^1}\mu_{a^0a^1} = \nabla_{c^0}\left[2\boldsymbol{g}_{c^1a^0}\rho_{a^1} - 2P_{c^1a^0}\sigma_{a^1} - C_{a^0a^1c^1}{}^p\sigma_p\right]$. □

3.2.5 Lemma. *Let us fix $k \geq 2$. If $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ is a solution of (3.8) then $(C\lozenge\sigma)_{c\mathbf{a}} = 0$.*

Proof. We shall prove the lemma using the prolongation (3.11). Applying ∇_{c^1} to both sides of the equation for $\nabla_{c^2}\sigma_{\mathbf{a}}$, we obtain

$$\nabla_{c^1}\nabla_{c^2}\sigma_{\mathbf{a}} = \nabla_{c^1}\mu_{c^2\mathbf{a}} + \boldsymbol{g}_{c^2a^1}\nabla_{c^1}\nu_{\dot{\mathbf{a}}}.$$

The left-hand side is equal to

$$\frac{k}{2}R_{c^1c^2a^1}{}^p\sigma_{p\dot{\mathbf{a}}} = \frac{k}{2}C_{c^1c^2a^1}{}^p\sigma_{p\dot{\mathbf{a}}} + k\boldsymbol{g}_{c^1a^1}P_{c^2}{}^p\sigma_{p\dot{\mathbf{a}}} + kP_{c^1a^1}\sigma_{c^2\dot{\mathbf{a}}}$$

according to (1.16). On the other hand, from (3.11) the right-hand side is equal to

$$\Big(-k\boldsymbol{g}_{c^1a^1}\rho_{c^2\dot{\mathbf{a}}} + kP_{c^1a^1}\sigma_{c^1\dot{\mathbf{a}}} - \frac{1}{2}\big(2C_{c^2a^1c^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} - (k-1)C_{a^2a^1c^1}{}^{p}\sigma_{pc^2\ddot{\mathbf{a}}}\big)\Big)$$
$$+\boldsymbol{g}_{c^2a^1}\Big(-k\rho_{c^1\dot{\mathbf{a}}} - kP_{c^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}(C\blacklozenge\sigma)_{c^1\dot{\mathbf{a}}}\Big).$$

Now equating these two displays and using $C_{c^2a^1c^1}{}^{p} = -\frac{1}{2}C_{c^1c^2a^1}{}^{p}$ we obtain an identity which holds for solutions. Comparing the expression with the definition of $(C\lozenge\sigma)$ in (3.9), we see the identity is

$$(k-1)(C\lozenge\sigma) = 0.$$

□

Note that a curvature condition, equivalent to that in Lemma 3.2.5, is in [38]. There the identity for solutions is stated in terms of the Riemann tensor R, rather than in terms of the Weyl tensor C. In this form it has also been derived in [45] (although I could not find the necessary restriction $k \geq 2$ in that source). Expressing the identity via the Weyl curvature, as we do, emphasises that this is a conformally invariant condition.

Next we observe that (3.12) defines a conformally invariant differential splitting operator. We define a differential operator $\mathbb{D}$ on $\mathcal{E}_{\mathbf{a}^k}[k+1]$ by

$$\sigma_{\mathbf{a}} \mapsto \sigma_{A^0\mathbf{A}} := \mathbb{Y}_{A^0\mathbf{A}}^{\mathbf{a}}\sigma_{\mathbf{a}} + \frac{1}{k+1}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}}\mu_{a^0\mathbf{a}} + \mathbb{W}_{A^0A^1\dot{\mathbf{A}}}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}} - \mathbb{X}_{A^0\mathbf{A}}^{\mathbf{a}}\rho_{\mathbf{a}}, \qquad (3.14)$$

where $\sigma_{\mathbf{a}}$, $\mu_{a^0\mathbf{a}}$, $\nu_{\dot{\mathbf{a}}}$ and $\rho_{\mathbf{a}}$ are given by (3.12). Then we have the following.

3.2.6 Lemma. $\mathbb{D}$ *is a conformally invariant operator*

$$\mathbb{D} : \mathcal{E}_{\mathbf{a}^k}[k+1] \longrightarrow \mathcal{E}_{A^0\mathbf{A}^k} \quad \textit{for} \quad 1 \leq k \leq n-1.$$

Proof. Let us compare $\mathbb{D}$ and $T_{A^0\mathbf{A}}{}^{\mathbf{a}}$ from Example 2.1.6 for $\sigma \in \mathcal{E}_{\mathbf{a}^k}[k+1]$. It follows from the formula (2.54) with $w = k+1$ that $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{W}$ slots of

$T_{A^0\mathbf{A}}{}^{\mathbf{a}}$ and $\mathbb{D}$ are equal up to the multiple $n(k+1)(n-k+1)$. Let us compute the $\mathbb{X}$–slot of $\mathbb{D}$. Clearly

$$\begin{aligned}\frac{1}{nk}\nabla^p\nabla_{\{p}\sigma_{\mathbf{a}\}_0} &= \frac{1}{nk}\nabla^p\Big[\nabla_p\sigma_{\mathbf{a}} - \nabla_{[p}\sigma_{\mathbf{a}]} - \frac{k}{n-k+1}\boldsymbol{g}_{pa^1}\nabla^q\nu_{q\dot{\mathbf{a}}}\Big] \\ &= \frac{1}{n(k+1)(n-k+1)}\Big[(n-k+1)\Delta\sigma_{\mathbf{a}} + (n-2k)\nabla_{a^1}\nabla^q\nu_{q\dot{\mathbf{a}}} \\ &\qquad + (n-k+1)R^p{}_{a^1}\sharp\sigma_{p\dot{\mathbf{a}}}\Big]\end{aligned}$$

where the second equality follows after a simple calculation. The $\mathbb{X}$–slot of $\mathbb{D}$ is given by $\rho_{\mathbf{a}}$ from (3.12). Using the $\mathbb{X}$–slot of (2.54), a short computation (namely the decomposition of the term $R^p{}_{a^1}\sharp\sigma_{p\dot{\mathbf{a}}}$ in the last display according to (1.16)) reveals that

$$\begin{aligned}T_{A^0\mathbf{A}}{}^{\mathbf{a}}\sigma_{\mathbf{a}} &= \mathbb{D}(\sigma)_{A^0\mathbf{A}} + \frac{1}{n(k+1)}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}C^p{}_{a^1}\sharp\sigma_{p\dot{\mathbf{a}}} \\ &= \mathbb{D}(\sigma)_{A^0\mathbf{A}} - \frac{k-1}{2n(k+1)}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}C_{a^1a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}.\end{aligned}$$

Since $T_{A^0\mathbf{A}}{}^{\mathbf{a}}$ is conformally invariant, the Lemma follows. □

Remark. 1. For $k=1$, $\mathbb{D}$ is just the $w=1$ and special case of the operator $\mathbb{D}^{\beta a}$ from section 5.1 of [10].
2. Note that the operator $\mathbb{D}$ is not unique as an splitting operator "putting" $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ into the top slot of $F_{A^0\mathbf{A}} \in \mathcal{E}_{A^0\mathbf{A}^k}$. $\mathbb{D}$ can be obviously modified by any multiple of $\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}C_{a^1a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}$.

Assume $k \geq 2$. We define a 1st order differential operator

$$\Phi_c : \mathcal{E}_{A^0\mathbf{A}^k} \longrightarrow \mathcal{E}_{cA^0\mathbf{A}^k}$$

that will turn up in our later calculations. Given a section $F_{A^0\mathbf{A}} \in \mathcal{E}_{[A^0\mathbf{A}^k]}$ which, for a choice $g \in [g]$ of the metric in the conformal class, is convenient to take to be in the form

$$F_{A^0\mathbf{A}} = \mathbb{Y}_{A^0\mathbf{A}}{}^{\mathbf{a}}\sigma_{\mathbf{a}} + \frac{1}{k+1}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\,\mathbf{a}}\mu_{a^0\mathbf{a}} + \mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}} - \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\rho_{\mathbf{a}}, \qquad (3.15)$$

we set

$$\begin{aligned}\Phi_c(F_{A^0\mathbf{A}}) := &-\frac{1}{2}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\,\mathbf{a}}C_{a^0a^1c}{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}\\ &+\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\Big[A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} - \frac{1}{2}A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} - \frac{1}{2}C_{a^1a^2c}{}^p\nu_{p\ddot{\mathbf{a}}}\\ &+\frac{k}{2(n-k)}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}\Big] \ .\end{aligned} \tag{3.16}$$

Our aim is to construct a connection ${}^k\nabla$ on $\mathcal{E}_{A^0\mathbf{A}^k}$ such that solutions $\sigma_{\mathbf{a}}$ of (3.8) correspond to sections of $\mathcal{E}_{A^0\mathbf{A}^k}$ that are parallel according to ${}^k\nabla$. Let us start with the normal tractor connection ∇. Using the previous proposition, it is a short and straightforward calculation to show that if $\sigma_{\mathbf{a}}$ is a solution of (3.8), $k\geq 2$ then $\nabla_c\mathbb{D}(\sigma)_{A^0\mathbf{A}} = \Phi_c(\mathbb{D}(\sigma)_{A^0\mathbf{A}})$. Also, it is easy to verify (or see [32]) that for $k=1$, if σ_{a^1} is a solution of (3.8) then $\nabla_c\mathbb{D}(\sigma)_{A^0A^1} = \Omega_{pcA^0A^1}\sigma^p$. This leads us to the following.

3.2.7 Lemma. (i) *Given a metric g from the conformal class, the mapping*

$$\sigma_{\mathbf{a}} \mapsto \mathbb{D}(\sigma)_{A^0\mathbf{A}}, \quad \textit{with inverse} \quad F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}}\ ,$$

gives a bijective mapping between sections of $\sigma_{\mathbf{a}}\in\mathcal{E}_{\mathbf{A}^k}[k+1]$ satisfying (3.8) and sections $F_{A^0\mathbf{A}}\in\mathcal{E}_{A^0\mathbf{A}^k}$ satisfying,

$$\begin{aligned}\nabla_cF_{A^0\mathbf{A}} &= \Phi_c(F_{A^0\mathbf{A}}) \qquad k\geq 2,\\ \nabla_cF_{A^0A^1} &= \Omega_{pcA^0A^1}\sigma^p \qquad k=1.\end{aligned}$$

(ii) *Upon a change of the metric $g\mapsto\widehat{g}=e^{2\Upsilon}g$, Φ_c transforms according to*

$$\widehat{\Phi}_c(F_{A^0\mathbf{A}}) = \Phi_c(F_{A^0\mathbf{A}}) - \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\Upsilon^p(C\lozenge\sigma)_{pc\mathbf{a}}$$

where $\Upsilon_a=\nabla_a\Upsilon$ and $\sigma_{\mathbf{a}}=(k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}}$.

Proof. We have already observed that $\nabla_c\mathbb{D}(\sigma)_{A^0\mathbf{A}} = \Phi_c(\mathbb{D}(\sigma)_{A^0\mathbf{A}})$ for solutions σ of (3.8) for $k\geq 2$, and the also the corresponding statement for $k = 1$. On the other hand, looking at the coefficients of $\mathbb{Y}$ on both sides of $\nabla_c F_{A^0\mathbf{A}} = \Phi_c(F_{A^0\mathbf{A}})$ we see this relation implies that the "top slot" $\sigma_{\mathbf{a}} := (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} F_{A^0\mathbf{A}}$ of F is a solution of (3.8). Thus the claimed bijective correspondence follows.

It remains to prove (ii). Let us consider a section $F_{A^0\mathbf{A}}$ of the form (3.15) and a conformal rescaling $g \mapsto \widehat{g}$ as above. Collecting together the conformal transformation formulae for all the relevant objects we have:

$$
\begin{aligned}
\widehat{\mu}_{a^{\mathbf{a}}} &= \mu_{a^{\mathbf{a}}} + (k+1)\Upsilon_{a^0}\sigma_{\mathbf{a}} \\
\widehat{\nu}_{\dot{\mathbf{a}}} &= \nu_{\dot{\mathbf{a}}} + k\Upsilon^p \sigma_{p\dot{\mathbf{a}}} \\
\widehat{\mathbb{Z}}_{A^0\mathbf{A}}^{a^0\,\mathbf{a}} &= \mathbb{Z}_{A^0\mathbf{A}}^{a^0\,\mathbf{a}} + (k+1)\Upsilon^{a^0}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} \\
\widehat{\mathbb{W}}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}} &= \mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}} - \Upsilon_{a^1}\mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}} \\
\widehat{A}_{ab^1b^2} &= A_{ab^1b^2} + \Upsilon^p C_{pab^1b^2} \\
\widehat{\nabla}_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} &= \nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} + (k-2)\Upsilon_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \\
&\quad + \boldsymbol{g}_{ca^1}\Upsilon^r(C\blacklozenge\sigma)_{r\dot{\mathbf{a}}}
\end{aligned}
\tag{3.17}
$$

The first two transformations are immediate from (1.47) since $F_{A^0\mathbf{A}}$ is (assumed to be) conformally invariant. The next two formulae are directly the properties of $\mathbb{Z}$– and $\mathbb{X}$–tractors from (1.47). The last but one is a simple calculation using the conformal transformation formulae from for example [29], and the last follows from Lemma 3.2.3 (i) and (3.7). Applying (3.17) to the formula (3.15) for Φ_c, we obtain

$$
\begin{aligned}
\widehat{\Phi}_c(F_{A^0\mathbf{A}}) - \Phi_c(F_{A^0\mathbf{A}}) = \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\Big[&-\frac{k+1}{2}\Upsilon^{a^0}C_{a^0a^1c}{}^p\sigma_{p\dot{\mathbf{a}}} \\
-\frac{k(k-1)}{2(n-k)}\Upsilon_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} &+ \Upsilon^q C_q{}^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} - \frac{1}{2}\Upsilon^q C_q{}^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} \\
-\frac{k}{2}C_{a^1a^2c}{}^p\Upsilon^q\sigma_{qp\ddot{\mathbf{a}}} + \frac{k(k-2)}{2(n-k)}\Upsilon_{a^1}&(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} + \frac{k}{2(n-k)}\boldsymbol{g}_{ca^1}\Upsilon^r(C\blacklozenge\sigma)_{r\dot{\mathbf{a}}}\Big]
\end{aligned}
$$

It is straightforward to verify that sum of the three terms involving $C\blacklozenge\sigma$ is equal to

$$-\frac{k}{n-k}\Upsilon^r \boldsymbol{g}_{a^1[r}(C\blacklozenge\sigma)_{c]\dot{\mathbf{a}}}\ . \tag{3.18}$$

