DEPARTMENT OF MATHEMATICS

MATHS 260 Tutorial 9

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1. Consider the linear system
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY},$$
where $\mathbf{A}$ is as given below. For each choice of $\mathbf{A}$:

   - Use the eigenvalues and eigenvectors provided for each system to sketch the phase portrait. DO NOT use *pplane* until you have sketched the phase portrait by hand.
   - Use *pplane* to draw the phase portrait and compare with your sketch.
   - Describe the long term behaviour of the solutions.
   - Write down the general solution of the system in terms of real-valued functions.

   (a) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix}$.
   Eigenvalues are $-1, -2$, eigenvectors are $(1, 2)^T$ and $(0, 1)^T$ respectively.

   (b) $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 1 & -3 \end{pmatrix}$.
   Eigenvalues are $-3 \pm i$, eigenvectors are $(1, -i)^T$ and $(1, i)^T$.

   (c) $\mathbf{A} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$.
   Eigenvalues are -3, -3, eigenvector is $(1, 1)^T$, generalised eigenvector is $(2, 1)^T$.

   (d) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$.
   Eigenvalues are 0 and -3, eigenvectors are $(2, -1)^T$ and $(-1, 2)^T$ respectively.

   (e) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$
   Eigenvalues are $2, -1$, eigenvectors are $(2, 1)^T$ and $(1, 2)^T$ respectively.

   (f) $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.
   One eigenvalue is $\frac{5}{2} + \frac{\sqrt{3}}{2}i$ with eigenvector $(5 + \sqrt{3}i, -4 + 2\sqrt{3}i)^T$.

2. For the system in Q1(e), let $\mathbf{Y} = (x(t), y(t))^T$.

   (a) Write down differential equations for the components $x(t)$ and $y(t)$.

   (b) Using the general solution in vector form that you obtained in Q1(e), write down the component solution functions $x(t)$ and $y(t)$.

3. Find the eigenvalues and eigenvectors (by hand) for the matrix
$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$
Check your answers using Matlab (see the procedure for finding eigenvalues and eigenvectors using Matlab on the last page).

4. **Challenge question:** Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

where

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -1 & -k \end{pmatrix}$$

and k is a parameter. Use *pplane* to investigate the type of equilibrium at the origin for a range of values of $k \in [-4, 5]$. Draw a sketch (by hand) for each different type that you find. *Hint: you should be able to find at least 5 different types of behaviour.* What happens when $k = -2$?

Procedure for finding eigenvalues and eigenvectors using Matlab

To find eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 1 & -3 \end{pmatrix}$$

(a) In the Matlab command window, i.e., the window in which you type `pplane` and other Matlab commands, define matrix $\mathbf{A}$ by typing:
`A=[-3 -1;1 -3]`

(b) Use Matlab to find the eigenvalues and eigenvectors by typing:
`[v,d]=eig(A)`
The output from Matlab will be

```
v =


   0.7071                  0.7071
        0 - 0.7071i             0 + 0.7071i



d =


  -3.0000 + 1.0000i             0
        0                 -3.0000 - 1.0000i
```

The columns of the matrix v are the eigenvectors, and so in this case they are

$$\begin{pmatrix} 0.7071 \\ -0.7071i \end{pmatrix} \text{ and } \begin{pmatrix} 0.7071 \\ 0.7071i \end{pmatrix}.$$

The diagonal entries of the matrix d are the eigenvalues of $\mathbf{A}$; in this case they are $-3+i$ and $-3-i$.

Notes:

- The eigenvector in the first column of $\mathbf{v}$ corresponds to the eigenvalue in the first column of $\mathbf{d}$, the second eigenvector in $\mathbf{v}$ corresponds to the second diagonal entry in $\mathbf{d}$, and so on.
- Matlab finds eigenvectors with length 1, i.e., the sum of the squares of the components of the reported eigenvectors is equal to 1. You may want to divide through by one of the components to make the eigenvectors easier to draw. For instance, you could try dividing the eigenvector by the smallest component. For matrix $\mathbf{A}$ above, you can rescale the eigenvector stored as the second column by typing `v(:,2)/0.7071` or `v(:,2)/min(abs(v(:,2)))`.