Einstein Metrics

and

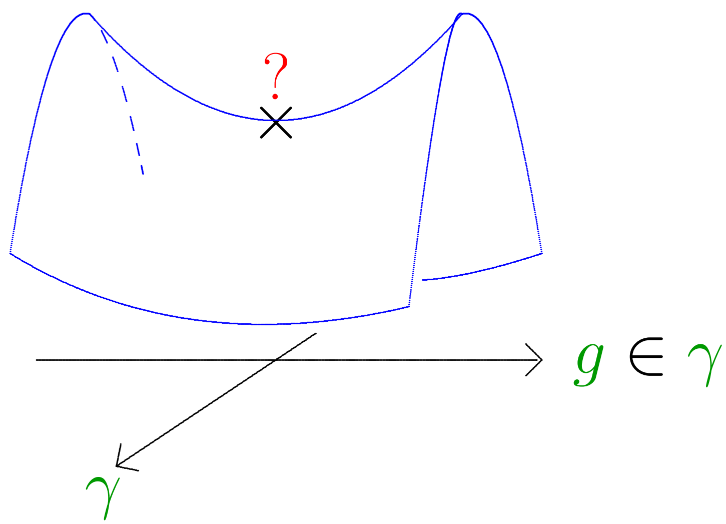
Global Conformal Geometry

II

Claude LeBrun
SUNY Stony Brook

Definition. *The Yamabe invariant of the smooth compact n-manifold M is given by*

$$\mathcal{Y}(M) = \sup_{\gamma} \inf_{g \in \gamma} V^{(2-n)/n} \int_M s_g \, d\mu_g$$



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Theorem (Gromov-Lawson/Stolz)**.** *For simply connected M^n, $n \geq 5$, index of Dirac operator is only obstruction to $s > 0$.*

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If $n = 2m$,

$$\mathbb{S}^* \otimes \mathbb{S} = \bigoplus_k \Lambda^k_{\mathbb{C}}$$

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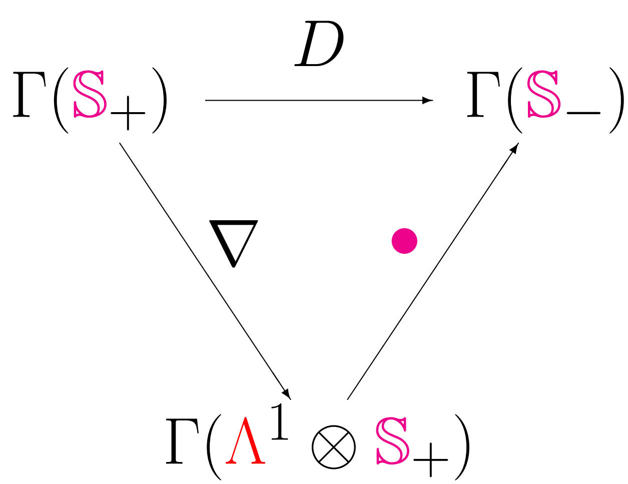
Compose to get Dirac operator D:



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Proposition (Lichnerowicz). *If M^{4k} compact spin, with $\hat{A}(M) \neq 0$, then $\nexists$ metric g on M with $s > 0$.*

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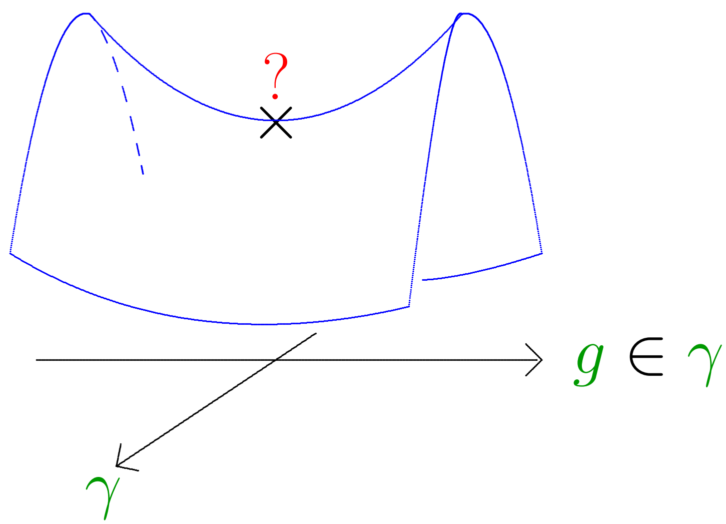

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Theorem. *Let* M *be a compact simply connected* n*-manifold,* $n \geq 3$*. If* $n \neq 4$*,* $\mathcal{Y}(M) \geq 0$*.*

Theorem. *There exist infinitely many compact simply connected* 4*-manifolds with* $\mathcal{Y}(M) < 0$*.*

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This is intimately tied to the fact that $\mathcal{Y}(M)$ depends strongly on the smooth structure in dimension four.

