Department of Mathematics

MATHS 260

Differential Equations

Mid-semester Test
30 April 2010

SURNAME:__________________

FORENAMES:__________________

Instructions

- This test contains **SIX** questions. Attempt **ALL** questions.
- Show **ALL** your working.
- Write your name and ID number at the top of each page.
- You have 50 minutes to do the test. Total marks = 45.
- Please read the following and sign below:

By signing this cover sheet I confirm that I am the student whose name appears above.

(Sign here): __________________

SURNAME: ____________________

FORENAMES: ____________________

(Capital letters please)

ID NUMBER: ____________________

(Official use only)

QUESTION 1 (6 marks)	
QUESTION 2 (7 marks)	
QUESTION 3 (12 marks)	
QUESTION 4 (6 marks)	
QUESTION 5 (12 marks)	
QUESTION 6 (2 marks)	
Total for 6 questions (45 marks)	

1. (6 marks) This question is about the differential equation

$$\frac{dx}{dt} = \frac{\cos t}{x^2}$$

(a) Find a one-parameter family of solutions to the differential equation.

(b) Find a solution to the IVP with $x(0) = 1$.

(c) What do the existence and uniqueness theorems tell you about solutions to the IVP with $x(0) = 0$?

2. (7 marks)

(a) Use Euler's method with stepsize $h = 1$ to compute an approximate value of the solution to the initial value problem

$$\frac{dy}{dt} = t^2 - y^2, \quad y(1) = -1$$

at final time $t = 3$. Show all your working.

(b) A different numerical method is used to find a solution at $t = 3$ to the same initial value problem (that is, with $y(1) = -1$), using different step-sizes. The following results are obtained:

number of steps	approximate solution at $t = 3$
8	3.92268017482
16	2.83410986275
32	2.80357184208
64	2.80263356201
128	2.80260255010
256	2.80260153916
512	2.80260153999

i. Use these results to estimate $y(3)$ accurate to 8 decimal places.

ii. Estimate the errors in the approximation using 32 and 64 steps.

iii. Hence calculate the effective order of the method at stepsize $h = 0.03125$.

3. (12 marks) This question is about the one-parameter family of differential equations

$$\frac{dy}{dt} = 2y^2 - k$$

where k is a parameter.

(a) Set $k = 2$.

i. Find all equilibrium solutions and determine their type (e.g., sink, source).

ii. Sketch the phase line.

(b) Repeat (a) for the case $k = -1$.

(c) Repeat (a) for the case $k = 0$.

(d) Now let k vary.

i. Locate the equilibrium solutions and determine their type for all values of k, including any bifurcation values.

ii. Sketch the bifurcation diagram. Be sure to label the main features of the bifurcation diagram.

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4. (6 marks) The following equation is used as a model of the number of possums living in a nature reserve.

$$\frac{dp}{dt} = kp(10 - p)$$

In the equation, p represents the number of possums in thousands, (so $p = 1$ means there are 1000 possums in the reserve) and t is time measured in years. The constant k is a parameter in the model.

(a) Briefly say what each term in the model might represent physically, i.e., say what physical phenomenon is being modelled by each term.

(b) An ecology graduate student sets traps in the nature reserve, and captures and removes 10 possums from the reserve each week. How could you modify the original equation to take account of this effect?

(c) Suppose you wanted to get some information about solutions to your new equation. Briefly explain (in two or three sentences) possible methods you could use to do this.

5. (12 marks) Consider the following system of differential equations

$$\frac{dY}{dt} = \begin{pmatrix} -2 & 5 \\ 0 & 3 \end{pmatrix} Y,$$

where

$$Y = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find all straight line solutions to this system of equations.

