MATHS 190 – SEMESTER 2 2009

SUMMARY FOR WEEKS 10, 11, 12

Relevant sections of textbook:
Chapter 5 "Contortions of Space"
All parts, <u>except</u> : §5.4 (knots) and pages 390 – 394 of §5.5 (Brouwer Fixed Point Theorem)

(Chapter 5.1)
<u>Lecture 19 : Rubber Sheet Geometry</u>

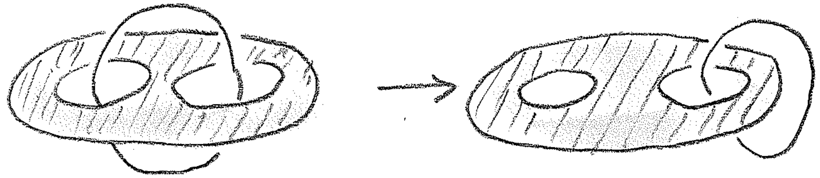
<u>Main Idea</u> : If we suppose that various 2 and 3-dimensional objects are made from an "infinitely flexible" material, then surprising feats are possible!

surprising things are possible.

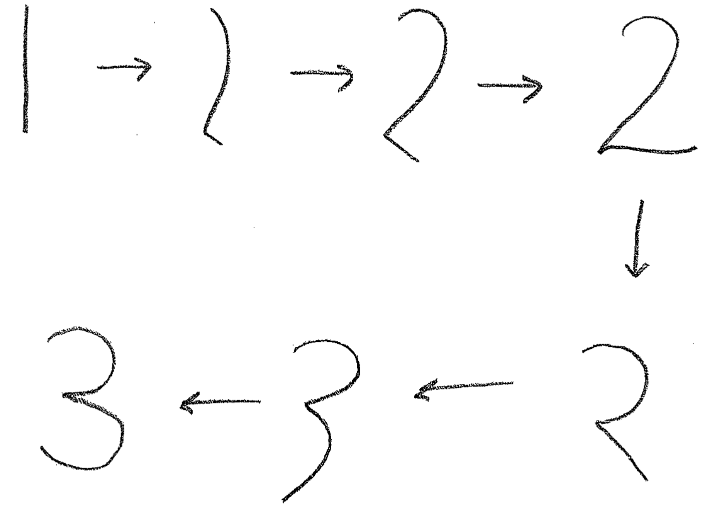
<u>Examples</u> :

(1) A punctured inner tube may be turned inside out

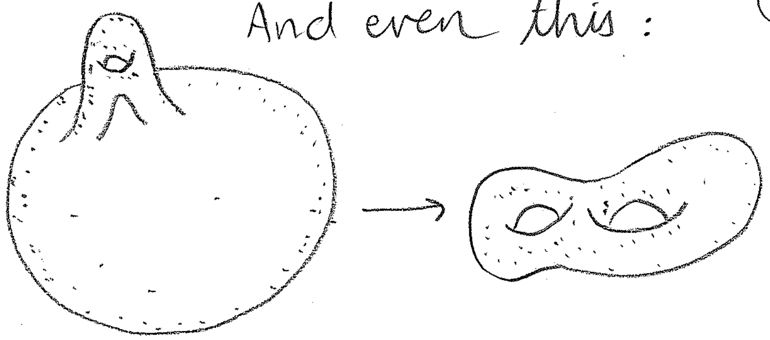
(2) We can do this :



(3) And this :



(4) And even this :



Rules of the game :

Two objects are <u>equivalent</u> <u>by</u> <u>distortion</u> provided the distortion involves :

(1) NO cutting
(2) NO gluing (2 distinct points collapsing onto 1)

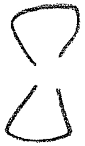
We are allowed to :

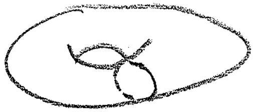
(1) Stretch (2) Bend
(3) Shrink (4) Twist.