Summing the remaining terms on the right hand side yields

$$\begin{aligned}&\big(-\Upsilon^q C_{qa^1c}{}^p\sigma_{p\dot{\mathbf{a}}} + \frac{k-1}{2}\Upsilon^q C_{a^2a^1c}{}^p\sigma_{pq\ddot{\mathbf{a}}}\big)\\ &+\Upsilon^q C_{ca^1q}{}^p\sigma_{p\dot{\mathbf{a}}} - \frac{1}{2}\Upsilon^q C_{a^1a^2q}{}^p\sigma_{pc\ddot{\mathbf{a}}} + \frac{k}{2}\Upsilon^q C_{a^1a^2c}{}^p\sigma_{pq\ddot{\mathbf{a}}}\\ &= -\Upsilon^r\left[C_{rca^1}{}^p\sigma_{p\dot{\mathbf{a}}} + C_{a^1a^2[r}{}^p\sigma_{|p|c]\ddot{\mathbf{a}}}\right]\ .\end{aligned} \tag{3.19}$$

Now summing the last two displays and comparing the result with the definition of $C\lozenge\sigma$ in (3.9), the Lemma (ii) follows. □

We have shown that, in contrast to $\Omega_{pcA^0A^1}\sigma^p$, Φ_c for $k \geq 2$ is not conformally invariant. Also note that it is not algebraic but is rather a first order differential operator. We would like to replace Φ_c with an operator which, in a suitable sense, has the same essential properties (including linearity) and yet is conformally invariant and algebraic. We deal with invariance first. For $k \geq 2$, we define the 1st order differential operator

$$\Psi_c : \mathcal{E}_{[A^0\mathbf{A}^k]} \longrightarrow \mathcal{E}_{c[A^0\mathbf{A}^k]},$$

for a given choice $g \in [g]$ of the metric and a section $F_{A^0\mathbf{A}} \in \mathcal{E}_{[A^0\mathbf{A}^k]}$ (taken to be of the form (3.15)), by

$$\Psi_c(F_{A^0\mathbf{A}}) := \Phi_c(F_{A^0\mathbf{A}}) + \frac{1}{n-2}\mathbb{X}_{A^0}{}^{\mathbf{a}}_{\mathbf{A}}\nabla^p(C\lozenge\sigma)_{pc\mathbf{a}}. \tag{3.20}$$

Recall that $(C\lozenge\sigma)_{[pq]\mathbf{a}} \in \mathcal{E}(2,k)_0[k+1]$ and is by construction conformally invariant. Hence we have the conformal transformation

$$\widehat{\nabla}^p(C\lozenge\sigma)_{pc\mathbf{a}} = \nabla^p(C\lozenge\sigma)_{pc\mathbf{a}} + (n-2)\Upsilon^p(C\lozenge\sigma)_{pc\mathbf{a}}$$

according to (3.7). From this and the previous Lemma (ii) it follows that Ψ_c is conformally invariant.

Now recall we have proved in Lemma 3.2.5 that $C\lozenge\sigma = 0$ for σ satisfying (3.8). Therefore $\Phi_c = \Psi_c$ in this case and we have

3.2.8 Lemma. *Lemma 3.2.7 part* (i) *holds if we replace the operator* Φ_c *by* Ψ_c *therein.* $\square$

Now we replace the operator Ψ_c with an algebraic alternative in the following way. From (3.20) and the formulae (3.16) for Φ_c, it is clear that in the operator Ψ_c, applied to $F_{A^0\mathbf{A}}$ in the form (3.15), only the coefficient of $\mathbb{X}$ contains terms of the first order. Recall that we have the decomposition $\mathcal{E}_{c\mathbf{a}^k}[k+1] \cong \mathcal{E}_{[c\mathbf{a}^k]}[k+1] \oplus \mathcal{E}_{\{c\mathbf{a}^k\}_0}[k+1] \oplus \mathcal{E}_{\mathbf{a}^{k-1}}[k-1]$. If $\sigma_{\mathbf{a}} = (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} F_{A^0\mathbf{A}}$ is a solution of (3.8), the parts of $\nabla_c\sigma_{\mathbf{a}}$ that lie in $\mathcal{E}_{[c\mathbf{a}^k]}[k+1]$ and $\mathcal{E}_{\mathbf{a}^{k-1}}[k-1]$ may be replaced by, respectively, $\mu_{a^0\mathbf{a}} \in \mathcal{E}_{a^0\mathbf{a}^k}[k+1]$ and $\nu_{\dot{\mathbf{a}}} \in \mathcal{E}_{\dot{\mathbf{a}}^k}[k-1]$, according to Proposition 3.2.4. Moreover, it is clear that in fact this replacement is conformally invariant for any $F_{A^0\mathbf{A}}$. Thus if we remove, from the $\mathbb{X}$–slot of the formulae for Ψ_c, all the terms depending on $\nabla_{\{c}\sigma_{\mathbf{a}\}_0}$, then the resulting operator $\widetilde{\Psi}_c$ will be algebraic, conformally invariant and will satisfy Lemma 3.2.8 (or rather the alternative version of this with $\widetilde{\Psi}_c$ replacing Ψ_c). We describe $\widetilde{\Psi}_c$ explicitly in the following Proposition.

3.2.9 Proposition. *The mapping*

$$\sigma_{\mathbf{a}} \mapsto \mathbb{D}(\sigma)_{A^0\mathbf{A}}, \quad \textit{with inverse} \quad F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}} F_{A^0\mathbf{A}} \ ,$$

gives a conformally invariant bijective mapping between sections of $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{A}^k}[k+1]$ *satisfying (3.8) and sections* $F_{A^0\mathbf{A}} \in \mathcal{E}_{A^0\mathbf{A}^k}$ *satisfying,*

$$\nabla_c F_{A^0\mathbf{A}} = \widetilde{\Psi}_c(F_{A^0\mathbf{A}}) \qquad 1 \le k \le n-1 \ .$$

For choice $g \in [g]$ *of a metric from the conformal class and a section* $F_{A^0\mathbf{A}} \in \mathcal{E}_{A^0\mathbf{A}^k}$*, expressed in the form (3.15), the conformally invariant alge-*

braic operator $\widetilde{\Psi}_c : \mathcal{E}_{A^0\mathbf{A}^k} \longrightarrow \mathcal{E}_{cA^0\mathbf{A}^k}$ *is given by the formula*

$$\begin{aligned}\widetilde{\Psi}_c(F_{A^0\mathbf{A}}) = & -\frac{1}{2}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}} C_{a^0a^1c}{}^{p}\sigma_{p\dot{\mathbf{a}}} + \frac{k(k-1)}{2(n-k)}\mathbb{W}_{A^0A^1\dot{\mathbf{A}}}{}^{\dot{\mathbf{a}}}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \\ & + \mathbb{X}_{A^0\mathbf{A}}{}^{\mathbf{a}}\Big[A_{a^1c}{}^{p}\sigma_{p\dot{\mathbf{a}}} + \frac{k-1}{2(n-k)}T(\sigma)_{c\mathbf{a}}\Big]\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}T(\sigma)_{c\mathbf{a}} = & \frac{1}{2}\big(\nabla_c C_{a^1a^2}{}^{pq}\big)\sigma_{pq\ddot{\mathbf{a}}} + 2A^{p}{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} - A^{p}{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} - \boldsymbol{g}_{ca^1}A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} \\ & - \big(C_{ca^1}{}^{pq}\mu_{pq\ddot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\mu_{pqc\dddot{\mathbf{a}}}\big) - \frac{n-k-1}{k}C_{a^1a^2c}{}^{p}\nu_{p\ddot{\mathbf{a}}} \\ & \in \mathcal{E}(1,k)[k-1].\end{aligned}$$

Proof. The case $k = 1$ is just reformulation of Lemma 3.2.7. Given Lemma 3.2.8, for the cases $k \geq 2$ this boils down to simply checking the formula for $\widetilde{\Psi}$. This is a direct computation of the formula (3.20) for Ψ_c and then in this formula, formally replacing each instance of $\nabla_c\sigma_{\mathbf{a}}$ by $\mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$. We need to compute only the non-algebraic terms $\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}$ from (3.16) and $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}}$ from (3.20). The latter is the subject of Lemma 3.2.10 below, while the the former is dealt with during the proof of that same Lemma, see (3.23). Combining these results with (3.16) and collecting terms yields the formula (3.21). □

It remains then to calculate $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}}$ as required in the proof of the Proposition above. For this we will need the following identities. They follow from the (second) Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ after a short computation.

$$\begin{aligned}\nabla_{a^1}C_{ca^2b^1b^2} &= \frac{1}{2}\nabla_c C_{a^1a^2b^1b^2} - \boldsymbol{g}_{cb^1}A_{b^2a^1a^2} + 2\boldsymbol{g}_{a^1b^1}A_{b^2ca^2} \\ \nabla_{a^1}C_{a^2a^3b^1b^2} &= 2\boldsymbol{g}_{a^1b^1}A_{b^2a^2a^3}.\end{aligned} \tag{3.22}$$

3.2.10 Lemma. *Assume* $2 \leq k \leq n-1$*. If the* $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ *then, up to the addition of (conformally invariant) terms involving the Weyl curvature* *contracted into* $\nabla_{\{c}\sigma_{\mathbf{a}\}_0}$, $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}} \in \mathcal{E}(1,k)_0[k-1]$ *is given by the formula*

$$\frac{n-2}{2(n-k)}\Big[\frac{1}{2}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} - \big(C_{ca^1}{}^{pq}\sigma_{pq\dot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\sigma_{pqc\dot{\ddot{\mathbf{a}}}}\big)$$
$$+ (n-k-1)\big(A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} + 2A^p{}_{a^1c}\sigma_{p\dot{\mathbf{a}}}\big) + \frac{(n-k+1)}{k}C_{a^1a^2c}{}^{p}\nu_{p\ddot{\mathbf{a}}}$$
$$+ \frac{(k-2)}{k}\boldsymbol{g}_{ca^1}C_{a^2a^3}{}^{pq}\nu_{pq\dot{\ddot{\mathbf{a}}}} - (k-1)\boldsymbol{g}_{ca^1}A_{a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}\Big] + (n-2)A_{a^1c}{}^{p}\sigma_{p\dot{\mathbf{a}}}.$$

Proof. Here we simply expand $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}}$ via the Leibniz rule and in the process we will formally replace each $\nabla_c\sigma_{\mathbf{a}}$ by $\mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$. We shall start with $\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}$. Recall $(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}}$ was given in (3.9) as a sum of two terms. Applying ∇_{a^1} to these, we obtain

$$\begin{aligned}\nabla_{a^1}C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} =& \frac{1}{2}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} - A^q{}_{a^1a^2}\sigma_{cq\ddot{\mathbf{a}}} + 2A^q{}_{ca^2}\sigma_{a^1q\ddot{\mathbf{a}}} \\ &+ C_{ca^2}{}^{pq}\big(\mu_{a^1pq\ddot{\mathbf{a}}} + \boldsymbol{g}_{a^1[p}\nu_{q\ddot{\mathbf{a}}]}\big) \\ \nabla_{a^1}C_{a^3a^2}{}^{pq}\sigma_{pqc\dot{\ddot{\mathbf{a}}}} =& 2A^q{}_{a^3a^2}\sigma_{a^1qc\dot{\ddot{\mathbf{a}}}} + C_{a^3a^2}{}^{pq}\big(\mu_{a^1pqc\dot{\ddot{\mathbf{a}}}} + \boldsymbol{g}_{a^1[p}\nu_{qc\dot{\ddot{\mathbf{a}}}]}\big).\end{aligned}$$

where we have also used (3.22). Now summing of the right-hand sides of the last displays yields

$$\begin{aligned}\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} =& \frac{k-2}{k}\Big[\frac{1}{2}(\nabla_c C_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} - A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} + 2A^p{}_{ca^1}\sigma_{p\dot{\mathbf{a}}} \\ &- \big(C_{ca^1}{}^{pq}\mu_{pq\dot{\mathbf{a}}} + C_{a^2a^1}{}^{pq}\mu_{pqc\ddot{\mathbf{a}}}\big) + \frac{1}{k}C_{a^1a^2c}{}^{p}\nu_{p\ddot{\mathbf{a}}} - \frac{1}{k}\boldsymbol{g}_{ca^1}C_{a^2a^3}{}^{pq}\nu_{pq\dot{\ddot{\mathbf{a}}}}\Big]\end{aligned} \tag{3.23}$$

where we have used $\frac{2}{k}C_{ca^2a^1}{}^{q} = \frac{1}{k}C_{a^1a^2c}{}^{q}$. Note $\nabla_{a^1}(C\blacklozenge\sigma)_{c\dot{\mathbf{a}}} \in \mathcal{E}(1,k)[k-1]$.

