Maths 260 Lecture 20

- **Topics for today:**
  Complex numbers:
  - Derivatives of complex-valued functions
  - The exponential of a complex number
  - Euler's formula
  - De Moivre's formula
  - Exponential polar form

- **Reading for this lecture:**
  Some notes on complex numbers
  BDH Appendix C

- **Suggested exercises:**
  Problems at the back of "Some notes on complex numbers"

Derivatives of complex valued functions

- Suppose t is real and $f(t)$ is a complex-valued function of t, i.e.

$$f(t) = u(t) + iv(t)$$

  for some real-valued functions u and v.
- Then, if u and v are differentiable with respect to t, we define the derivative of $f(t)$ to be

$$\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}$$

Example 1:

Find the derivative of the function $f(t) = \cos(t) + i\sin(t)$.

Properties of the function $f(t) = \cos(t) + i\sin(t)$:

- $f'(t) = if(t)$,
- $f(0) = 1$,
- $f(t_1)f(t_2) = f(t_1 + t_2)$.

Compare this to the function $g(t) = e^{at}$, where a is real:
Properties of $g(t)$:

- $g'(t) =$
- $g(0) =$
- $g(t_1)g(t_2) =$

Euler's formula

The similarities between the properties of f and g prompted Euler to make the definition:

Euler's Formula:

$$e^{it} = \cos t + i \sin t$$

Proof of Euler's formula

We can prove Euler's formula using the following power series expansions.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

Then we see that,

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \dots,$$

Remembering $i^2 = -1$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots,$$

Collecting the real and imaginary parts,

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

de Moivre's formula

- We can use Euler's formula to prove **de Moivre's formula:**

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

- Proof

  $(\cos(\theta) + i\sin(\theta))^n = (e^{i\theta})^n =$

Example 2: Use de Moivre's formula to express $\cos 2\theta, \sin 2\theta$ in terms of $\cos\theta, \sin\theta$.

Euler's Formula and Polar form

- **Example 3:** Rewrite $z = 1 + i$ using the complex exponential.

- In general, complex number $z = a + ib$ can be written in **polar form** as

$$z = re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$.

Example 4

- Evaluate $2i(1+i)$ using both polar and rectangular forms. Compare your answers.

Arithmetic in exponential form

- **Multiplication** and **division** are now easy in polar form.

Example 5: If $z_1 = 2e^{i\pi/6}$ and $z_2 = -e^{i\pi/4}$, compute $z_1 z_2$ and z_1/z_2

Arithmetic in exponential form

- We can easily calculate **powers** using complex exponentials.

Example 6: If $z = 3e^{i\pi/5}$, find z^2 and z^5.

Other properties of complex exponentials

- We have: $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

Example 7: Calculate $e^{(2+3i)t}$ and find the real and imaginary parts

Example 8: Calculate $e^{(-1-4i)t}$ and find the real and imaginary parts

Polar form and solving equations

- **Example 9:** Find all solutions to the equation $z^3 = 1$.
- $z = 1$ is obviously a solution. Any others? We expect three solutions from the Fundamental Theorem of Algebra.
- Write
$$z = re^{i\theta},$$
where $r = |z| > 0$.
- Then
$$z^3 = r^3 e^{3i\theta} = r^3(\cos 3\theta + i \sin 3\theta)$$
and therefore
$$r^3(\cos 3\theta + i \sin 3\theta) = 1$$
- So...

- Note that we get only three distinct solutions because of the periodicity of cosine and sine.
- Plot the solutions in the Argand plane:

Example 10

Find all solutions to the equation $z^3 = 1 + i$.

Derivatives in exponential form

- If λ is a complex number then

$$\frac{d}{dt}\left(e^{\lambda t}\right) = \lambda e^{\lambda t}.$$

- Proof:

Important ideas from today's lecture:

- Derivatives of complex-valued functions
- The complex exponential
- Euler's formula

$$e^{it} = \cos t + i \sin t$$

- de Moivre's formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

- Polar form of a complex number

$$z = re^{i\theta}$$