Stratergies :

To show two objects ARE equivalent normally involves an explicit description of the necessary distortions (drawing pictures).

To argue that two objects are NOT equivalent, find a property one of the objects has that is not shared by the other — a property that ought to be preserved under permissible distortions.

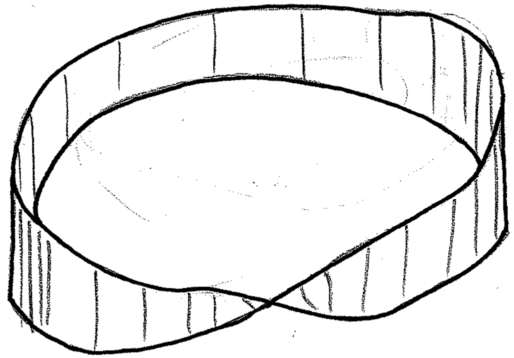
e.g. Removing the crossing point of the "8" figure divides it into two pieces "". The figure "4" has no such point. So "8" & "4" are not equiv.

e.g. Removing a simple closed loop from the torus  leaves it in one piece. Any such loop removed from the sphere creates two pieces. So the torus & sphere are not equiv.

(Chapter 5.2)

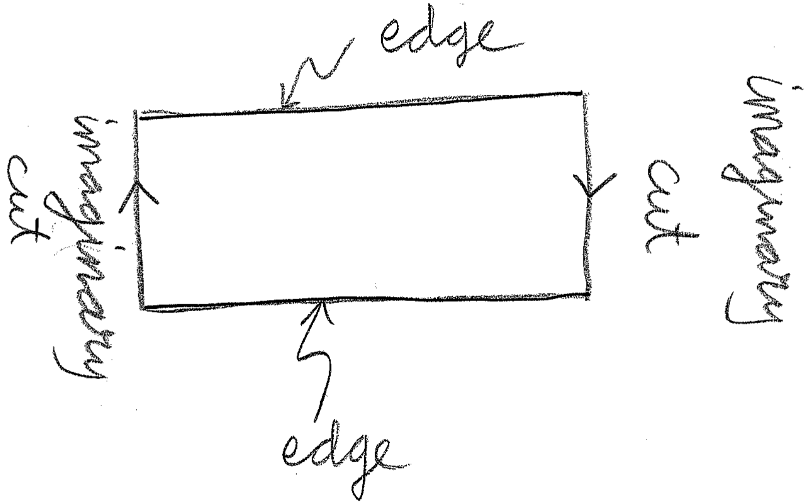
Lecture 20 : The band that wouldn't stop playing (Möbius Bands)

Main idea : The Möbius band is a surface with only ONE edge and ONE side :



Cutting the Möbius band in various ways has surprising results.