Now we shall compute the formula for $\nabla^q(C\lozenge\sigma)_{qc\mathbf{a}}$. According to (3.9), $(C\lozenge\sigma)$ is defined as sum of three terms. Applying ∇^q to the first of these, and using (1.17), we obtain

$$\nabla^q C_{qca^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} = (n-3)A_{ca^1}{}^{p}\sigma_{p\dot{\mathbf{a}}} + C^q{}_{ca^1}{}^{p}(\mu_{qp\dot{\mathbf{a}}} + \boldsymbol{g}_{q[p}\nu_{\dot{\mathbf{a}}]})\ .$$

Similarly for the second term, we obtain

$$\begin{aligned}\nabla^q C_{a^1a^2[q}{}^{p}\sigma_{|p|c]\ddot{\mathbf{a}}} =& \frac{1}{2}(n-3)A^p{}_{a^1a^2}\sigma_{pc\ddot{\mathbf{a}}} + \frac{1}{2}C_{a^1a^2}{}^{qp}(\mu_{qpc\ddot{\mathbf{a}}} + \boldsymbol{g}_{q[p}\nu_{c\ddot{\mathbf{a}}]}) \\ &- \frac{1}{2}(\nabla^q C_c{}^{p}{}_{a^1a^2})\sigma_{pq\ddot{\mathbf{a}}} + \frac{n-k+1}{2k}C_{a^1a^2c}{}^{p}\nu_{p\ddot{\mathbf{a}}}\ ,\end{aligned}$$

where we have used $\nabla^q \sigma_{q\dot{\mathbf{a}}} = \frac{n-k+1}{k} \nu_{\dot{\mathbf{a}}}$. Summing the right hand sides of the last two displays with the third term $\frac{k}{n-k} \nabla^q \boldsymbol{g}_{a^1[q} (C \blacklozenge \sigma)_{c]\dot{\mathbf{a}}}$ yields

$$\begin{aligned}
\nabla^q (C \lozenge \sigma)_{qc\mathbf{a}} = &\frac{1}{2} (\nabla^p C_c{}^q{}_{a^1 a^2}) \sigma_{pq\ddot{\mathbf{a}}} - \frac{1}{2} \big(C_{ca^1}{}^{pq} \mu_{pq\dot{\mathbf{a}}} + C_{a^2 a^1}{}^{pq} \mu_{pqc\ddot{\mathbf{a}}} \big) \\
&+ (n-3) \Big[A_{ca^1}{}^p \sigma_{p\dot{\mathbf{a}}} + \frac{1}{2} A^p{}_{a^1 a^2} \sigma_{pc\ddot{\mathbf{a}}} \Big] + \frac{n-1}{2k} C_{a^1 a^2 c}{}^p \nu_{p\ddot{\mathbf{a}}} \\
&+ \frac{k}{2(n-k)} \nabla_{a^1} (C \blacklozenge \sigma)_{c\dot{\mathbf{a}}} - \frac{k}{2(n-k)} \boldsymbol{g}_{ca^1} \nabla^q (C \blacklozenge \sigma)_{q\dot{\mathbf{a}}}
\end{aligned} \tag{3.24}$$

where we have used $C^{[q}{}_{ca^1}{}^{p]} = -\frac{1}{2} C_{ca^1}{}^{qp}$. In the last display, we need the term $\nabla^p (C \blacklozenge \sigma)_{p\dot{\mathbf{a}}}$. Using the definition (3.9) and applying the Leibniz rule for ∇^p, we obtain

$$\begin{aligned}
\nabla^p (C \blacklozenge \sigma)_{p\dot{\mathbf{a}}} = &\frac{k-2}{k} \Big[(n-3) A_{a^2}{}^{pq} \sigma_{pq\ddot{\mathbf{a}}} + C^r{}_{a^2}{}^{pq} \boldsymbol{g}_{r[p} \nu_{q\ddot{\mathbf{a}}]} \\
&+ (\nabla^r C^{pq}{}_{a^3 a^2}) \sigma_{pqr\dddot{\mathbf{a}}} - \frac{n-k+1}{k} C_{a^2 a^3}{}^{pq} \nu_{pq\dddot{\mathbf{a}}} \Big] \\
= &\frac{(k-2)(n-1)}{k} \Big[A_{a^2}{}^{pq} \sigma_{pq\ddot{\mathbf{a}}} - \frac{1}{k} C_{a^2 a^3}{}^{pq} \nu_{pq\dddot{\mathbf{a}}} \Big]
\end{aligned} \tag{3.25}$$

using (3.22). We will also need the identity

$$\frac{1}{2} (\nabla^p C_c{}^q{}_{a^1 a^2}) \sigma_{pq\ddot{\mathbf{a}}} = +\frac{1}{4} (\nabla_c C^{pq}{}_{a^1 a^2}) \sigma_{pq\ddot{\mathbf{a}}} - \frac{1}{2} \boldsymbol{g}_{ca^1} A_{a^2}{}^{pq} \sigma_{pq\ddot{\mathbf{a}}} + A_{a^1 c}{}^p \sigma_{p\dot{\mathbf{a}}}$$

which uses (3.22). Now we are ready to simplify (3.24) using (3.23), (3.25) and the last display. Collecting terms the result is

$$\begin{aligned}
\nabla^q (C \lozenge \sigma)_{qc\mathbf{a}} = &\frac{n-2}{4(n-k)} \Big[(\nabla_c C_{a^1 a^2}{}^{pq}) \sigma_{pq\ddot{\mathbf{a}}} - 2 \big(C_{ca^1}{}^{pq} \mu_{pq\dot{\mathbf{a}}} + C_{a^2 a^1}{}^{pq} \mu_{pqc\ddot{\mathbf{a}}} \big) \\
&+ 2(n-k-1) A^p{}_{a^1 a^2} \sigma_{pc\ddot{\mathbf{a}}} + \frac{2(n-k+1)}{k} C_{a^1 a^2 c}{}^p \nu_{p\ddot{\mathbf{a}}} \\
&+ \frac{2(k-2)}{k} \boldsymbol{g}_{ca^1} C_{a^2 a^3}{}^{pq} \nu_{pq\dddot{\mathbf{a}}} - 2(k-1) \boldsymbol{g}_{ca^1} A_{a^2}{}^{pq} \sigma_{pq\ddot{\mathbf{a}}} \Big] \\
+ &\frac{1}{(n-k)} \Big[(n-k) A_{a^1 c}{}^p + (k-2) A^p{}_{ca^1} + (n-3)(n-k) A_{ca^1}{}^p \Big] \sigma_{p\dot{\mathbf{a}}}
\end{aligned}$$

Now the final step is to simplify the last line using the relation $A_{ca^1}{}^p = A^p{}_{a^1 c} + A_{a^1 c}{}^p$ which follows directly from the definition $A_{pa^1 c} := 2\nabla_{[a^1} P_{c]p}$. A short computation reveals that the last line is equal to

$$(n-2) A_{a^1 c}{}^p + (n-2) \frac{n-k-1}{n-k} A^p{}_{a^1 c}.$$

The Lemma now follows from the last two displays. □

Summarising our results we have the following.

3.2.11 Theorem. *For $1 \leq k \leq n-1$, the mapping $\mathcal{E}_{\mathbf{a}^k}[k+1] \longrightarrow \mathcal{E}_{A^0\mathbf{A}^k}$ given by $\sigma \mapsto \mathbb{D}(\sigma)$ defined by (3.14) is a conformally invariant differential operator. Upon restriction it gives a bijective mapping from solutions of the conformal Killing equation (3.8) onto sections of $\mathcal{E}_{A^0\mathbf{A}^k}$ that are parallel with respect to the connection ${}^k\nabla_c := \nabla_c - \widetilde{\Psi}_c$ where ∇_c is the normal tractor connection and $\widetilde{\Psi}_c$ is given by (3.21). The connection ${}^k\nabla_c$ is a conformally invariant connection on the form-tractor bundle $\mathcal{E}_{A^0\mathbf{A}^k}$. The inverting map from sections of $\mathcal{E}_{A^0\mathbf{A}^k}$, parallel for ${}^k\nabla_c$, to solutions of (3.8) is $F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}}$.*

Sections of $\mathcal{E}_{A^0\mathbf{A}^k}$ which are parallel for the normal tractor connection ∇_c are mapped injectively to solutions of (3.8) by

$$F_{A^0\mathbf{A}} \mapsto (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}} \ ,$$

and $\tilde{\Psi}_c$ annihilates the range of this map.

Proof. Everything has been established in the previous Lemmas except for the last claim. That parallel sections are mapped injectively to conformal Killing forms is an immediate consequence of the formula (1.48) for the normal tractor connection on form-tractors. (Note that the equation from the first slot of $\nabla_c F_{A^0\mathbf{A}} = 0$ is $\nabla_c\sigma_{\mathbf{a}^k} - (k+1)\mu_{c\mathbf{a}^k} + \boldsymbol{g}_{ca^1}\varphi_{\dot{\mathbf{a}}^k} = 0$. This is the same equation as from the first slot for a $(k+1)$-form-tractor parallel for ${}^k\nabla_c$, as $\widetilde{\Psi}_c$ does not affect this top slot – the coefficient of $\mathbb{Y}$.) Next it is an elementary exercise using the formula (1.48) to verify that if $F_{A^0\mathbf{A}}$ is parallel for the normal tractor connection, then necessarily $F_{A^0\mathbf{A}} = \mathbb{D}(\sigma)$ where $\sigma_{\mathbf{a}} = (k+1)\mathbb{X}^{A^0\mathbf{A}}{}_{\mathbf{a}}F_{A^0\mathbf{A}}$. On the other hand from the first part of the Theorem it follows that $\mathbb{D}(\sigma)$ is parallel for ${}^k\nabla$. So $\tilde{\Psi}_c(\sigma)$ vanishes everywhere. □

Remark. Let us say (as suggested in [41]) that a conformal Killing form σ is *normal* if it has the property that $\mathbb{D}(\sigma)$ is parallel for the normal tractor connection. It follows immediately from the Theorem that the operator $\tilde{\Psi}_c$ detects exactly the failure of conformal Killing forms to be normal; a conformal Killing form is normal if and only if $\tilde{\Psi}_c(\sigma)$ is zero.

3.2.12. Coupled conformal Killing equations In this section we show that solutions $\sigma \in \mathcal{E}^k[k+1]$ of the original equation (3.8) are in bijective correspondence with solutions of the coupled conformal Killing equation $\tilde{\nabla}_{(a}\overline{\sigma}_{b)_0\mathbf{B}^{k-1}} = 0$ on $\mathcal{E}_{a\mathbf{B}^{k-1}}[2]$ for a certain conformally invariant connection $\tilde{\nabla}$. Along the way we obtain some related preliminary results that should be of independent interest.

We defined the operator $\overline{M}$ in 2.1.9 by the formula

$$\overline{M}_{\mathbf{B}^l}^{\mathbf{a}^{k-l,l}} : \mathcal{E}_{\mathbf{a}^k}[k+1] \longrightarrow \mathcal{E}_{\mathbf{a}^{k-l}\mathbf{B}^l}[k-l+1]$$
$$\overline{M}_{\mathbf{B}^l}^{\mathbf{a}^{k-l,l}} \sigma_{\mathbf{a}^k} = (n-k+1)\mathbb{Z}_{\mathbf{B}^l}^{\mathbf{b}^l}\sigma_{\mathbf{a}^{k-l}\mathbf{b}^l} - l\mathbb{X}_{B^1\dot{\mathbf{B}}^l}^{\dot{\mathbf{b}}^l}\nabla^{b^1}\sigma_{\mathbf{a}^{k-l}\mathbf{b}^l}$$

for $1 \leq l \leq k$. This is similar to the formula for the middle operator M in (2.14). Here we define also the operator $\underline{M}$ by the formula

$$\underline{M}_{\mathbf{a}^{k,l}\mathbf{B}^l} : \mathcal{E}_{\mathbf{a}^k}[k+1] \longrightarrow \mathcal{E}_{\mathbf{a}^{k+l}\mathbf{B}^l}[k+l+1]$$
$$\underline{M}_{\mathbf{a}^{k,l}\mathbf{B}^l}\sigma_{\mathbf{a}^k} = (k+1)\mathbb{Z}_{\mathbf{B}^l}^{\mathbf{b}^l}\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}}\sigma_{\mathbf{a}^k} - l\mathbb{X}_{B^1\dot{\mathbf{B}}^l}^{\dot{\mathbf{b}}^l}\boldsymbol{g}_{\dot{\mathbf{b}}^l\dot{\mathbf{a}}^{k,l}}\nabla_{a^{k+1}}\sigma_{\mathbf{a}^k}$$

for $1 \leq l \leq n-k$, where we use multi-indices

$$\mathbf{a}^{k,l} = [a^{k+1} \cdots a^{k+l}]$$
$$\dot{\mathbf{a}}^{k,l} = [a^{k+2} \cdots a^{k+l}] \;.$$

The conformal invariance of $\overline{M}$ and $\underline{M}$ may be verified directly via the formulae (1.47). Applying these to a form $\sigma \in \mathcal{E}^k[k+1]$, $1 \leq k \leq n-1$, we obtain the tractor-valued forms

$$\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} = \overline{M}_{\mathbf{B}^l}^{\mathbf{a}^{k-l,l}}\sigma_{\mathbf{a}^k} \quad \text{and} \quad \underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} = \underline{M}_{\mathbf{a}^{k,l}\mathbf{B}^l}\sigma_{\mathbf{a}^k}. \tag{3.26}$$

Although $\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ and $\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$, as defined in (3.26), are invariant for the stated ranges of l, in the sequel we shall only need the tensor valence of $\overline{\sigma}$ and $\underline{\sigma}$ to be in the interval $[1, n-1]$. Therefore we shall henceforth assume that for $\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ we have $1 \leq l \leq k-1$ and for $\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$ we have $1 \leq l \leq n-k-1$, respectively.

Let us next describe $\nabla_{\{c}\overline{\sigma}_{\mathbf{a}^{k-l}\}_0\mathbf{B}^\mathbf{l}}$ and $\nabla_{\{c}\underline{\sigma}_{\mathbf{a}^{k+l}\}_0\mathbf{B}^\mathbf{l}}$ when σ is a solution of (3.8). (Recall that ∇ denotes the coupled Levi–Civita–normal tractor connection.) This is explicitly formulated in the proposition below. First we need the following lemma.

3.2.13 Lemma. *Let us suppose that σ is a solution of (3.8). Then*

$$\nabla_c \nabla^p \sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l} \overset{\{c\mathbf{a}^{k-l}\}_0}{=} (n-k+1)\left[-\frac{k-1}{n-k} C_c{}^p{}_{[a^1}{}^q \sigma_{|p|\dot{\mathbf{a}}^{k-l}|q|\dot{\mathbf{b}}^l]} - P_c{}^p \sigma_{\dot{\mathbf{a}}^{k-l}p\dot{\mathbf{b}}^l}\right] \tag{a}$$

$$\nabla_c \nabla_{a^{k+1}} \sigma_{\mathbf{a}^k} \overset{\{c\mathbf{a}^{k+1}\}_0}{=} (k+1)\left[C_{ca^{k+1}a^1}{}^p \sigma_{p\dot{\mathbf{a}}^k} - P_{ca^{k+1}} \sigma_{\mathbf{a}^k}\right]. \tag{b}$$

In reading (b) here recall the convention that sequentially labelled indices (at a given level) are assumed to be skewed over.

Proof. First let us note that the trace part in the first case, and skew–symmetrisation $[c\mathbf{a}^{k+1}]$ in the second case, is zero on both sides. In the subsequent discussion we use Proposition 3.2.4 and the notation therein.