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Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

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where

$$s = \text{scalar curvature}$$
$$\mathring{r} = \text{trace-free Ricci curvature}$$
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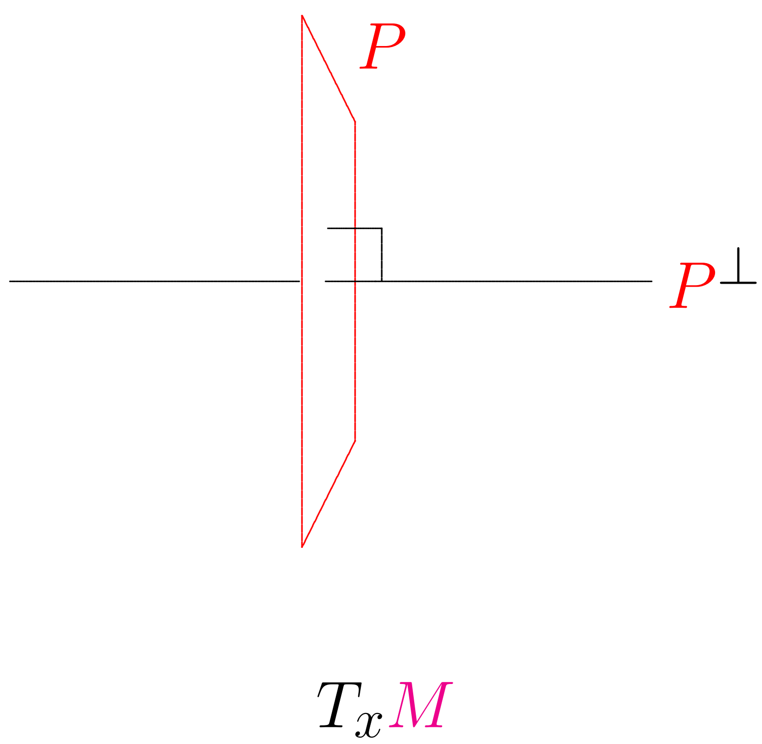

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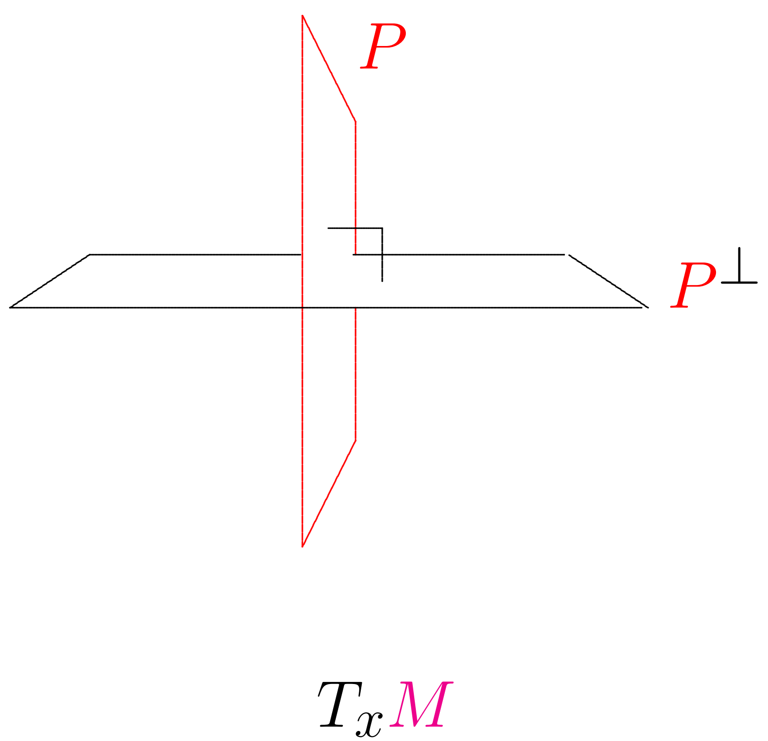