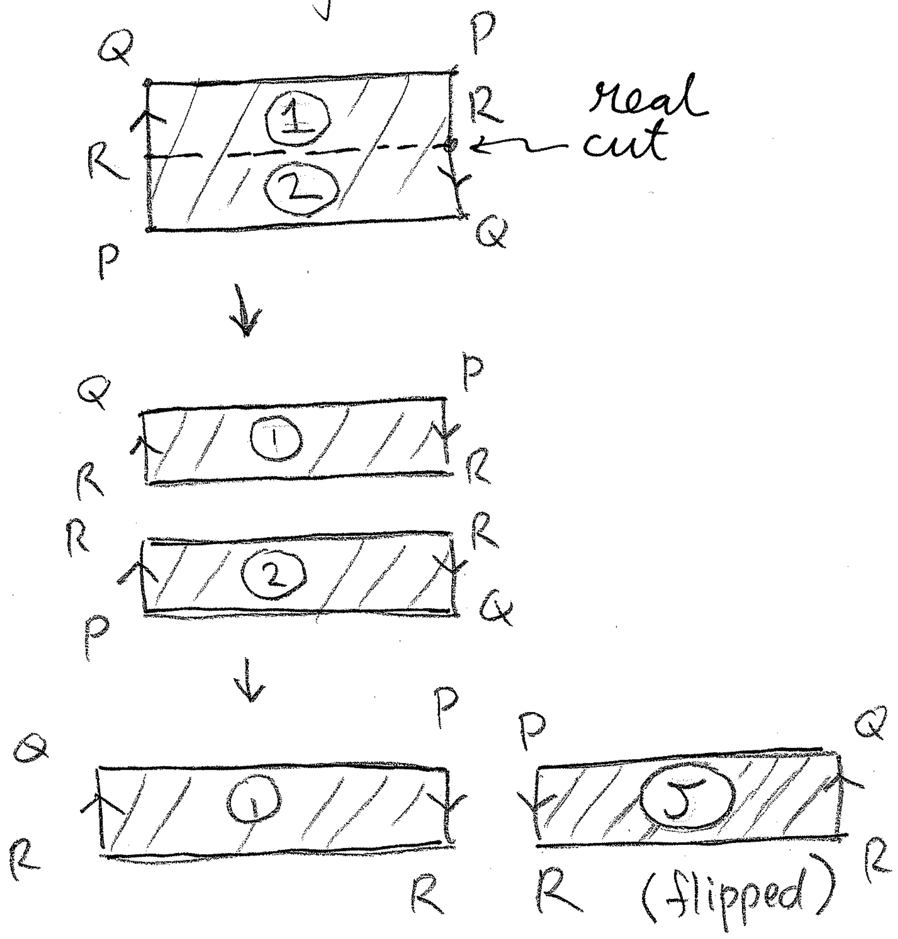
One way to understand these results is with a rectangular model of the Möbius band:

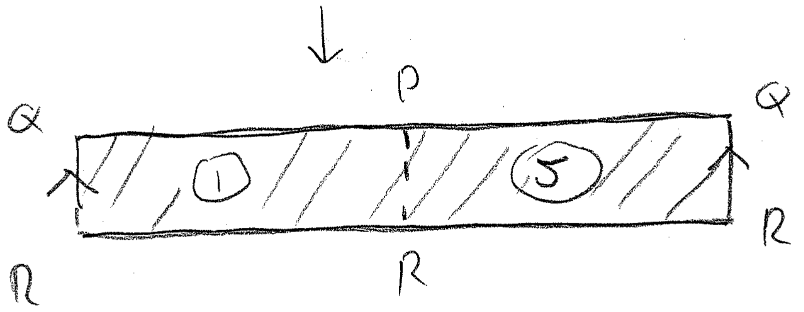


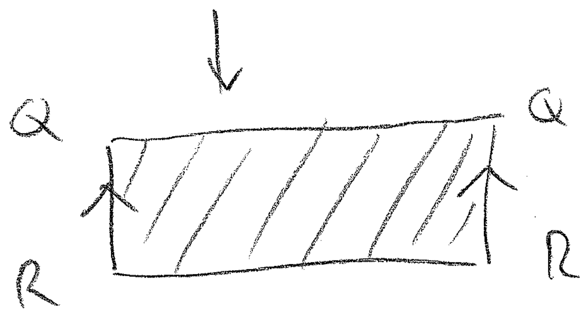
One imagines gluing back

the cuts with matching arrows to recover the Möbius band.

Example Cutting a Möbius band along centreline:







We get a SINGLE band
TWO edges and TWO sides!