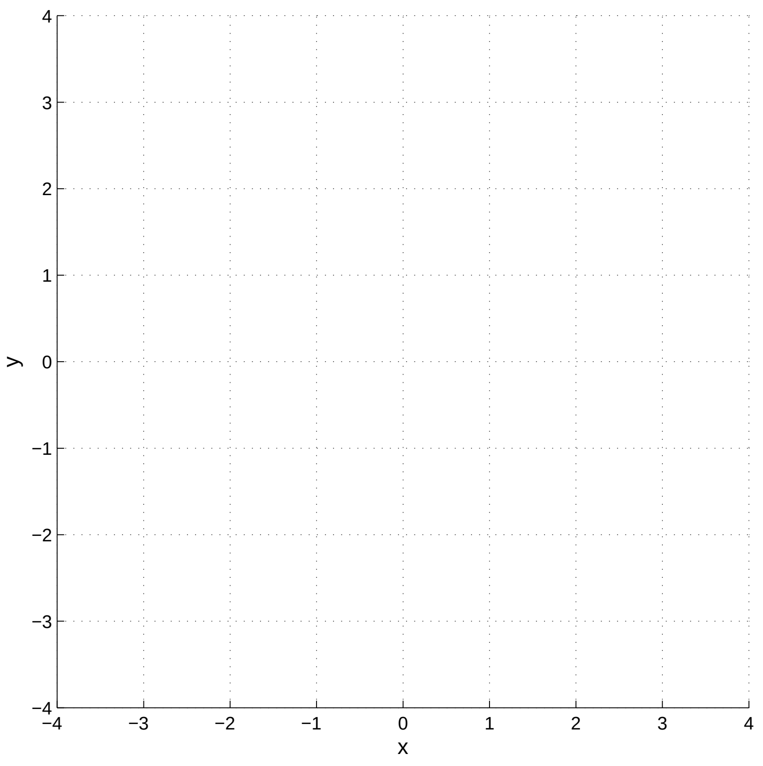
(b) Find the general solution to this system of equations. Your answer should contain two arbitrary constants.

(c) Find the solution that passes through $(x, y) = (0, 2)$ when $t = 0$.

(d) On the grid provided, sketch the phase portrait showing:

- all equilibrium solutions.
- all straight line solutions.
- the solution curve you found in part (c) above, for $t < 0$ as well as $t > 0$. Indicate on your sketch where $t = 0$.
- at least three other representative solution curves.

Grid for part (d):



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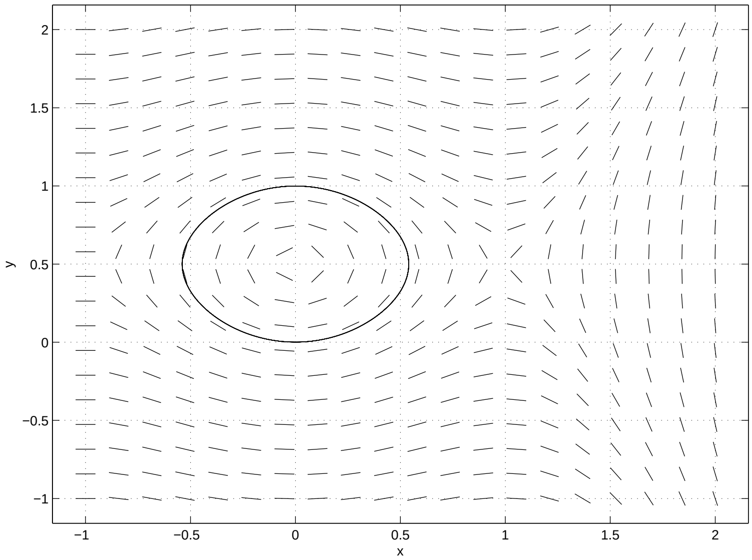
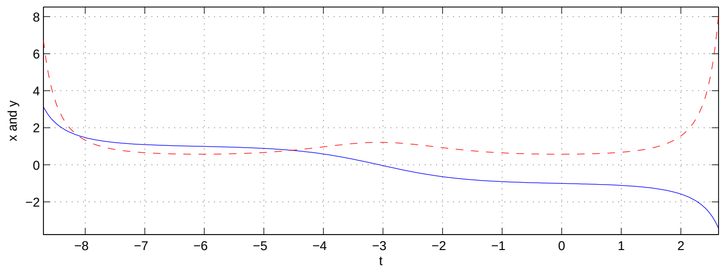
6. (2 marks) The following figure shows the direction field and a solution for a two-dimensional system of autonomous ODEs.



Figures (a) and (b) below show plots of $x(t)$ and $y(t)$ against t. Which of (a) or (b) corresponds to the solution drawn in the direction field above? CIRCLE your answer.

(a)



(b)