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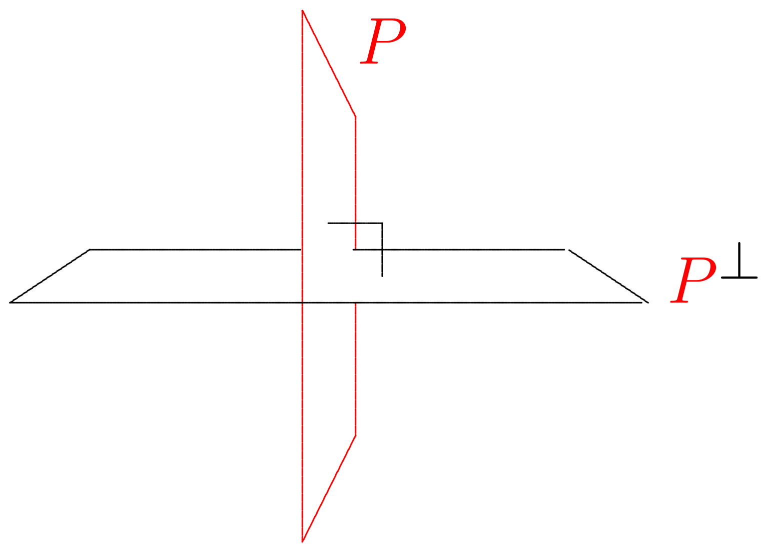

$$K(P) = K(P^{\perp})$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

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for Euler-characteristic $\chi(M) = \sum_j (-1)^j b_j(M)$.

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Here $b_\pm(M)$ = max dim subspaces $\subset H^2(M, \mathbb{R})$ on which intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ ([\varphi], [\psi]) & \mapsto & \int_M \varphi \wedge \psi \end{array}$$

is positive (resp. negative) definite.

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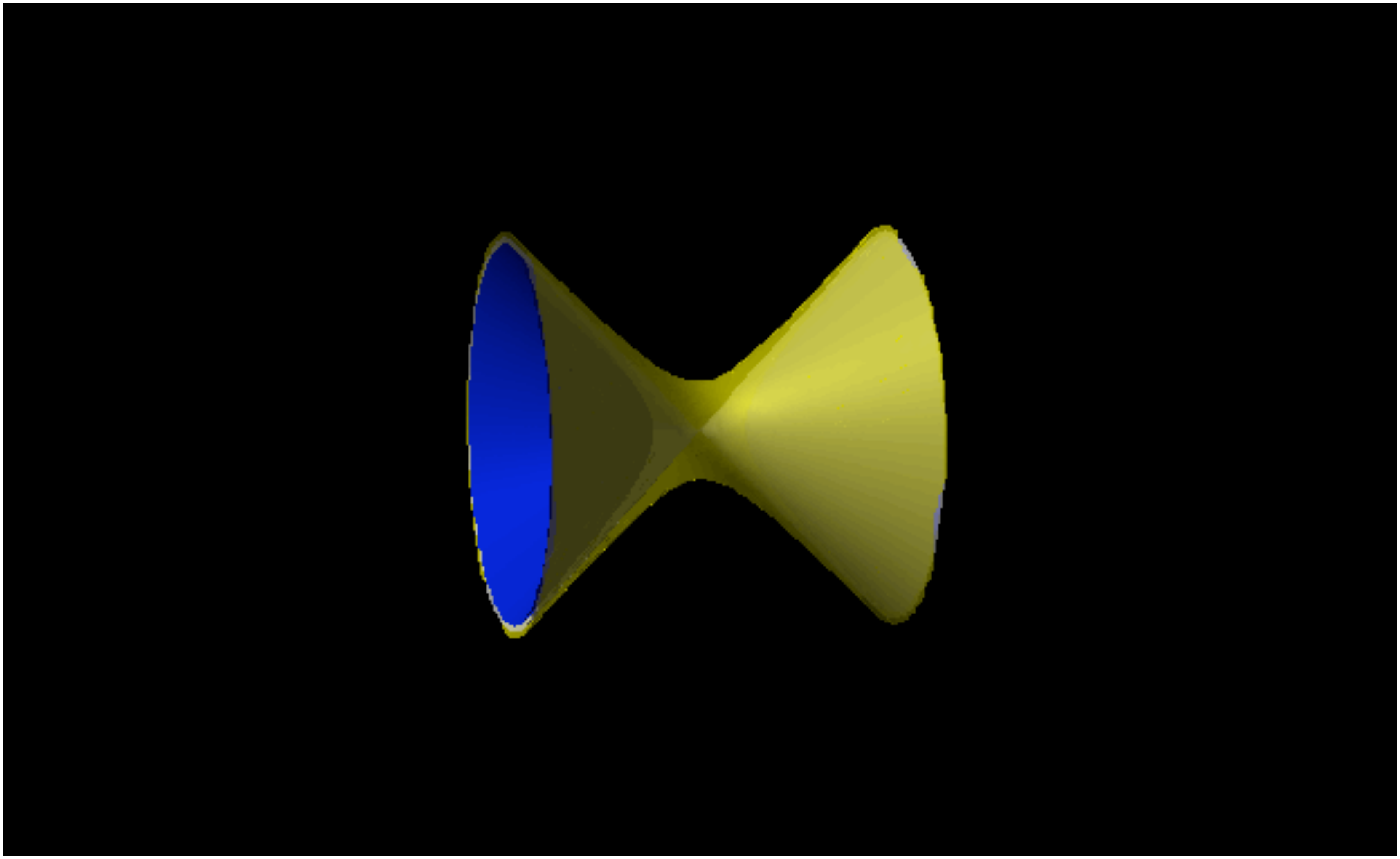