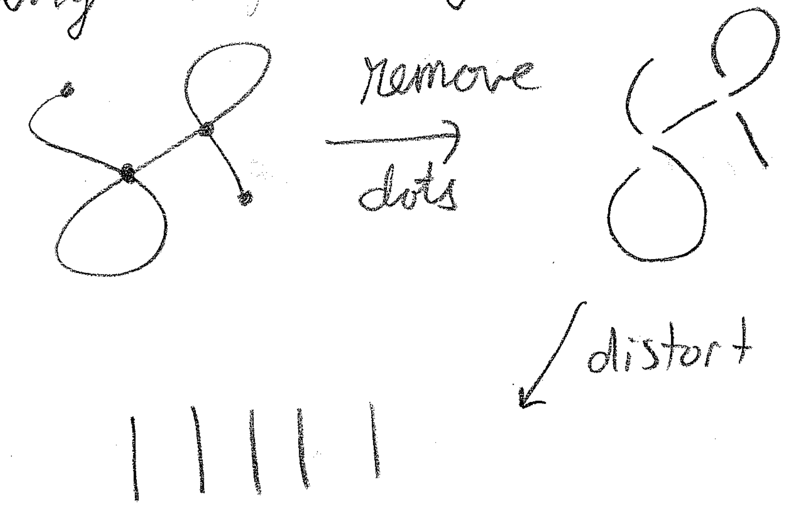
Beyond the band

We also discussed the existence of more abstract surfaces such as the Klein bottle and Boy's Surface (Projective plane).
NOT EXAMINABLE.

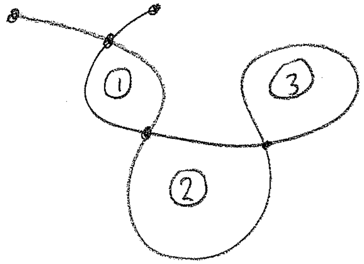
(Chapter 5.3)

Lecture 21: Feeling Edgy (The Euler Characteristic)

Graphs: A *graph* is a "doodle" with dots on it, such that removing the dots leaves only curves, each equivalent by distortion to a single line segment:



Vertices, Edges and Regions :



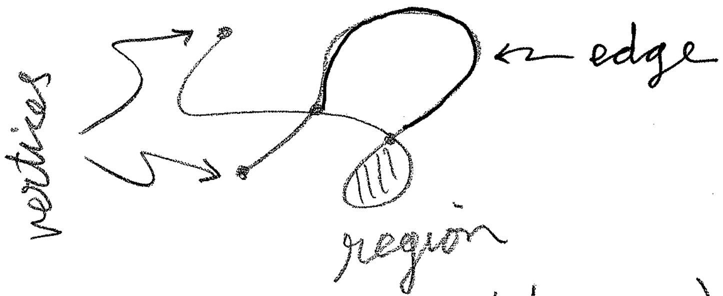
4 (outside counts as region)

How to count regions

V = number of vertices (dots)

E = " " edges

F = " " regions ("faces")

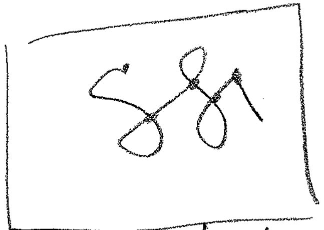


Euler Characteristic := V - E + F

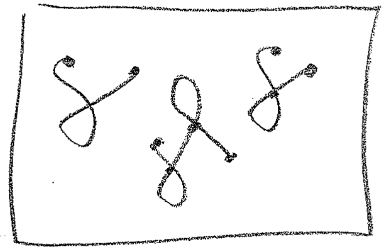
Euler Characteristic Theorem

> The Euler Characteristic of any connected planar graph is two :
>
> $$V - E + F = 2$$

"connected" means in one piece :



connected



not connected (3 pieces or components)