The left-hand side of (a) is equal to $\frac{n-k+1}{k}\nabla_c \nu_{\mathbf{a}^{k-l}\dot{\mathbf{b}}^l}$ up to the sign $(-1)^{k-l}$. Now the Lemma (a) follows using $C_c{}^{[p}{}_{a^1}{}^{q]} = \frac{1}{2} C_{ca^1}{}^{pq}$ and the equation for $\nabla_c \nu_{\mathbf{a}^{k-l}\dot{\mathbf{b}}^l}$ in (3.11) where $(C \blacklozenge \sigma)_{c\mathbf{a}^{k-l}\dot{\mathbf{b}}^l}$ is given by Lemma 3.2.3 (i). Note that the projection $\{..\}$ over indices in the latter lemma exactly removes the completely skew–symmetric part of $C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}$ (see (3.10)). Since the projection $\{c\mathbf{a}^{k-l}\}_0$ annihilates the completely skew–symmetric part $C_{[ca^2}{}^{pq}\sigma_{|pq|\ddot{\mathbf{a}}]}$ we have $(C \blacklozenge \sigma)_{c\mathbf{a}^{k-l}\dot{\mathbf{b}}^l} =_{\{c\mathbf{a}^{k-l}\}_0} C_{ca^1}{}^{pq}\sigma_{pq\dot{\mathbf{a}}^{k-l}\dot{\mathbf{b}}}$. The part (b) follows similarly from the expression for $\nabla_c \mu_{a^{k+1}\mathbf{a}^k}$ in (3.11). □

3.2.14 Proposition. *The form $\sigma \in \mathcal{E}^k[k+1]$, $1 \le k \le n-1$ is a solution of (3.8) if and only if either of the following conditions is satisfied:*

$$\nabla_c \overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} \overset{\{c\mathbf{a}^{k-l}\}_0}{=} \frac{l(k-1)(n-k+1)}{n-k} \mathbb{X}_{B^1\dot{\mathbf{B}}^l}{}^{\dot{\mathbf{b}}^l} C_c{}^p{}_{[a^1}{}^q \sigma_{|p|\dot{\mathbf{a}}^{k-l}|q|\dot{\mathbf{b}}^l]}$$

$$\nabla_c \underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} \overset{\{c\mathbf{a}^{k+l}\}_0}{=} -l(k+1) \mathbb{X}_{B^1\dot{\mathbf{B}}^l}{}^{\dot{\mathbf{b}}^l} C_{c[a^{k+1}a^1}{}^p \sigma_{|p|\dot{\mathbf{a}}^k} \boldsymbol{g}_{\dot{\mathbf{a}}^{k,l}]\dot{\mathbf{b}}^l}.$$

Proof. The expressions on the left-hand-side can be computed by directly differentiating the expressions (3.26) defining $\underline{\sigma}$ and $\overline{\sigma}$ and expanding in terms of the $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{W}$, $\mathbb{Z}$ splitting operators introduced in 1.2.5. The resulting "$\mathbb{Y}$–slot" (i.e. the coefficient of $\mathbb{Y}$) on the left-hand-side is zero order, as an operator on σ, and is killed by the symmetrisation $\{c\mathbf{a}^{k-l}\}$ in the case of $\nabla_c \overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ and by taking the trace-free part in the case of $\nabla_c \underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$. Essentially the same argument shows (in both cases) that also the operator in the $\mathbb{W}$ slot vanishes. The $\mathbb{Z}$ slot is of the first order as an operator on σ. To show this vanishes requires some computation. We will need the relation

$$k\boldsymbol{g}_{c[a^1} \nabla^p \sigma_{|p|\dot{\mathbf{a}}^{k-l}\mathbf{b}^l]} = (k-l)\boldsymbol{g}_{ca^1} \nabla^p \sigma_{|p|\dot{\mathbf{a}}^{k-l}\mathbf{b}^l} + l\boldsymbol{g}_{cb^1} \nabla^p \sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l}. \qquad (3.27)$$

(Recall our convention that all sequentially labelled indices are implicitly skewed over. So the b-indices are skewed and also the a-indices are skewed.) To prove this first observe the projection to the completely skew part of the right-hand-side obviously yields exactly the left-hand-side. On the other hand the right-hand-side is manifestly skew over the b–indices and also over the a–indices. A trivial calculation verifies that that it is also skew–symmetric in the index pair a^1b^1 and so the result follows.

Using (3.11) for $\nabla_c \sigma_{\mathbf{a}}$, it is straightforward to compute the $\mathbb{Z}$ slot of $\nabla_c \overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ is

$$(n-k+1)\nabla_{[c}\sigma_{\mathbf{a}^{k-l}\mathbf{b}^l]} + k\boldsymbol{g}_{c[a^1}\nabla^p \sigma_{|p|\dot{\mathbf{a}}^{k-l}\mathbf{b}^l]} - l\boldsymbol{g}_{cb^1}\nabla^p \sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l}.$$

The first term is killed by the projection $\mathcal{P}_{\{c\mathbf{a}^{k-l}\}}$ and the remaining part is in the trace part over $\{c\mathbf{a}^{k-l}\}$ (i.e. in particular is annihilated by $\mathcal{P}_{\{c\mathbf{a}^{k-l}\}_0}$)

due to (3.27). The $\mathbb{Z}$ slot of $\nabla_c \underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$ is

$$\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}} \nabla_{[c}\sigma_{\mathbf{a}^k]} - l\boldsymbol{g}_{cb^1}\boldsymbol{g}_{\dot{\mathbf{b}}^l\dot{\mathbf{a}}^{k,l}} \nabla_{a^{k+1}}\sigma_{\mathbf{a}^k} + \frac{k(k+1)}{n-k+1}\boldsymbol{g}_{ca^1}\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}} \nabla^p \sigma_{p\dot{\mathbf{a}}^k}$$

(also using (3.11)). The last term is killed by taking the trace–free part and it is easy to show the sum of the first two terms is $\boldsymbol{g}_{\mathbf{b}^l\mathbf{a}^{k,l}} \nabla_c \sigma_{\mathbf{a}^k}$ (up to a scalar multiple) which vanishes after the symmetrisation $\{c\mathbf{a}^{k+l}\}$.

At this point it is worthwhile noting that if the projection $\mathcal{P}_{\{c\mathbf{a}^{k-l}\}_0}$ kills $\nabla_c \overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ or the projection $\mathcal{P}_{\{c\mathbf{a}^{k+l}\}_0}$ kills $\nabla_c \underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$ then σ is a solution of (3.8); the vanishing of the $\mathbb{Z}$–slots implies $\nabla_c \sigma_{\mathbf{a}} = \mu_{c\mathbf{a}} + \boldsymbol{g}_{ca^1}\nu_{\dot{\mathbf{a}}}$ in (3.11) since $\mathcal{P}_{\{c\mathbf{a}^k\}_0} \circ \mathcal{P}_{\{c\mathbf{a}^{k-l}\}_0}$ is a non-zero multiple of $\mathcal{P}_{\{c\mathbf{a}^k\}_0}$.

It remains to evaluate the $\mathbb{X}$–slots. This can be done easily using the rules for $\nabla_c \mathbb{W}$ and $\nabla_c \mathbb{X}$ from 1.2.5. We get

$$\begin{aligned} &-l\mathbb{X}_{B^1\dot{\mathbf{B}}^l}{}^{\dot{\mathbf{b}}^l} \left[(n-k+1)P_c{}^p \sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l} + \nabla_c \nabla^p \sigma_{\mathbf{a}^{k-l}p\dot{\mathbf{b}}^l}\right] \\ &-l\mathbb{X}_{B^1\dot{\mathbf{B}}^l}{}^{\dot{\mathbf{b}}^l} \left[(k+1)P_{c[a^{k+1}}\sigma_{\mathbf{a}^k} g_{\dot{\mathbf{a}}^{k,l}]\dot{\mathbf{b}}^l} + \nabla_c \nabla_{[a^{k+1}}\sigma_{\mathbf{a}^k} g_{\dot{\mathbf{a}}^{k,l}]\dot{\mathbf{b}}^l}\right] \end{aligned}$$

for $\nabla_c \overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l}$ and $\nabla_c \underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l}$, respectively. Now the proposition follows using Lemma 3.2.13. □

For our next construction we will especially need the first case of the proposition above for $l = k-1$, that is for $\overline{\sigma}_{a^1\dot{\mathbf{B}}^k}$. We will construct a connection $\tilde{\nabla}$ on $\mathcal{E}_{a^1\dot{\mathbf{B}}^k}$ such that the equation $\tilde{\nabla}_{(c}\overline{\sigma}_{a^1)_0\dot{\mathbf{B}}^k} = 0$ is equivalent to the equation (3.8). Reformulating the Proposition for $\overline{\sigma}_{a^1\dot{\mathbf{B}}^k}$, we get that σ is a solution of (3.8) if and only if

$$\nabla_{(c}\overline{\sigma}_{a^1)_0\dot{\mathbf{B}}^k} = \frac{(k-1)(k-2)(n-k+1)}{n-k} \mathbb{X}_{B^2\ddot{\mathbf{B}}^k}{}^{\ddot{\mathbf{b}}^k} C_{b^3}{}^p{}_{(c}{}^q \sigma_{a^1)_0 pq\ddot{b}^k}. \tag{3.28}$$

This shows that $\nabla_{(c}\overline{\sigma}_{a^1)_0\dot{\mathbf{B}}^k} = 0$ is equivalent to (3.8) in the flat case. In the curved case we modify the connection ∇ in the following way. Let us consider the tensor-tractor field

$$\begin{aligned}\kappa_{cE^0E^1F^0F^1} &:= \mathbb{X}_{E^0}{}^{e^1}_{E^1}\Omega_{ce^1F^0F^1} \\ &= \mathbb{X}_{E^0}{}^{e^1}_{E^1}\mathbb{Z}^{f^0f^1}_{F^0F^1}C_{ce^1f^0f^1} - 2\mathbb{X}_{E^0}{}^{e^1}_{E^1}\mathbb{X}_{F^0}{}^{f^1}_{F^1}A_{f^1ce^1}\ ,\end{aligned}$$

where $\Omega_{ce^1F^0F^1}$ is the curvature of the normal tractor connection. By construction this is conformally invariant. We will show that the required connection $\tilde{\nabla}$ can be written in the form

$$\tilde{\nabla}_c = \nabla_c + x\kappa_c\sharp\sharp,\ x \in \mathbb{R}$$

where (via the tractor metric) we view $\kappa_{cE^0E^1F^0F^1}$ as a 1-form taking values in $\mathrm{End}(\mathcal{E}^A) \otimes \mathrm{End}(\mathcal{E}^A)$ and $\sharp$ indicates the usual action of tractor-bundle endomorphisms (i.e. it is the tractor bundle analogue of the $\mathrm{End}(TM)$ action defined in section 3.2.2 and we use the same notation as for that case). To determine the parameter $x \in \mathbb{R}$, let us compute the double action:

$$\begin{aligned}\kappa_c\sharp\sharp(\overline{\sigma}_{a^1\dot{\mathbf{B}}^k}) &= \mathbb{X}^{e^1}\mathbb{Z}^{f^0f^1}C_{ce^1f^0f^1}\sharp\sharp\left[(n-k+1)\mathbb{Z}^{\dot{\mathbf{b}}^k}_{\dot{\mathbf{B}}^k}\sigma_{\mathbf{a^1}\dot{\mathbf{b}}\mathbf{k}^k}\right] \\ &= (k-1)(n-k+1)\mathbb{X}^{e^1}\sharp\mathbb{Z}^{\dot{\mathbf{b}}^k}_{\dot{\mathbf{B}}^k}C_{ce^1b^2}{}^{q}\sigma_{a^1q\ddot{\mathbf{b}}^k} \\ &= -\frac{1}{2}(k-1)(k-2)(n-k+1)\mathbb{X}_{B^2}{}^{\ddot{\mathbf{b}}^k}_{\ddot{\mathbf{B}}^k}C_{c}{}^{p}{}_{b^3}{}^{q}\sigma_{a^1qp\dddot{\mathbf{b}}^k}.\end{aligned}$$

The form of the right-hand-side shows that $\tilde{\nabla}$ is the required connection for a suitable parameter $x \in \mathbb{R}$, and comparing with (3.28) yields the explicit value for x. The resulting connection is

$$\tilde{\nabla}_c = \nabla_c + \frac{2}{n-k}\kappa_c\sharp\sharp, \tag{3.29}$$

where on the right-hand side ∇ is the usual tractor connection. Note that this connection is obviously conformally invariant (since both κ and the the tractor connection are conformally invariant). This might seem inevitable, since from its derivation (or otherwise) it is clear that the equation (3.28) is conformally invariant. However (3.29) is an invariant connection which may turn out to have applications in other circumstances.

Let us summarise the last result.

3.2.15 Proposition. *A weighted k-form* $\sigma \in \mathcal{E}^k[k+1]$ *is a conformal Killing k-form (i.e. solution of (3.8)) if and only if*

$$\tilde{\nabla}_{(a}\overline{\sigma}_{b)_0} = 0 \tag{3.30}$$

where $\tilde{\nabla}$ *is the Levi-Civita connection coupled with (3.29) and* $\overline{\sigma}$ *is the conformally invariant tractor extension of* σ *given by (3.26) with* $l = k-1$.

Although we shall not directly need it below it is interesting to observe at this point that the last result generalises. First observe that as well as the action $\kappa_c\sharp\sharp$ used in (3.29), we can consider also the action $\omega_c\sharp\sharp$ where we view the tensor-tractor field

$$\omega_{cE^0E^1f^0f^1} := \mathbb{X}_{E^0}{}^{e^1}_{E^1}C_{ce^1f^0f^1}$$

as a one form taking values in $\text{End}(\mathcal{E}^A) \otimes \text{End}(\mathcal{E}^a)$ and $\sharp$ indicates the usual action of tensor/tractor-bundle endomorphisms. Now for any real or complex parameter x we obtain a connection on tensor tractor fields via the formula,

$$\nabla^x_c = \nabla_c + x(\omega_c\sharp\sharp + \kappa_c\sharp\sharp). \tag{3.31}$$

where ∇ indicates the usual coupled tractor-Levi Civita connection.