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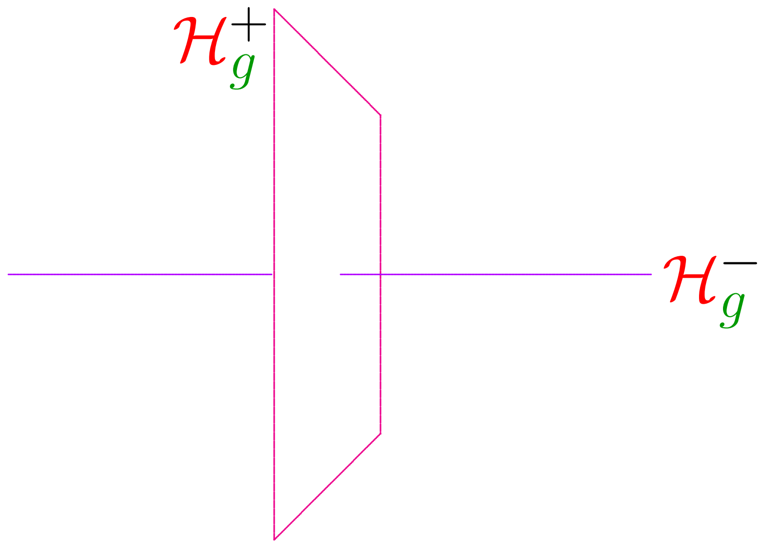

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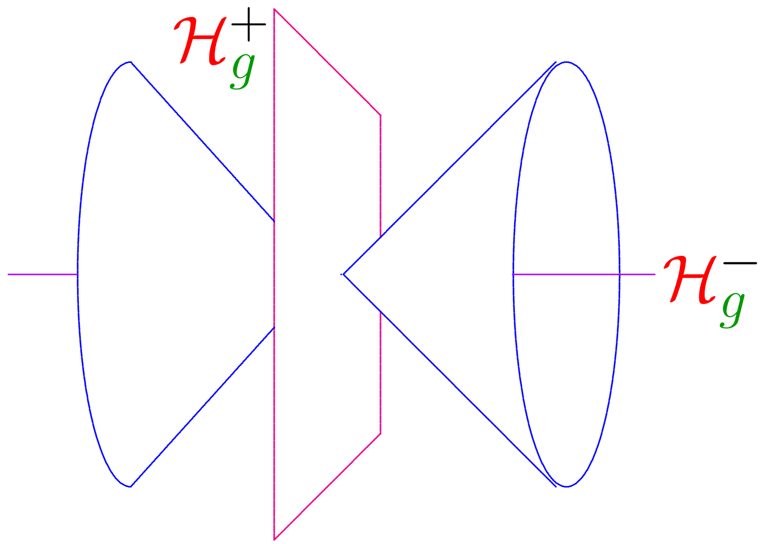