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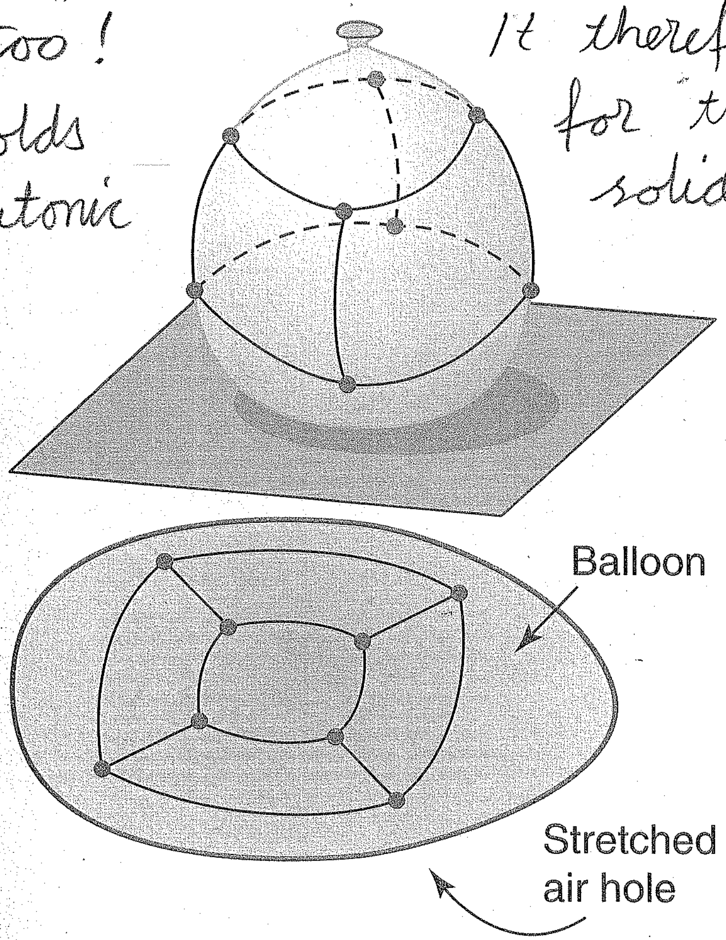
Euler Ch. Theorem holds for graphs on spheres too! It therefore holds for the Platonic solids!

Template 5.3.1 Square projected onto balloon



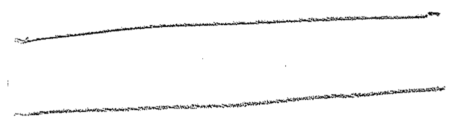
(chapter 5.5) 14

Lecture 22 : Fixed points, hot loops and rainy days

Result 1 : ("Human Chain" experiment) Consider two identical pieces of string :

A

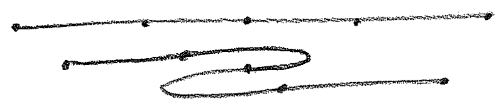
B



Now stretch, shrink and/or fold B such that each point is still opposite some point of A :

A

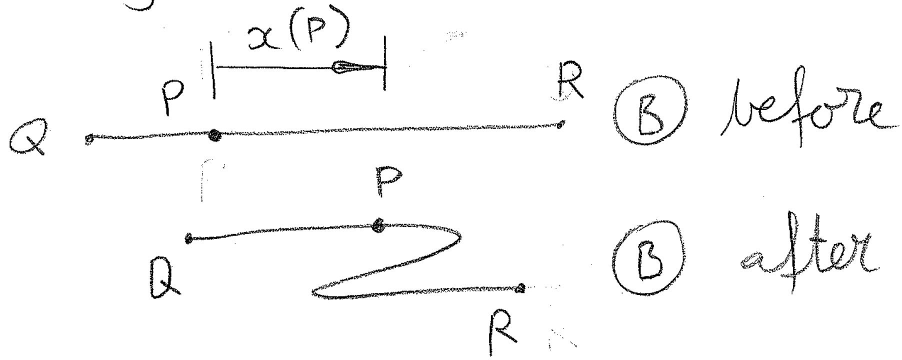
B



Then :

There is always a point on string B that is directly opposite the same point on A that it was lined up with before it was folded.