3.2.16 Theorem. *A weighted k-form* $\sigma \in \mathcal{E}^k[k+1]$ *is a conformal Killing k-form (i.e. solution of (3.8)) if and only if either of the following conditions holds:*

$$\nabla^x_{\{c}\overline{\sigma}_{\mathbf{a}^{k-l}\}_0\mathbf{B}^l} = 0 \quad or \quad \nabla^y_{\{c}\underline{\sigma}_{\mathbf{a}^{k+l}\}_0\mathbf{B}^l} = 0$$

where $x = \frac{2}{n-k}$ *and* $y = \frac{2}{k}$, *and* $\overline{\sigma}$, $\underline{\sigma}$ *are the conformally invariant tractor extensions of* σ *given by (3.26).*

Proof. First let us compute the actions $\omega_c\sharp\sharp$ and $\kappa_c\sharp\sharp$ on $\overline{\sigma}$ and $\underline{\sigma}$. The result is

$$\begin{aligned}
\omega_c\sharp\sharp\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} &= -\frac{1}{2}l(k-l)(n-k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}C_c{}^p{}_{a^1}{}^q\sigma_{p\dot{\mathbf{a}}^{k-l}q\dot{\mathbf{b}}^l}\\
\kappa_c\sharp\sharp\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} &= -\frac{1}{2}l(l-1)(n-k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}C_c{}^p{}_{b^2}{}^q\sigma_{\mathbf{a}^{k-l}qp\ddot{\mathbf{b}}^l}\\
\omega_c\sharp\sharp\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} &= \frac{1}{2}l(k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}\Big[(l-1)C_{ca^{k+2}b^2a^{k+1}}\boldsymbol{g}_{\ddot{\mathbf{a}}^{k,l}\ddot{\mathbf{b}}^l}\sigma_{\mathbf{a}^k}\\
&\qquad + kC_{ca^{k+1}a^1}{}^p\boldsymbol{g}_{\dot{\mathbf{a}}^{k,l}\dot{\mathbf{b}}^l}\sigma_{p\dot{\mathbf{a}}^k}\Big]\\
\kappa_c\sharp\sharp\underline{\sigma}_{\mathbf{a}^{k+l}\mathbf{B}^l} &= -\frac{1}{2}l(l-1)(k+1)\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}C_{ca^{k+2}b^2a^{k+1}}\boldsymbol{g}_{\ddot{\mathbf{a}}^{k,l}\ddot{\mathbf{b}}^l}\sigma_{\mathbf{a}^k}.
\end{aligned}$$

Now the value $y=\frac{2}{k}$ follows immediately from Proposition 3.2.14. In the case of $\overline{\sigma}$, we can reformulate Proposition 3.2.14 in the following way: σ is a solution of (3.8) if and only if

$$\begin{aligned}
\nabla_c\overline{\sigma}_{\mathbf{a}^{k-l}\mathbf{B}^l} \overset{\{c\mathbf{a}^{k-l}\}_0}{=} \frac{l(n-k+1)}{n-k}\mathbb{X}_{B^1}{}^{\dot{\mathbf{b}}^l}_{\dot{\mathbf{B}}^l}\Big[&(k-l)C_c{}^p{}_{a^1}{}^q\sigma_{p\dot{\mathbf{a}}^{k-l}q\dot{\mathbf{b}}^l}\\
&+(l-1)C_c{}^p{}_{b^2}{}^q\sigma_{\mathbf{a}^{k-l}qp\dot{\mathbf{b}}^l}\Big],
\end{aligned}$$

cf. (3.27). Thus the value $x=\frac{2}{n-k}$ follows. □

Remark. Note that the connections (3.31) preserve the SO(p,q) symmetry type (over tensor indices) and SO$(p+1,q+1)$ symmetry type of the any tensor-tractor field they act on. The coupled tractor-Levi Civita connection ∇ has this property. Then the $\omega_c\sharp\sharp$ action preserves these symmetries since ω_c is a 1-form taking values in the tensor product of orthogonal tractor endomorphisms tensor with orthogonal tensor endomorphisms. Similarly κ_c is a 1-form taking values in the tensor square of orthogonal tractor endomorphisms.

Note also that the action $C_{ab}\sharp$ of the Weyl tensor on tensors may in a natural way be viewed as a conformal action of the tractor curvature $\Omega_{ab}\sharp$ on *tensors*. (For example contract each tensor index "c" into a $Z_C{}^c$ and then apply the usual action of $\Omega_{ab}\sharp$ on these tractor indices. Finally remove each the new tractor index by contracting with $Z^C{}_e$. The result is conformally invariant since $\Omega_{ab}{}^C{}_D X^D = 0$.) If we extend the action $\Omega_{ab}\sharp$ to tensors in this way, then the connections ∇^x and ∇^y become simply $\nabla^x_c = \nabla_c + x\kappa_c\sharp\sharp$ and $\nabla^y_c = \nabla_c + y\kappa_c\sharp\sharp$ with x and y as above.

3.2.17. Applications: Helicity raising and lowering and almost Einstein manifolds. In the first part here we will assume the structure is almost Einstein in the sense of [31]. This is a manifold with a conformal structure and a section $\alpha \in \mathcal{E}[1]$ satisfying $\left[\nabla_{(a}\nabla_{b)_0} + P_{(ab)_0}\right]\alpha = 0$. Equivalently there is a standard tractor I_A that is parallel with respect to the normal tractor connection ∇. It follows that $I_A := \frac{1}{n} D_A \alpha = Y_A\alpha + Z^a_A\nabla_a\alpha - \frac{1}{n}X_A(\Delta + P)\alpha$, for some section $\alpha \in \mathcal{E}[1]$, and so $X^A I_A = \alpha$ is non-vanishing on an open dense subset of M and on this subset $g = \alpha^{-2}\boldsymbol{g}$ is an Einstein metric (where, recall $\boldsymbol{g}$ is the conformal metric). In particular any conformally Einstein manifold is almost Einstein but in general the converse is not true.

In this setting we immediately have the Theorem which follows. Recall that in a particular choice of metric and using ν and μ from Proposition 3.2.4, a k-form σ is a Killing form if it is a solution of (3.8) with $\nabla_{a^1}\nu_{\dot{\mathbf{a}}^k}$ identically 0. We will term a k-form σ a *dual-Killing form* if it is a solution of (3.8) where instead $\nabla_{a^0}\mu_{\mathbf{a}^k}$ is identically 0. (On oriented manifolds the Hodge dual of a Killing form is a dual-Killing form and vice versa.)

3.2.18 Theorem. *Let us consider a k–form $\sigma_{\mathbf{a}^k} \in \mathcal{E}^k[k+1]$. Then, for $k \in \{1, \cdots, n\}$,*

$$\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}} := \alpha\nabla^p\sigma_{\mathbf{a}^{k-1}p} - (n-k+1)(\nabla^p\alpha)\sigma_{\mathbf{a}^{k-1}p} \quad \in \mathcal{E}^{k-1}[k]$$

is conformally invariant. For $k \in \{0, \cdots, n-1\}$,

$$\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}} := \alpha\nabla_{a^{k+1}}\sigma_{\mathbf{a}^k} - (k+1)(\nabla_{a^{k+1}}\alpha)\sigma_{\mathbf{a}^k} \quad \in \mathcal{E}^{k+1}[k+2]$$

is conformally invariant. If σ is a solution of (3.8) then we have the following equivalences:

$$\begin{aligned} \nabla_{\{c}\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}\}_0} = 0 &\Longleftrightarrow C_{ca^1}{}^{pq}\sigma_{\dot{\mathbf{a}}^{k-1}pq} \overset{\{c\mathbf{a}^{k-1}\}_0}{=} 0 \\ \nabla_{\{c}\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}\}_0} = 0 &\Longleftrightarrow C_{ca^{k+1}a^1}{}^{p}\sigma_{\dot{\mathbf{a}}^{k}p} \overset{\{c\mathbf{a}^{k+1}\}_0}{=} 0 \end{aligned} \tag{3.32}$$

for $2 \leq k \leq n-1$ and $1 \leq k \leq n-2$, respectively. In the case that the first curvature condition is satisfied then the corresponding conformal Killing form $\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}}$ is a Killing form away from the zero set of α, and in the Einstein scale $g = \alpha^{-2}\boldsymbol{g}$. In the case that the second curvature condition is satisfied then the corresponding conformal Killing form $\underline{\underline{\sigma}}_{\mathbf{a}^{k-1}}$ is a dual-Killing form away from the zero set of α, and in the Einstein scale $g = \alpha^{-2}\boldsymbol{g}$.

Proof. The first part of the theorem follows from relations $\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}} = I^B\overline{\sigma}_{\mathbf{a}^{k-1}B}$ and $\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}} = I^B\underline{\sigma}_{\mathbf{a}^{k+1}B}$ where the forms $\overline{\sigma}_{\mathbf{a}^{k-1}B}$ and $\underline{\sigma}_{\mathbf{a}^{k+1}B}$ are defined by (3.26) in Section 3.2.12. The result (3.32) follows from Proposition 3.2.14 and continuity, since the tractor I^B is parallel and I^BX_B is non-vanishing on an open dense set in the manifold. For the final points note that, from the formulae for $\overline{\overline{\sigma}}_{\mathbf{a}^{k-1}}$ and $\underline{\underline{\sigma}}_{\mathbf{a}^{k+1}}$ given in the first part of the theorem, it is clear that these are, respectively, coclosed and closed in the Einstein scale $g = \alpha^{-1}\boldsymbol{g}$ given off the zero set of α. □

Remark. 1. Note that the first curvature condition on the right-hand side of (3.32) is that $(C\blacklozenge\sigma) = 0$. That is that the projection of $C\sharp\sigma$ to $\mathcal{E}(1, k-1)[k-1]$ should vanish everywhere. Similarly the second is simply that the (unique up to a multiple) projection of $C\sharp\sigma$ to $\mathcal{E}(1, k+1)_0[k+1]$ should vanish everywhere. Note that in the case that the manifold is oriented then the second curvature condition is exactly that the Hodge dual of σ satisfies the first condition (as applied to $(n-k)$-form solutions of (3.8)).

2. Note that on an almost Einstein manifold with a conformal Killing k-form such that $(C\blacklozenge\sigma) = 0$ then, according to the Theorem, on the open

dense set where α is non-vanishing there is a scale so that $\overline{\overline{\sigma}}$ is a Killing form. But the section α does not necessarily give a global metric whereas the form $\overline{\overline{\sigma}}$ is a globally defined conformal Killing form. A similar comment applies to $\underline{\underline{\sigma}}$.

3.2.19 Corollary. *If σ_{ab} is a conformal Killing 2-form then*

$$\overline{\overline{\sigma}}_a = \alpha \nabla^p \sigma_{ap} - (n-1)(\nabla^p \alpha)\sigma_{ap}$$

is a conformal Killing vector field (i.e. solution of (3.8) with $k = 1$). If $\sigma'_{\mathbf{a}^{n-2}}$ is a conformal Killing $(n-2)$-form then

$$\underline{\underline{\sigma}}'_{\mathbf{a}^{n-1}} := \alpha \nabla_{a^{n-1}} \sigma'_{\mathbf{a}^{n-2}} - (n-1)(\nabla_{a^{n-1}}\alpha)\sigma'_{\mathbf{a}^{n-2}} \quad \in \mathcal{E}^{n-1}[n]$$

is a conformal Killing $(n-1)$-form. Away from the zero set of α, $\overline{\overline{\sigma}}_a$ is a Killing vector for the Einstein metric $g = \alpha^{-2}\boldsymbol{g}$, while in this scale $\underline{\underline{\sigma}}'_{\mathbf{a}^{n-1}}$ is a dual-Killing form.

Proof. This is just the Theorem above for $k = 2$. The condition $C_{(ab)_0}{}^{pq}\sigma_{pq}$ is trivially satisfied, and, hence, so too is the dual condition (cf. point 1. of the Remark above). □

Note that a weaker form of the first part of the Corollary has been proved (by a direct computation) in [45, 7.2].

Remark. Note that according to the Corollary, on Einstein 4-manifolds a non-parallel conformal Killing 2-form implies the existence of either a non-trivial Killing vector field or a non-trivial dual-Killing 3-form. Thus if the 4-manifold is also oriented then, in any case, a non-parallel conformal Killing 2-form determines a non-trivial Killing vector field.

The first part of the theorem is valid also for $k = 1$ in the sense, that if σ_a satisfies (3.8) then $\overline{\overline{\sigma}} := \alpha\nabla^p\sigma_p - n(\nabla^p\alpha)\sigma_p$ is (conformally invariant and) another almost Einstein scale. This is easily seen as follows. Let us write $\sigma_{CD} := \mathbb{D}^a_{CD}\sigma_a$, where $\mathbb{D}$ was defined for Lemma 3.2.6. Then

$$\nabla_a \sigma_{CD} = \Omega^p{}_{aCD}\sigma_p \ . \tag{3.33}$$

by Lemma 3.2.7. Note that $I^D\sigma_{CD}$ is parallel with respect to the normal tractor connection ∇ since

$$\nabla_a I^D \mathbb{D}^a_{CD}\sigma_a = (\nabla_a \sigma_{CD})I^D = \sigma^p \Omega_{paCD} I^D = 0.$$

Then the result follows from Theorem 3.1 of [34] since $\overline{\overline{\sigma}} = X^C I^D \sigma_{CD}$.

Some related results follow. Following [34] we term a metric (or conformal structure) *weakly generic* if the Weyl curvature is injective as bundle map $TM \longrightarrow \otimes^3 TM$.

3.2.20 Proposition. *(i) If σ_a is a non-homothetic conformal Killing vector field (i.e. a $k=1$ solution of (3.8) with non-constant $\nabla_a\sigma^a$) on an Einstein manifold then there exists a non-trivial conformal gradient field. That is a non-trivial solution $\tilde{\sigma}_a$ of (3.8) which is exact for the Einstein scale.*
(ii) If a weakly generic conformally Einstein manifold M admits a conformal Killing vector field σ^a, then σ^a is a homothety for any Einstein metric in the conformal class.

Proof. Let us write $I^1_D := I_D$ and $I^2_C := \sigma_{CP}I^P$, where $\sigma_{CP} = \mathbb{D}^a_{CP}\sigma_a$. These parallel tractors determine a parallel tractor 2-form tractor $I^1_{[C}I^2_{D]}$. Let us write $\tilde{\sigma}_a := \frac{1}{2}\mathbb{X}^{CD}{}_a I^1_{[C}I^2_{D]}$. (Note that from the last part of Theorem 3.2.11 it follows immediately that $\tilde{\sigma}_a$ is a conformal Killing field hence $\Omega^p{}_{aCD}\tilde{\sigma}_p = 0$ by (3.33). Thus $C_{abc}{}^p\tilde{\sigma}_p = 0$.)

Since I^1_D and I^2_C are parallel and the top slot of I^2_C is $\overline{\overline{\sigma}} = X^C I^D \sigma_{CD}$ it follows (Theorem 3.1 of [34]) that $I^2_C = \frac{1}{n}D_C\overline{\overline{\sigma}}$. To compute $\tilde{\sigma}_a$ let us write explicitly

$$I_D^1 = Y_D\alpha + Z_D^d\nabla_d\alpha - \frac{1}{n}X_D(\Delta + P)\alpha$$
$$I_C^2 = Y_C\overline{\overline{\sigma}} + Z_C^c\nabla_c\overline{\overline{\sigma}} - \frac{1}{n}X_C(\Delta + P)\overline{\overline{\sigma}}.$$

Here we have used the tractor D operator given by the formula (1.32). Now it follows easily that $\tilde{\sigma}_a$ is $(\nabla_a\alpha)\overline{\overline{\sigma}} - \alpha(\nabla_a\overline{\overline{\sigma}})$ up to a (nonzero) scalar multiple. (From this formula, it is also easy to verify by a direct computation that $\tilde{\sigma}_a$ satisfies (3.8).) In the Einstein scale α we have $\nabla\alpha = 0$, whence $\tilde{\sigma}_a = -\nabla_a(\alpha\overline{\overline{\sigma}}) = -\nabla_a(\alpha^2\nabla^p\sigma_p)$.