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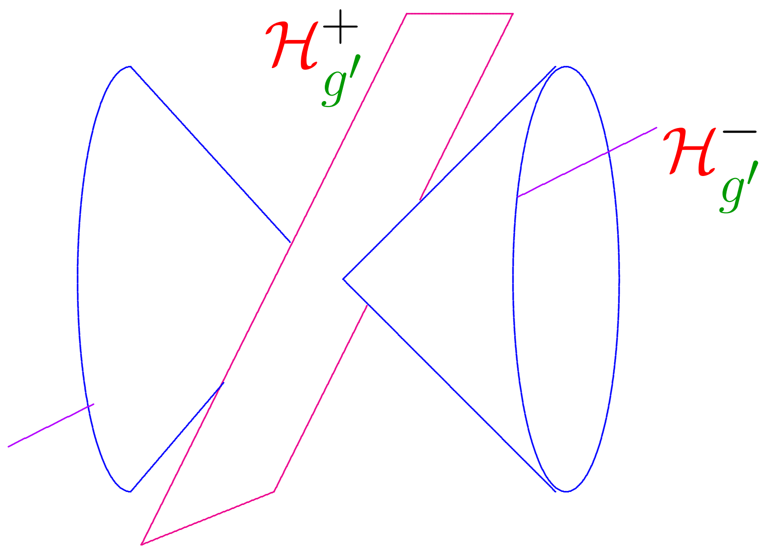

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Warning: "Exotic differentiable structures!"

No diffeomorphism classification currently known!

Typically, one homeotype $\longleftrightarrow$ ∞ many diffeotypes.

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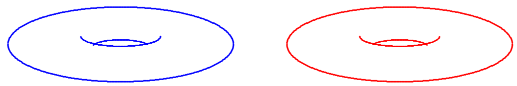

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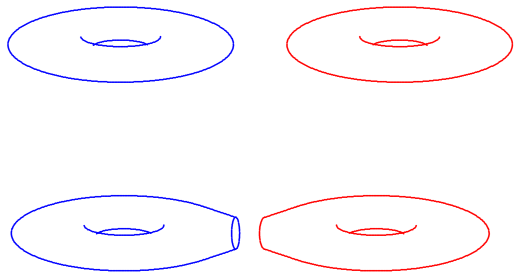

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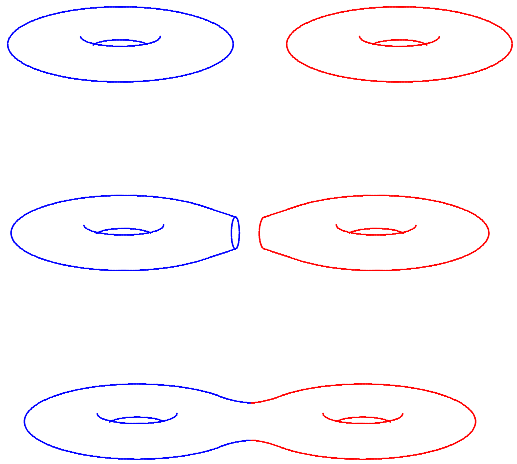

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so $\nexists$ Einstein metric if $k \geq 4 + 5j$.

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Both inequalities strict unless finitely covered by flat T^4, Calabi-Yau $K3$, or Calabi-Yau $\overline{K3}$.

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Theorem (Yau). *$K3$ admits Ricci-flat metrics.*

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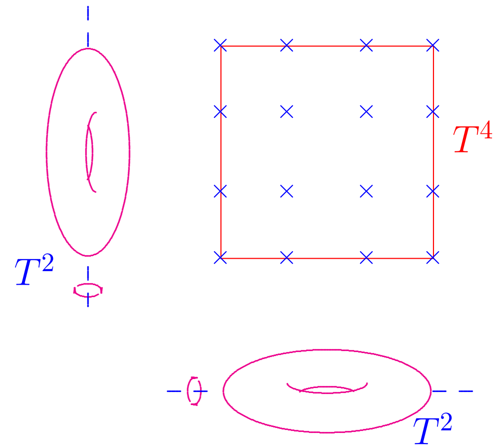
Begin with $T^4/\mathbb{Z}_2$:



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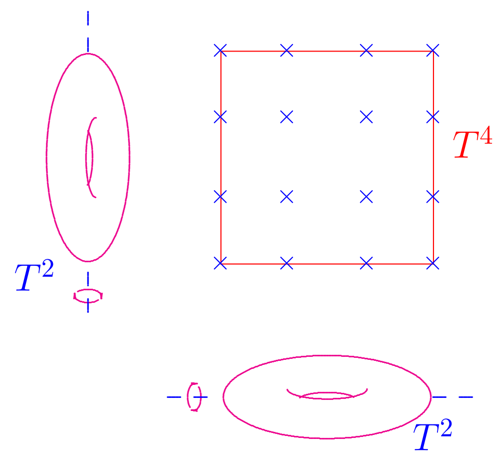

Replace $\mathbb{R}^4/\mathbb{Z}_2$ neighborhood of each singular point with copy of T^*S^2.

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Corollary. *Any smooth compact* simply connected *non-spin 4-manifold M is homeomorphic to a connect sum*

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where $j = b_+(M)$ *and* $k = b_-(M)$.

Conjecture (11/8 Conjecture). *Any smooth compact* simply connected *spin* 4*-manifold* M *is (unorientedly) homeomorphic to either* S^4 *or a connected sum* $jK3\#k(S^2 \times S^2)$.

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such that $c_1(M)$ *is negative multiple of* $j^* c_1(\mathbb{CP}_k)$.

Remark. This happens $\Leftrightarrow -c_1(M)$ is a Kähler class. Short-hand: $c_1(M) < 0$.

Remark. When $m = 2$, such M are necessarily minimal complex surfaces of general type.

Corollary. *For any* $\ell \geq 5$, *the degree* ℓ *surface*

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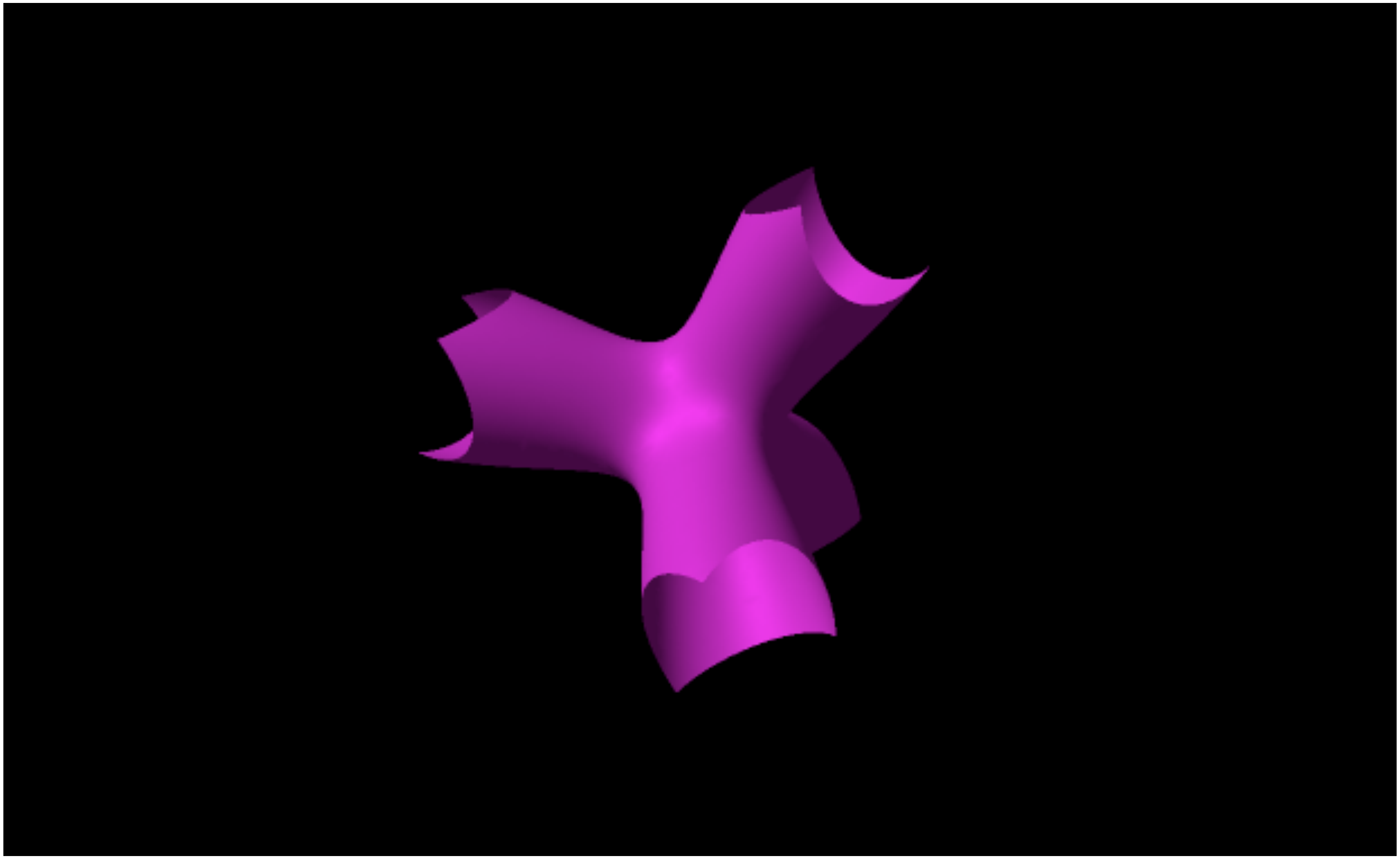

Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

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Any complex surface M can be obtained from a minimal surface X by blowing up a finite number of times:

$$M \approx X \# k \overline{\mathbb{CP}}_2$$

One says that X is minimal model of M.

Compact complex surface (M^4, J) general type if

$$\dim \Gamma(M, \mathcal{O}(K^{\otimes \ell})) \sim a\ell^2, \qquad \ell \gg 0,$$

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If $\ell \geq 5$, then $\Gamma(M, \mathcal{O}(K^{\otimes \ell}))$ gives holomorphic map

$$f_\ell : M \to \mathbb{CP}_N$$

which just collapses each $\mathbb{CP}_1$ with self-intersection -1 or -2 to a point.

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But $\bar\partial + \bar\partial^*$ does generalize:

spin^c Dirac operator, preferred connection on L.

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Every unitary connection A on L induces spin^c Dirac operator

$$D_A : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

generalizing $\bar{\partial} + \bar{\partial}^*$.

Seiberg-Witten equations:

$$\begin{aligned} D_A\Phi &= 0 \\ F_A^+ &= -\frac{1}{2}\Phi \odot \overline{\Phi} \end{aligned}$$

Unknowns:

both Φ and A.

Here F_A^+ = self-dual part of curvature of A.
Non-linear, but elliptic once 'gauge-fixing'

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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$\Longrightarrow \nexists g$ with $s > 0$.

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Moreover, equality holds in either case iff $M = X$, and g is Kähler-Einstein with $\lambda < 0$.

Theorem. *Up to rescaling and diffeomorphisms, there is only one Einstein metric on a complex-hyperbolic manifold* $\mathbb{C}\mathcal{H}_2/\Gamma$.

Similar theorem in real hyperbolic case:

Besson-Courtois-Gallot.

Theorem. *Let X be a minimal surface of general type, and let*

$$M = X \# k \overline{\mathbb{CP}_2}.$$

Then M cannot admit an Einstein metric if

$$k \geq {c_1}^2(M)/3.$$

(Better than Hitchin-Thorpe by a factor of 3.)

So being "very" non-minimal is an obstruction.

Theorem. *Let M be the 4-manifold underlying a non-minimal surface of general type. Then M does not admit a supreme Einstein metric.*

Theorem. *Let M be the 4-manifold underlying a complex surface of general type. Then any supreme Einstein metric. on M is Kähler, with* $\lambda < 0$.

Theorem. *Let* M *be the* 4*-manifold underlying a compact complex surface of general type, with minimal model* X*:*

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Then the Yamabe invariant of M *is independent of* k*, and is given by*

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Similar results for certain connected sums of complex surfaces.

Question. *Are there any non-minimal* M *of general type which actually admit Einstein metrics?*

If so, very different from Kähler-Einstein metrics!