Proof: Let P be a point on B and $x(P)$ the distance travelled by P, to the right, during the folding/stretching:



Then $x(Q) \geq 0$

but $x(R) \leq 0$

Since $x(P)$ changes "continuously" as we change point P (no cutting of string allowed) there is some point P such that

$$x(P) = 0.$$

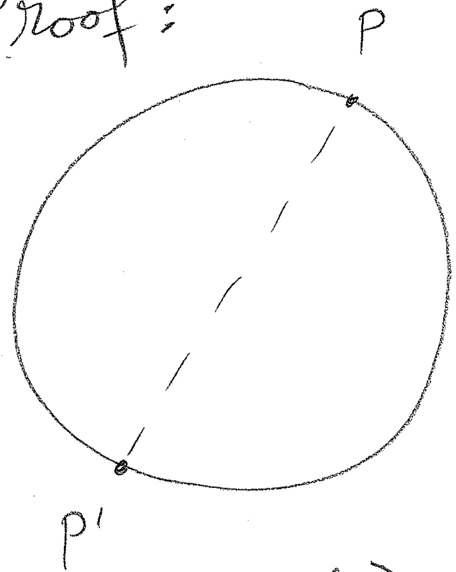
This proves the claim.

Result 2 (Hot Loop Theorem)

If we have a circle of variably heated wire, then there is a pair of opposite points having the same temperature.

We are assuming the temperature varies "continuously" (no sudden jumps).

Proof: Let P be a point on the wire and P' the opposite point.



Define $x(P) := ($temperature at $P) - ($temperature at $P')$.

Then suppose $x(P) \neq 0$. Then either $x(P) > 0$ or $x(P) < 0$.

If $x(P) > 0$ then $x(P') < 0$. By "continuity" of temperature variations, there is some point Q between P & P' with $x(Q) = 0$. But then Q has desired property.

The case $x(P) < 0$ is similar

Result 3 (The Meteorology theorem)

By the hot loop theorem, there are always two places on the equator of the

Earth having the same temperature.
The following is also true:

> At every instant, there are two diametrically opposite points on the earth (antipodes) with identical temperatures and identical barometric pressures.

Proof: Beyond scope of Maths 190.

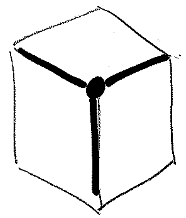
Lecture 23 (Chapter 5.3 again)

Why there are only five Platonic solids

[Refer to Slide 1 (over the page)]

Let Mysterydron be any solid equivalent by distortion to the sphere having:

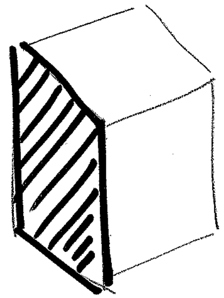
- a constant number of edges per vertex, ev, say.

E.g.  ev = 3

- a constant number of edges per face, ef, say.

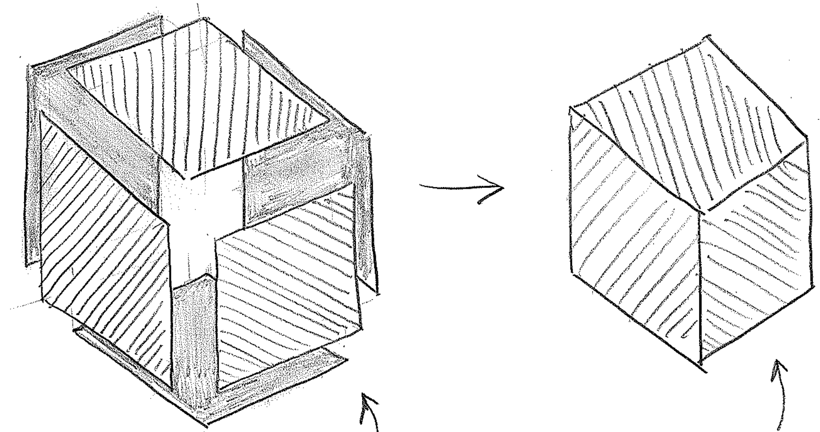
	Number of Vertices	Number of Edges	Number of Faces	$V - E + F$	Edges per Vertex	Edges per Face
Tetrahedron	4	6	4	2	3	3
Cube	8	12	6	2	3	4
Octahedron	6	12	8	2	4	3
Dodecahedron	20	30	12	2	3	5
Icosahedron	12	30	20	2	5	3
Mysteryhedron	V	E	F	2	ev	ef

E.g.