(ii) This is an immediate consequence of part (i) since a conformal it is well known (and an easy exercise to verify) that any conformal gradient field $\tilde{\sigma}^a$ necessarily satisfies $C_{ab}{}^c{}_p\tilde{\sigma}^p = 0$. □

Theorem 3.2.18 exploited the standard tractor I_A which (corresponds to an almost Einstein scale α and) is parallel with respect to the normal tractor connection ∇. Here we drop the assumption that the manifold is almost Einstein and assume instead that the manifold is equipped with a conformal Killing field σ^a. Then we use the tractor $\sigma_{AB} := \mathbb{D}^p_{AB}\sigma_p$ (given by (3.14)) provided by the conformal Killing form σ_a. This is not, in general, parallel with respect to the normal tractor connection ∇. Rather, we obtained (3.33) in Lemma 3.2.7.

3.2.21 Theorem. *For each pair $\sigma \in \mathcal{E}^1[2]$ and $\tau \in \mathcal{E}^k[k+1]$*

$$\check{\overline{\tau}}_{\mathbf{a}^{k-2}} := 2\sigma^p\nabla^q\tau_{\mathbf{a}^{k-2}pq} + (n-k+1)(\nabla^p\sigma^q)\tau_{\mathbf{a}^{k-2}pq} \quad k \in \{2, \cdots, n\}$$

is a conformally invariant section of $\mathcal{E}^{k-2}[k-1]$, and

$$\check{\underline{\tau}}_{\mathbf{a}^{k+2}} := 2\sigma_{a^{k+1}}\nabla_{a^{k+2}}\tau_{\mathbf{a}^k} + (k+1)(\nabla_{a^{k+1}}\sigma_{a^{k+2}})\tau_{\mathbf{a}^k}, \quad k \in \{0, \cdots, n-2\}$$

is a conformally invariant section of $\mathcal{E}^{k+2}[k+3]$. *If* σ *and* τ *are solutions of (3.8) then the following is satisfied: for* $3 \leq k \leq n-1$ $\check{\overline{\tau}}_{\mathbf{a}^{k-2}}$, *is a solution of (3.8) if and only if*

$$(n-k+1)C^{r}{}_{c}{}^{pq}\tau_{\mathbf{a}^{k-2}pq}\sigma_r + (k-2)C_{ca^1}{}^{pq}\tau_{p\dot{\mathbf{a}}^{k-2}qr}\sigma^r \overset{\{c\mathbf{a}^{k-2}\}_0}{=} 0$$

and, for $1 \leq k \leq n-3$, $\check{\underline{\tau}}_{\mathbf{a}^{k+2}}$, *is a solution of (3.8) if and only if*

$$2C_{ca^{k+1}a^1}{}^{p}\tau_{p\dot{\mathbf{a}}^k}\sigma_{a^{k+2}} - C^{p}{}_{ca^{k+1}a^{k+2}}\tau_{\mathbf{a}^k}\sigma_p \overset{\{c\mathbf{a}^{k+2}\}_0}{=} 0.$$

Proof. Similarly as in the proof of Theorem 3.2.18, the first part follows from relations $\check{\overline{\tau}}_{\mathbf{a}^{k-2}} = \overline{\tau}_{\mathbf{a}^{k-2}RS}\sigma^{RS}$ and $\check{\underline{\tau}}_{\mathbf{a}^{k+2}} = \underline{\tau}_{\mathbf{a}^{k+2}RS}\sigma^{RS}$. The second part is a result of a direct computation. Using Proposition 3.2.14 and (3.33) we obtain the following:

$$\begin{aligned}
\nabla_c\overline{\tau}_{\mathbf{a}^{k-2}RS}\sigma^{RS} &\overset{\{c\mathbf{a}^{k-2}\}_0}{=} (\nabla_c\overline{\tau}_{\mathbf{a}^{k-2}RS})\sigma^{RS} + \overline{\tau}_{\mathbf{a}^{k-2}RS}\nabla_c\sigma^{RS} \\
&\overset{\{c\mathbf{a}^{k-2}\}_0}{=} \frac{2(n-k+1)}{n-k}\mathbb{X}_{RS}^{s}\left[(k-2)C_{c}{}^{p}{}_{a^1}{}^{q}\tau_{p\dot{\mathbf{a}}^{k-2}qs} - C_{c}{}^{p}{}_{s}{}^{q}\tau_{p\dot{\mathbf{a}}^{k-2}qa^1}\right]\sigma^{RS} \\
&\quad + \tau_{\mathbf{a}^{k-2}RS}\Omega^{p}{}_{c}{}^{RS}\sigma_p \\
&\overset{\{c\mathbf{a}^{k-2}\}_0}{=} \frac{n-k+1}{n-k}\left[(n-k+1)C^{s}{}_{c}{}^{pq}\tau_{\mathbf{a}^{k-2}pq}\sigma_s + (k-2)C_{ca^1}{}^{pq}\sigma_{p\dot{\mathbf{a}}^{k-2}qs}\sigma^s\right], \\
\nabla_c\underline{\tau}_{\mathbf{a}^{k+2}RS}\sigma^{RS} &\overset{\{c\mathbf{a}^{k+2}\}_0}{=} (\nabla_c\underline{\tau}_{\mathbf{a}^{k+2}RS})\sigma^{RS} + \underline{\tau}_{\mathbf{a}^{k+2}RS}\nabla_c\sigma^{RS} \\
&\overset{\{c\mathbf{a}^{k+2}\}_0}{=} -2(k+1)\mathbb{X}_{RS}^{s}C_{ca^{k+1}a^1}{}^{p}\tau_{p\dot{\mathbf{a}}^k}\boldsymbol{g}_{a^{k+2}s}\sigma^{RS} + \underline{\tau}_{\mathbf{a}^{k+2}RS}\Omega^{p}{}_{c}{}^{RS}\sigma_p \\
&\overset{\{c\mathbf{a}^{k+2}\}_0}{=} -(k+1)\left[2C_{ca^{k+1}a^1}{}^{p}\tau_{p\dot{\mathbf{a}}^k}\sigma_{a^{k+2}} - C^{p}{}_{ca^{k+1}a^{k+2}}\tau_{\mathbf{a}^k}\sigma_p\right].
\end{aligned}$$

□

Note for the cases of a conformal Killing 3-form τ the first curvature condition of the Theorem is satisfied by any conformal gradient vector field σ.

Now it is obvious how to obtain more general results for couples of conformal Killing forms $\sigma \in \mathcal{E}^l[l+1]$ and $\tau \in \mathcal{E}^k[k+1]$ where $1 \leq k, l \leq n-1$. We set $\sigma_{\mathbf{A}^{l+1}} := \mathbb{D}\sigma$ and define $\check{\overline{\tau}}_{\mathbf{a}^{k-l-1}} := \overline{\tau}_{\mathbf{a}^{k-l-1}\mathbf{A}^{l+1}}\sigma^{\mathbf{A}^{l+1}}$ and $\check{\underline{\tau}}_{\mathbf{a}^{k+l+1}} :=$

$\underline{\tau}_{\mathbf{a}^{k+l+1}\mathbf{A}^{l+1}}\sigma^{\mathbf{A}^{l+1}}$ for $0 \leq k-l-1 \leq n$ and $0 \leq k+l+1 \leq n$, respectively. The case $l = 1$ is described in the previous Theorem and in general, the obstructions for $\check{\bar{\tau}}_{\mathbf{a}^{k-l-1}}$ and $\check{\underline{\tau}}_{\mathbf{a}^{k+l+1}}$ to be solutions of (3.8) are very similar to the cases $l = 1$. (In the proof of these new cases, we replace $\nabla_c\sigma^{RS}$ by $\nabla_c\sigma^{\mathbf{A}^{l+1}}$. The latter is, in general quite complicated but we actually need only '$\mathbb{Z}$–slot' and '$\mathbb{Y}$–slot' which are similar to the case $l = 1$.)

3.2.22 Corollary. *Let $\sigma_a \in \mathcal{E}_a[2]$ be a solution of (3.8) and write $\mu_{bc} := \nabla_{[b}\sigma_{c]}$ (in a choice of scale). Then the section*

$$\sigma_{a^0}\mu_{a^1a^2}\cdots\mu_{a^{2p-1}a^{2p}} \in \mathcal{E}^{2p+1}[2p+2],\ p \leq \lfloor\frac{n-2}{2}\rfloor$$

is conformally invariant. If $\sigma_{a^0}C_{a^1a^2c}{}^d\sigma_d = 0$ then it is a solution of (3.8) for any $1 \leq p \leq \lfloor\frac{n-2}{2}\rfloor$.

Proof. For $p = 1$, this is Theorem 3.2.21 applied to $\tau := \sigma \in \mathcal{E}^1[2]$. If the curvature condition is satisfied then it is easily checked that applying the same Theorem to σ_a and $\tau := \sigma_{a^0}\mu_{a^1a^2}$, we obtain the case $p = 2$. Repeating this procedure, the general case follows. □

Let us note there are several results in [46] related to those in this section, see Propositions 3.4 and 3.5 in [46]. These concern a special case satisfying that $\nabla_c\mu_{a^0a^1}$ is pure trace (which implies that σ_a is an eigensection for the Rho–tensor $P_a{}^b$ viewed as a section of $\mathrm{End}(TM)$). This immediately yields $\sigma_{a^0}C_{a^1a^2c}{}^p\sigma_p = 0$ using (3.11).

Our last application concerns conformal Killing m-tensors. These are valence m symmetric trace-free tensors $t_{b\cdots c} \in \mathcal{E}_{(b\cdots c)_0}[2m]$ which are solutions of the conformally invariant equation $\nabla_{(a}t_{b\cdots c)_0} = 0$. Obviously, any conformal Killing form $\sigma_a \in \mathcal{E}_a[2]$ yields a conformal Killing tensor $\sigma_{(a}\cdots\sigma_{b)_0}$. Note that generalising the $m = 2$ version of this observation we have the following. If $\sigma_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}^k}[k+1]$ is conformal Killing form then $\sigma_{(a}{}^{\dot{\mathbf{c}}}\sigma_{b)_0\dot{\mathbf{c}}} \in \mathcal{E}_{(ab)_0}[4]$, is a conformal Killing 2-tensor. (The special case of this where σ is a conformal Killing 2-form appeared in [52, 4.1(4)].) This follows from (3.11) by a direct computation or from the relation $\sigma_{(a}{}^{\dot{\mathbf{e}}}\sigma_{b)_0\dot{\mathbf{e}}} = \frac{1}{(n-k+1)^2}\overline{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\overline{\sigma}_{b)_0\dot{\mathbf{E}}}$ (which holds since X_A and Z_a^A are orthogonal), and Propositions 3.2.14 and 3.2.15. The point here is that one applies the normal tractor ∇_c connection to $\overline{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\overline{\sigma}_{b)_0\dot{\mathbf{E}}}$ to obtain $2\overline{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\nabla_b\,\overline{\sigma}_{c)_0\dot{\mathbf{E}}}$ after the projection to $\mathcal{E}_{(abc)_0}[4]$. Then from Proposition 3.2.14 and again the orthogonality of X_A and Z_a^A we may replace ∇ by $\tilde{\nabla}$ to obtain $2\overline{\sigma}_{(a}{}^{\dot{\mathbf{E}}}\tilde{\nabla}_b\,\overline{\sigma}_{c)_0\dot{\mathbf{E}}}$. But then by Proposition 3.2.15 the last expression vanishes. It is clear this example generalises and so we have the following Theorem.

3.2.23 Theorem. *Suppose $\sigma^1, \cdots, \sigma^m$ is a collection of conformal Killing forms of respective ranks $r_1, \cdots, r_m$ where $(\sum^m r_i) - m$ is an even number. Then*

$$\sigma^1_{(a} \cdot \sigma^2_b \cdot \ \cdots \ \cdot \sigma^m_{c)_0}$$

is a conformal Killing m-tensor, where $\sigma^1_a \cdot \sigma^2_b \cdot \ \cdots \ \cdot \sigma^m_c$ indicates any contraction of the collection $\sigma^1, \cdots, \sigma^m$ over the suppressed indices.

Of course it will often be the case that a given contraction $\sigma^1_a \cdot \sigma^2_b \cdot \ \cdots \ \cdot \sigma^m_c$ vanishes upon projection to the trace-free part. However it is easy to proliferate non-trivial examples.

Appendix A

gBGG splitting operator in the flat case

We shall prove uniqueness of the gBGG splitting operator in the conformally flat case, see Defined 1.3.7. This concerns regular patterns. Recall that for a given $\mathfrak{so}_n(\mathbb{C})$–dominant weight Λ, such a pattern consists of bundles $V^{w.\Lambda}$, $w \in W^{\mathfrak{p}}$, see details in 1.3.3 and 1.1.1. Here we show that regular patterns can be interpreted, on the level of $\mathfrak{p}$–representations, as certain cohomology spaces. We start in a more general setting.

For a representation $\pi : \mathfrak{a} \longrightarrow \mathfrak{gl}(\mathbb{V})$ of an arbitrary Lie algebra $\mathfrak{a}$ we have the differential $\partial : \mathrm{Hom}\ (\bigwedge^k \mathfrak{a}; \mathbb{V}) \longrightarrow \mathrm{Hom}\ (\bigwedge^{k+1} \mathfrak{a}; \mathbb{V})$ defined by the formula

$$\begin{aligned}(\partial p)(X_0 \wedge \cdots \wedge X_k) = &\sum_{i<j} (-1)^{i+j} p([X_i, X_j] \wedge X_0 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_k) \\ &+ \sum_i (-1)^i \pi(X_i) p(X_0 \wedge \cdots \hat{X}_i \cdots \wedge X_k).\end{aligned}$$

It is straightforward to show $\partial^2 = 0$. Thus a differential ∂ induces a cohomology space $H^k(\mathfrak{a}; \mathbb{V})$, called the cohomology of $\mathfrak{a}$ with coefficients in $\mathbb{V}$. (We set $\mathrm{Hom}\ (\bigwedge^k \mathfrak{a}; \mathbb{V}) = 0$ for $k < 0$ and $k > \dim \mathfrak{a}$). Here we follow the notation usual for parabolic geometries i.e. ∂ denotes the coboundary operator.

