Finding Poincaré Constants for Non-convex Domains in $\mathbb{R}^n$

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In its most general form, Poincaré's inequality gives a bound on the L_p-norm of a function in terms of the L_p-norm of its derivatives.
Its natural setting is in Sobolev spaces of functions defined on suitably well-behaved domains $\Omega \subset \mathbb{R}^n$. It is intimately associated with the following Sturm-Liouville Problem with Neumann boundary conditions.

$$\begin{array}{ll} \nabla^2 u(x) - \lambda u(x) = 0 & \text{on } \Omega \\ \nabla u(x) \cdot \mathbf{n}(x) = 0 & \text{on } \partial\Omega \end{array}$$

The statement of Poincaré's inequality in the Hilbert space $H^1 (= W^{1,2})$ is as follows.

For any $\Omega \subset \mathbb{R}^n$ connected open bounded with Lipschitz boundary there exists a $C \in \mathbb{R}$, such that for all $u \in H^1(\Omega)$:

$$\|u - \langle u \rangle\|_{L_2} \leq C \, \|\nabla u\|_{L_2} \, ,$$

where $\langle u \rangle$ denotes the mean of u over Ω

The mere *existence* of such a C can be proven by a compactness argument but this gives no clue as to how the C depends on the shape and size of the domain Ω.

In my talk I will discuss my work of finding estimates of C for rectangles, annuli and a certain class of non-convex, trouser-shaped domains in $\mathbb{R}^2$. I have found an optimal constant for the rectangle, an almost-optimal constant for the annulus and some intial estimates for the trousers. As this is my dissertation project it is still very much a work in progress. I will discuss the subtleties and anxieties of working out the Poincaré constants, making use of some of your favourite theorems from the theory of integration and functional analysis along the way.