$ef = 4.$

Calculating E from F



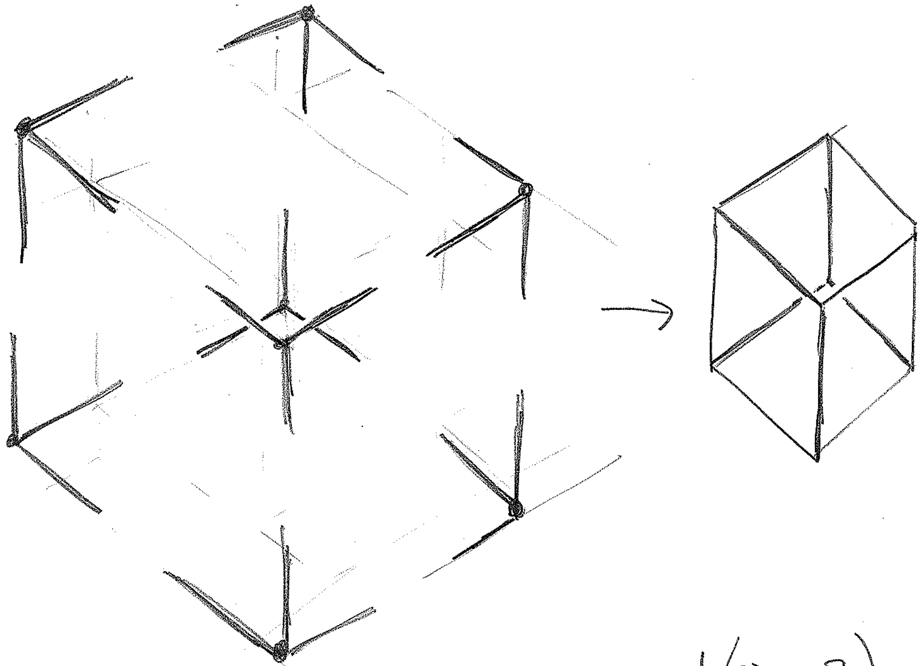
$E = 6 \times 4$

$E = \frac{1}{2}(6 \times 4)$

More generally: $E = \frac{1}{2}(F \times ef)$

$$\Rightarrow \boxed{F = \frac{2E}{ef}}$$

Calculating E from V



$$E = \tfrac{1}{2}(8 \times 3)$$

$$E = 8 \times 3$$

More generally : $E = \frac{1}{2}(V \times ev)$

$$\Rightarrow \boxed{V = \frac{2E}{ev}}$$

Apply Euler Characteristic Theorem :

$$V - E + F = 2$$

$$\Rightarrow \frac{2E}{ev} - E + \frac{2E}{ef} = 2$$

$$\boxed{\underbrace{\left(\frac{2}{ev} + \frac{2}{ef} - 1\right)}_{\text{Must be positive !!}} E = 2}$$

So $\boxed{\frac{2}{ev} + \frac{2}{ef} > 1}$ — $(*)$

ev	ef	$\frac{2}{ev} + \frac{2}{ef}$		
3	3	$\frac{2}{3} + \frac{2}{3} = \frac{4}{3} \geqslant 1$	✓	tetrahedron
3	4	$\frac{2}{3} + \frac{1}{2} = \frac{7}{6} \geqslant 1$	✓	cube
4	3	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6} \geqslant 1$	✓	tetrahedron
4	4	$\frac{1}{2} + \frac{1}{2} = 1$	×	
3	5	$\frac{2}{3} + \frac{2}{5} = \frac{16}{15} \geqslant 1$	✓	dodecahedron
5	3	$\frac{2}{5} + \frac{2}{3} = \frac{16}{15} \geqslant 1$	✓	isocahedron
3	6	$\frac{2}{3} + \frac{1}{3} = 1$	×	
6	3	$\frac{1}{3} + \frac{2}{3} = 1$	×	

No more possibilities consistent with (*) exist!