We are interested only in the case, where $\mathfrak{a} = \mathfrak{p}_+$ and $\pi = \nu|\mathfrak{p}_+$ for some representation $\nu : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathbb{V})$ and where $\mathfrak{p}_+$ is the nilpotent part of a (real or complex) parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$. It follows from the structure of parabolic subalgebras that we have the natural action of the elements from $\mathfrak{p}$ on Hom $(\bigwedge \mathfrak{p}_+; \mathbb{V})$ (it is the adjoint action on $\bigwedge \mathfrak{p}_+$ and the action given by π on $\mathbb{V}$). This induces the representation of $\mathfrak{p}$ on Hom $(\bigwedge \mathfrak{p}_+; \mathbb{V})$ which descends to the representation $\beta : \mathfrak{p} \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+, \mathbb{V}))$ on the cohomology. This representation is completely reducible hence we need only the restriction $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H(\mathfrak{p}_+, \mathbb{V}))$. This is shown in [40] for the complex case (see the Theorem below), the real case follows from the complexification.

In the complex case, $\mathfrak{g}$ and $\mathbb{V}$ are over $\mathbb{C}$. Then structure of β is described by the Theorem below. (See [48] for a real version of this Theorem.) This uses the notation $\Phi_w = w(\Delta_-) \cap \Delta_+$, see 1.1.1 for the notation. The set Φ_w contains only roots of $\mathfrak{p}_+$ i.e. the positive roots of $\mathfrak{g}$ which do not lie in the semisimple part of $\mathfrak{g}_0$ (see [40]).

A.1.1 Theorem. [40] **Kostant's result.** *Assume the complex case. For a finite dimensional representation $\nu : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathbb{V})$ with highest weight Λ and restriction $\pi = \nu|\mathfrak{p}$, the irreducible components of β are in bijective correspondence with the set $W^{\mathfrak{p}}$ and the multiplicity of each component is one. The highest weight of the irreducible component of the representation β corresponding to $w \in W^{\mathfrak{p}}$ is $w.\Lambda = w(\Lambda + R) - R$ and it occurs at degree $|w|$. The generator of this component (the vector of the highest weight) is $\bigwedge_{\alpha \in \Phi_w} \mathfrak{g}_\alpha \longrightarrow v_{w\Lambda}$ where $v_{w\Lambda} \in \mathbb{V}$ is a weight vector of the weight $w\Lambda$.*

Henceforth we assume the conformal setting. That is, $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{so}_{p,q}$. Using the notation from 1.3.1, the irreducible $\mathfrak{p}$–representations $\mathbb{V}^{w.\Lambda}$, $w \in W^{\mathfrak{p}}$ are components of $H\left(\mathfrak{p}_+; (\mathbb{V}^\Lambda)^*\right)^* \cong H(\mathfrak{g}_-; \mathbb{V}^\Lambda)$. The short operators from the regular pattern with the weight Λ can be constructed as operators between irreducible bundles corresponding to components of the $\mathfrak{p}$–module $H(\mathfrak{g}_-;\mathbb{V}^\Lambda)$ [20, 13]. The space of cochains of degree k is denoted by Hom $(\bigwedge^k \mathfrak{g}_-;\mathbb{V}^\Lambda)$ and we have the isomorphisms

$$\text{Hom }(\bigwedge^k \mathfrak{g}_-;\mathbb{V}^\Lambda) \cong \bigwedge^k \mathfrak{p}_+ \otimes \mathbb{V}^\Lambda \cong \mathbb{E}_{\mathbf{a}^k} \otimes \mathbb{V}^\Lambda$$

since $\mathfrak{p}_+ \cong \mathfrak{g}_-^* \cong T_x^*M$ for each $x \in M$ and we use the notation $E_{\mathbf{a}^k} = \bigwedge^k T^*M$. It follows from Theorem A.1.1 that there is exactly one copy of $\mathbb{V}^{w.\Lambda}$ in $\bigoplus_{i=1}^n \mathbb{E}_{\mathbf{a}^i} \otimes \mathbb{V}^\Lambda$, both viewed as $\mathfrak{g}_0$–modules.

Now recall the definition of the gBGG splitting operator, see 1.3.7. This is an (invariant) differential splitting operator for bundles from regular patterns. In the complex case and for a $\mathfrak{g}$–dominant weight Λ and $w \in W^{\mathfrak{p}}$, $|w| = k$, this is $\mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{E}_{\mathbf{a}^k} \otimes \mathcal{V}^\Lambda$. In the real case, this is an operator $\Psi : \mathcal{V} \longrightarrow \mathcal{V}'$ for $\mathcal{V}$ irreducible such that $\mathcal{V}^{w.\Lambda} \subseteq \mathcal{V}(\mathbb{C})$ and $\Psi(\mathbb{C})|_{\mathcal{V}^{w.\Lambda}}$ is the gBGG splitting operator.

A.1.2 Theorem. *Let V be an (irreducible) bundle from a regular pattern. Then the gBGG splitting operator on $\mathcal{V}$ exists uniquely in the conformally flat case.*

Proof. The existence is established in [20, 13] and in Section 2.2 for both scalars. Let us consider the complex case and suppose we have two splitting operators $\Psi_1, \Psi_2 : \mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{E}_{\mathbf{a}^k} \otimes \mathcal{V}^\Lambda$ in the flat case. Suppose the difference $\Psi_1 - \Psi_2 : \mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{E}_{\mathbf{a}^k} \otimes \mathcal{V}^\Lambda$ is non–trivial. Recall we defined $\mathfrak{g}_0$–components and projecting parts of bundles, sections and differential operators in 1.2.2. If $\Psi_1 - \Psi_2$ is nontrivial then there is an irreducible projecting part $pr : \mathcal{V}^\Gamma \hookrightarrow \mathcal{E}_{\mathbf{a}^k} \otimes \mathcal{V}^\Lambda$ of the operator $\Psi_1 - \Psi_2$ such that the (invariant) operator $\Phi := pr^*(\Psi_1 - \Psi_2) : \mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{V}^\Gamma$ is nontrivial. (Recall $pr^* : \mathcal{E}_{\mathbf{a}^k} \otimes \mathcal{V}^\Lambda \twoheadrightarrow \mathcal{V}^\Gamma$.) Thus Φ is an operator between irreducibles which means $\Gamma = w'.\Lambda$,

$w' \in W^{\mathfrak{p}}$ according the classification in 1.3.3. Since Ψ_1 and Ψ_2 have the same projecting part and Φ is nontrivial, clearly $\mathcal{V}^{\Gamma} \neq \mathcal{V}^{w.\Lambda}$. On the other hand, $|w'| = k$ due to the uniqueness of $\mathbb{V}^{\Gamma} \subseteq \bigoplus_{i=1}^{n} \mathbb{E}_{\mathbf{a}^i} \otimes \mathbb{V}^{\Lambda}$ (and because $\mathbb{V}^{\Gamma} \hookrightarrow \mathbb{E}_{\mathbf{a}^k} \otimes \mathbb{V}^{\Lambda}$).

Summarizing we have a nontrivial invariant operator $\Phi : \mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{V}^{w'.\Lambda}$ for $|w| = |w'| = k$. But Φ cannot be a multiple of identity as the source and target spaces are different. Hence $w \neq w'$ which implies that n is even and $k = \frac{n}{2}$. But there is no operator between two components of degree $\frac{n}{2}$ in the pattern in 1.3.3 for n even so there is no nontrivial operator $\mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{V}^{\Gamma}$ in the flat case.

Now let us consider the real case and two different gBGG splittings $\Psi_1, \Psi_2 : \mathcal{V} \longrightarrow \mathcal{V}'$. Then $(\Psi_1 - \Psi_2)(\mathbb{C}) \neq 0$ which again yields a nontrivial operator $\Phi : \mathcal{V}^{w.\Lambda} \longrightarrow \mathcal{V}^{\Gamma}$ as above. Thus the result follows from the complex case. $\square$

Appendix B

Transformation of connections

For a given pseudometric g on the manifold M, the Levi–Civita connection ∇ on TM is the unique torsion free connection satisfying $\nabla g = 0$. Then the Clifford section $\beta_a \in \mathcal{E}_a \otimes \text{End}\ (\mathcal{E}_\lambda)$ determines the spin connection ∇ on the dual spin bundle E_λ such that $\nabla\beta = 0$ where ∇ denotes the coupled connection. (We use the dual here but recall $E_\lambda \cong E^\lambda$.) We need to know how these connections change if we multiply g by $e^{2\Upsilon}$ for a smooth positive function $\Upsilon \in \mathcal{E}$ on M. We shall consider only the real case here but the same results apply in the complex setting.

We shall use the notation from 1.2.1. In particular, a metric g from the conformal class corresponds to a nonzero section $\sigma \in \mathcal{E}[1]$, using $g = \sigma^{-2}\boldsymbol{g}$. Consider a section $f \in \mathcal{E}[w]$. The exterior derivative d is defined on $\sigma^{-w}f \in \mathcal{E}[0]$ and we put $\nabla f := \sigma^w d(\sigma^{-w}f)$. Consider another metric $\hat{g} = e^{2\Upsilon}g$. This correspond to $\hat{\sigma} = e^{-\Upsilon}\sigma \in \mathcal{E}[1]$. Then $\hat{\nabla} f = \hat{\sigma}^w d(\hat{\sigma}^{-w}f) = \nabla f + w(d\Upsilon)f$ after a short computation. Using the 1–form $\Upsilon_a := \nabla_a\Upsilon$, we obtain the usual formula

$$\hat{\nabla}_a f = \nabla_a f + w\Upsilon_a f \quad \text{for } f \in \mathcal{E}[w]. \tag{B.1}$$

B.1.1. Transformation of the Levi–Civita connection. To compute an analogue of (B.1) for $\nabla_i U^a$ for $U^b \in \mathcal{E}^b$, we need to know how the Christoffel symbols Γ of ∇ change if we multiply the pseudometric g by $e^{2\Upsilon}$. We shall compute this for a coordinate frame $e_{\underline{a}} = (e_1, \cdots, e_n)$ of TM and the dual coframe $\varepsilon^{\underline{b}} = (\varepsilon^1, \cdots, \varepsilon^n)$ of T^*M. That is, 'concrete' indices are underlined. The Christoffel symbols $\Gamma_{\underline{ia}}{}^{\underline{b}} = \varepsilon^{\underline{b}}(\nabla_{\underline{i}} e_{\underline{a}})$ can be expressed as derivatives of the pseudometric g and we obtain

$$\hat{\Gamma}_{\underline{ia}}{}^{\underline{b}} = \frac{1}{2}\hat{g}^{\underline{br}}\left(\hat{g}_{\underline{ir},\underline{a}} + \hat{g}_{\underline{ra},\underline{i}} - \hat{g}_{\underline{ia},\underline{r}}\right) = \Gamma_{\underline{ia}}{}^{\underline{b}} + \Upsilon_{\underline{a}}\delta_{\underline{i}}^{\underline{b}} + \Upsilon_{\underline{i}}\delta_{\underline{a}}^{\underline{b}} - \Upsilon^{\underline{b}} g_{\underline{ai}}$$

where the indices after comma denote the values of partial derivatives and $\hat{g}_{\underline{ab}} = e^{2\Upsilon} g_{\underline{ab}}$ and $\hat{g}^{\underline{ab}} = e^{-2\Upsilon} g^{\underline{ab}}$. The coordinate expression for the covariant derivative is $\nabla_{\underline{i}} U^{\underline{b}} = \frac{\partial U^{\underline{b}}}{\partial x^{\underline{i}}} + \Gamma_{\underline{ia}}{}^{\underline{b}} U^{\underline{a}}$. If we insert the transformed Christoffel symbols $\hat{\Gamma}$ and use the general abstract index notation, we get the result

$$\hat{\nabla}_a U^b = \nabla_a U^b + \Upsilon_p \delta_a^b U^p + \Upsilon_a U^b - \Upsilon^b g_{pa} U^p \quad \text{for } U^b \in \mathcal{E}^b. \tag{B.2}$$

From this and (B.1), we easily obtain the corresponding formula for $U^b \in \mathcal{E}^b[w]$. Note this yields the transformation of $\nabla_a \omega_b$ for a 1–form $\omega_b \in \mathcal{E}_b$ because $\mathcal{E}_b \cong \mathcal{E}^b[-2]$.

B.1.2. Transformation of the spin connection. Let us start with the spin structure (M, g, β) where the Clifford section β satisfies the Clifford relation

$$\beta_a\beta_b + \beta_b\beta_a = -g_{ab}\mathrm{id}, \quad \beta_a \in \mathcal{E}_a \otimes \mathrm{End}\,(\mathcal{E}_\lambda). \tag{B.3}$$

To compute Christoffel symbols of the spin connection, we need an orthonormal frame $e_{\underline{a}}$ of TM and the dual coframe $\varepsilon^{\underline{b}}$ of T^*M. By this we mean $g(e_{\underline{p}}, e_{\underline{q}}) = \pm\delta_{\underline{pq}}$. That is, $+$ or $-$ may depend on p, q. (For example, $p = 1$ yields $+$, $p = 2$ yields $-$ etc.). Then the spin connection for $f \in \mathcal{E}_\lambda$ is of the form

$$\nabla_i f = \frac{\partial f}{\partial x^i} + r\bar{\Gamma}_i f \quad \text{where } \bar{\Gamma}_i = \Gamma_{i\underline{a}}{}^{\underline{b}}\beta^{\underline{a}}\beta_{\underline{b}} \in \mathcal{E}_i \otimes \mathrm{End}\,(\mathcal{E}_\lambda),\ r \in \mathbb{R}. \tag{B.4}$$

Here $\Gamma_{i\underline{a}}{}^{\underline{b}}$ are Christoffel symbols of the Levi–Civita connection i.e., $\Gamma_{i\underline{a}}{}^{\underline{b}} = \varepsilon^{\underline{b}}(\nabla_i e_{\underline{a}})$. We require $\beta_{\underline{a}}$ to be parallel i.e. $\nabla_i \beta_{\underline{a}} = 0$ with respect to the coupled connection. This will determine the parameter $r \in \mathbb{R}$. $\nabla_i \beta_{\underline{a}} = 0$ means $\nabla_i \beta_{\underline{a}} f = \beta_{\underline{a}} \nabla_i f$ for every $f \in \mathcal{E}_\lambda$ hence

$$\frac{\partial \beta_{\underline{a}} f}{\partial x^i} - \Gamma_{i\underline{a}}{}^{\underline{b}} \beta_{\underline{b}} f + r \bar{\Gamma}_i \beta_{\underline{a}} f = \beta_{\underline{a}} \frac{\partial f}{\partial x^i} + r \beta_{\underline{a}} \bar{\Gamma}_i f$$

using (B.4). Since $\beta_{\underline{a}}$ is constant along fibres of the bundle E_λ i.e., $\frac{\partial \beta_{\underline{a}} f}{\partial x^i} = \beta_{\underline{a}} \frac{\partial f}{\partial x^i}$, it follows from the last display that $-\Gamma_{i\underline{a}}{}^{\underline{b}} \beta_{\underline{b}} = r\left(\beta_{\underline{a}} \bar{\Gamma}_i - \bar{\Gamma}_i \beta_{\underline{a}}\right)$. If we insert the form of $\bar{\Gamma}_i$ from (B.4) and use (B.3) for simplification, we will obtain

$$-\Gamma_{i\underline{a}}{}^{\underline{b}} \beta_{\underline{b}} = r\left(\Gamma_{i\underline{b}}{}^{\underline{c}} g_{\underline{ca}} - \Gamma_{i\underline{a}}{}^{\underline{c}} g_{\underline{cb}}\right) \beta^{\underline{b}}$$

after a short computation. Now recall $\Gamma_{i\underline{a}}{}^{\underline{b}} = \varepsilon^{\underline{b}}(\nabla_i e_{\underline{a}}) = -(\nabla_i \varepsilon^{\underline{b}})(e_{\underline{a}})$ because $\varepsilon^{\underline{b}}(e_{\underline{a}}) = \delta_{\underline{a}}^{\underline{b}}$. From this, $\Gamma_{i\underline{ac}} = \Gamma_{i\underline{a}}{}^{\underline{b}} g_{\underline{bc}}$ is skew symmetric on the indices $\underline{ac}$ hence the right hand side of the last display is equal to $-2r\Gamma_{i\underline{a}}{}^{\underline{b}} \beta_{\underline{b}}$. Therefore $r = \frac{1}{2}$.

Now let us consider conformal transformation $\hat{g} := e^{2\Upsilon} g$. The orthonormal frame with respect to $\hat{g}$ is $\hat{e}_{\underline{a}} = e^{-\Upsilon} e_{\underline{a}}$. In the other words, we have to consider the frame and the coframe $e_{\underline{a}}$ of $TM[-1]$ and $\varepsilon^{\underline{b}}$ of $T^*M[1]$, respectively. Using the abstract indices, this is $e_{\underline{a}} = e_{\underline{a}}^j \in \mathcal{E}^j[-1]$ and $\varepsilon^{\underline{b}} = \varepsilon_j^{\underline{b}} \in \mathcal{E}_j[1]$. Then $\Gamma_{i\underline{a}}{}^{\underline{b}} = \varepsilon^{\underline{b}}(\nabla_i e_{\underline{a}}) = \varepsilon_c^{\underline{b}} \nabla_i e_{\underline{a}}^c$ and using (B.2) and (B.1) for $\hat{\nabla}_i e_{\underline{a}}^c$, we obtain the transformation

$$\hat{\Gamma}_{i\underline{a}}{}^{\underline{b}} = \Gamma_{i\underline{a}}{}^{\underline{b}} + \Upsilon_{\underline{a}} \varepsilon_i^{\underline{b}} - \Upsilon^{\underline{b}} g_{i\underline{a}} \qquad \text{(B.5)}$$

where $g_{i\underline{a}} = g_{ic} e_{\underline{a}}^c$. Note inserting this to $\nabla_{\underline{i}} U^{\underline{b}} = \frac{\partial U^{\underline{b}}}{\partial x^{\underline{i}}} + \Gamma_{\underline{ia}}{}^{\underline{b}} U^{\underline{a}}$ (with abstract indices), we obtain an analogue of (B.2) for $U^b \in \mathcal{E}^b[-1]$. Similarly, if we apply the transformation (B.5) to a spinor section f_λ, we have to consider the latter one to be appropriately weighted, i.e. $f_\lambda \in \mathcal{E}_\lambda[\frac{1}{2}]$. (Note $E_\lambda[\frac{1}{2}]$ is a self dual (conformal) bundle.) Now if we insert (B.5) to (B.4) with $r = \frac{1}{2}$

and use abstract indices, we obtain the final result

$$\hat{\nabla}_a f = \nabla_a f - \frac{1}{2}\Upsilon_a f - \Upsilon^p \beta_a \beta_p f \quad \text{for } f \in \mathcal{E}_\lambda[1/2]. \tag{B.6}$$

From this and (B.1), we immediately obtain the formula (1.21) for $f \in \mathcal{E}_\lambda[w]$. Let us note that $E_\lambda[1/2]$ is the bundle $\sum$ from [8] where transformation of the spinor connection is used but not explicitly mentioned. See also [7] and references therein for more details.

Bibliography

[1] Bailey, T.N.; Eastwood, M.G.; Gover, A.R.: *Thomas's structure bundle for conformal, projective and related structures*, Rocky Mountain J. Math. 24, 1191–1217 (1994)

[2] Baston, R.; Eastwood, M.G.: *The Penrose Transform: its interaction with representation Theory*, Oxford University Press, (1989)

[3] Bernstein, I.N.; Gelfand, I.M.; Gelfand, S.I.: *Structure of representations generated by vectors of highest weight*, Funct. Annal. Appl. 5:1 (1971), pp. 1–8

[4] Bernstein, I.N.; Gelfand, I.M.; Gelfand, S.I.: *Differential operators on the base affine space and a study of* $\mathfrak{g}$*–modules*, in: Lie groups and their representations, Proc. Summer School in Group Represenations. Bolyai Janos Math. Soc., Budapest 1971, pp.21-64, New York: Halsted 1975

[5] Boe, B.D.; Collingwood, D.H. *A comparision theory for the structure of induced representations I.*, J. of Algebra 94 (1985), 511–545

[6] Boe, B.D.; Collingwood, D.H. *A comparision theory for the structure of induced representations II.*, Math. Z. 190 (1985), 1–11

[7] Branson, T.: *Interwining Differential Operators for Spinor–Form Representations of the Conformal Group*, Adv. Math. 54 (1984), 1–21

[8] Branson, T.: *Conformal structure and spin geometry*, in: "Dirac operators: Yesterday and Today", pp.163–191, J.-P. Bourguignon, T. Branson, A. Chamsedinne, O. Hijazi, R.J. Stanton, eds., International Press, Somerville MA, 2005

[9] Branson, T.; Čap, A.; Eastwood, M.G.; Gover, A.R.; *Prolongations of Geometric Overdetermined Systems*, to appear: Int. J. Math.

[10] Branson, T.; Gover, A.R.: *Electromagnetism, metric deformations, ellipticity and gauge operators on conformal 4-manifolds*, Diff. Geom. and its Applications, **17**, (2002), 229–249.

[11] Branson, T.; Gover, A.R.: *Conformally invariant operators, differential forms, cohomology and a generalisation of Q curvature*, Comm. Partial Diff., to appear.

[12] Budinich, P.; Trautman, A.: *The Spinorial Chessboard*, Trieste Notes in Physics, Springer–Verlag, 1988

[13] Calderbank, M.J.; Diemer, T.: *Differential invariants and curved Bernstein–Gelfand–Gelfand sequences*, J. Reine Angew. Math. 537 (2001) 67-103

[14] Calderbank, M.J.; Diemer, T.; Souček: *Ricci-corrected derivatives and invariant differential operators.* Differential Geom. Appl. 23 (2005), no. 2, 149–175

[15] Čap, A.; Gover, A.R.: *Tractor calculi for parabolic geometries*, Trans. Amer. Math. Soc. 354, 1511–1548 (2002)

[16] Čap, A.; Gover, A.R.: *Tractor bundles for irreducible parabolic geometries*, in: Global analysis and harmonic analysis (Marseille–Luminy, 1999), 129–154. Semin. Congr., 4, Soc. Math. France, Paris 2000

[17] Čap, A.; Gover, A.R.: *Standard tractors and the conformal ambient metric construction.* Ann. Global Anal. Geom. 24 (2003), no. 3, 231–259.

[18] Čap, A.; Slovák, J.: *Parabolic geometries*, prepared monography

[19] Čap, A.; Slovák, J.; Souček, V.: *Invariant operators on manifolds with almost Hermitian symmetric structures. III. Standard operators.* Differential Geom. Appl. 12 (2000), no. 1, 51–84

[20] Čap, A.; Slovák, J.; Souček, V.: *Bernstein–Gelfand–Gelfand sequences*, Ann. of Math. 154, no 1 (2001), 97-113.

[21] Eastwood, M.G.: *Notes on conformal differential geometry*, in: The Proceedings of the 15th Winter School "Geometry and Physics" (Srní1995), 57–76, Rend. Circ. Mat. Palermo (2) Suppl. No. 43, 1996

[22] Eastwood, M.G.; Graham, C.R.: *Invariants of conformal densities*, Duke Math. Jour. 63 (1991), 633–671

[23] Eastwood, M.G.; Rice, J.W.: *Conformally invariant differential operators on Minkowski space and their curved analogues*, Commun. Math. Phys. 109 (1987), 207–228. Erratum, Commun. Math. Phys. 144 (1992), 213

[24] Eastwood, M.G.; Slovák, J.: *Semi-holonomic Verma Modules*, Journal of Algebra 197 (1997), 424-448

[25] Fegan, H.D.: *Conformally invariant first order differential operators*, Quart. J. Math 27 (1976), 371–378

[26] Fulton, W.; Harris, J.: "Representation theory", Springer-Verlag, GTM 129, 1991, pp. 551

[27] Gover, A.R.: *Aspects of parabolic invariant theory.* The 18th Winter School "Geometry and Physics" (Srn, 1998). Rend. Circ. Mat. Palermo (2) Suppl. No. 59 (1999), 25–47

[28] Gover, A.R.: *Conformally invariant operators of the standard type*, Quart. Jour. Math. 41 (1989), 197–207

[29] Gover, A.R.: *Invariant theory and calculus for conformal geometries*, Adv. Math. 163 (2001), 206–257

[30] Gover, A.R.: *Conformal de Rham Hodge theory and operators generalising the Q–curvature*, in: The Proceedings of the 24th Winter School "Geometry and Physics" (Srní2004), 109–137, Rend. Circ. Mat. Palermo (2) Suppl. 75, 2005

[31] Gover, A.R.: *Almost conformally Einstein manifolds and obstructions*, in Differential Geometry and its Applications, proceedings of the 9th International Conference on Differential Geometry and its Applications, Prague 2004, Charles University, Prague, 2005, pp243–255. Preprint math.DG/0412393, `www.arxiv.org`

[32] Gover, A.R.: *Laplacian operators and Q–curvature on conformally Einstein manifolds*. Preprint math.DG/0506037, `www.arxiv.org`

[33] Gover, A.R.; Hirachi, K.: *Conformally invariant powers of Laplacian – A complete nonexistence theorem*, J. Amer. Math. Soc. 17 (2004), no.2, 389–405

[34] Gover, A.R.; Nurowski, P.: *Obstructions to conformally Einstein metrics in n dimensions*, J. Geom. Phys. **56**, (2006), 450–484.

[35] Gover, A.R.; Peterson, L.,J.: *Conformally invariant powers of Laplacian, Q–curvature and tractor calculus*, Comm. Math. Phys. 235 (2003), no. 2, 339-378

[36] Graham, R.; Mason, R.; Jenne, L.J.; Sparling, G.A.: *Conformally invariant powers of the Laplacian I. Existence*, J. London Math. Soc. 46 (1992), 557–565

[37] Humphreys, J.E.: "Introduction to Lie Algebras and Representation Theory." Grad. Texts. Math. vol. 9, Springer, 1972

[38] Kashiwada, T., *On conformal Killing tensor*, Natur. Sci. Rep. Ochanomizu Univ. **19** (1986), 67-74.

[39] Kolář, I.; Michor, P.W.; Slovák, J.: "Natural operations in differential geometry ", Springer–Verlag, 1993, pp. 434

[40] Kostant, B.: *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. 74, No.2 (1961), pp. 329-387

[41] Leitner, F.: *Conformal Killing forms with normalisation condition* Rend. Circ. Mat. Palermo (2) Suppl. No. 75 (2005), 279–292.

[42] Onishchik, A.L.: *Lectures on real semisimple Lie algebras and their representations*, ESI Lecture Notes, Vol. 1, 86p, 2004

[43] Penrose, R., Rindler, W.: "Spinors and space time, Vol.1", Cambridge University Press, Cambridge London New York, 1984

[44] Penrose, R., Rindler, W.: "Spinors and space time, Vol.2", Cambridge University Press, Cambridge London New York, 1984

[45] Semmelmann, U., *Conformal Killing forms on Riemannian manifolds*, habilitation, Ludwig Maximilians University in Munchen, 2001.

[46] Semmelmann, U.: *Conformal Killing forms on Riemannian manifolds*, Math. Z., Vol 245, No 3, (2003), 503–527.

[47] Semmelmann, U.: *Killing forms on G2 and Spin7 manifolds.* Preprint math.DG/0410065, `www.arxiv.org`

[48] Šilhan, J.: *A real analog of Kostant's version of the Bott–Borel–Weil theorem*, Journal of Lie Theory 14 (2004)

[49] Slovák, J.: "Invariant operators on conformal manifolds". Research Lecture Notes, University of Vienna, 1992, pp.138

[50] Slovák, J.: "Parabolic geometries". Research Lecture Notes, University of Adelaide, 1997, pp.70

[51] Stein, E. M.; Weiss, G.: *Generalization of the Cauchy-Riemann equations and representations of the rotation group.* Amer. J. Math. 90 1968 163–196

[52] Stepanov, S.E., *On conformal Killing 2–form of electromagnetic field*, J. Geom. and Phys., **33**, (2000), 191–209.

[53] Thomas, T.Y.: *On conformal geometry*, Proc. Natl. Acad. Sci. USA 12 (1926), 352–359

[54] Varadarajan, V.S.: *Supersymmetry for Mathematicians: An Introduction*, Courant Lecture Notes, Vol. 11, pp. 300

[55] Verma, D.N.: *Structure of certain induced representations of complex semisimple Lie algebras*, Bull. Amer. Math. Soc. 74 (1968), pp. 160-166

[56] Zuckerman, G.: *Tensor products of finite and infinite dimensional representations of semisimple Lie groups*, Ann. Math. 106 (1977), 295–308