# Maths 260 Lecture 25

- **Topics for today:**
  - Non-linear systems: linearisation near equilibria
  - Classification of equilibria in nonlinear systems
- **Reading for this lecture:** BDH Section 5.1
- **Suggested exercises:** BDH Section 5.1; 1, 3, 7, 9, 11
- **Reading for next lecture:** BDH Section 5.2
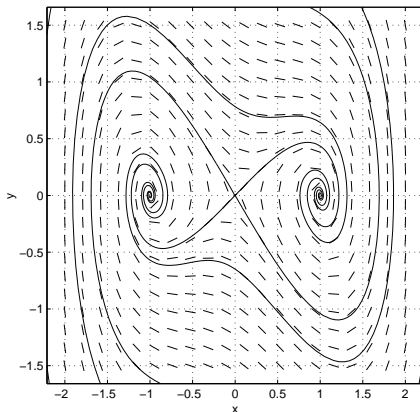- **Today's handouts:** None

## Example 1:

Consider the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - \frac{1}{2}y$$

Equilibrium solutions:

## Slope field and some solutions:



$$dx/dt = y$$
$$dy/dt = x - x^3 - 0.5\,y$$

The equilibrium at the origin looks like a saddle in a linear system, and the other equilibria look like spiral sinks.

# An approximation

We can understand the saddle-like nature of $(0, 0)$ if we approximate the nonlinear system by a linear system.

For $x$, $y$ very close to zero, $x^3$ is much smaller than $x$ or $y$. So we can ignore the $x^3$ term in the nonlinear system, and approximate the nonlinear system near $(0, 0)$ by the linear system

$$\frac{dx}{dt} = y$$
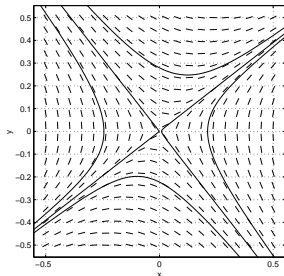$$\frac{dy}{dt} = x - \frac{1}{2}y$$

i.e.,

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$
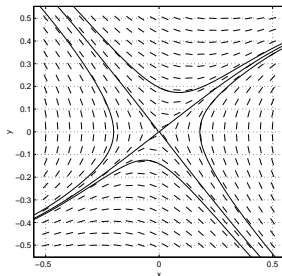
## Comparison

The eigenvalues of matrix **A** are 0.78 and $-1.28$, so the equilibrium at the origin of the approximate linear system is a saddle.

The following pictures show the slope field and solutions for the linear system (on left) and an enlargement of the nonlinear system near the origin (on right).

# Linearisation

Note that the linear system is a good approximation to the linear system near the equilibrium, but is hopeless away from the equilibrium (see the earlier phase portrait).

The procedure used above is called linearisation:

- ▶ Near an equilibrium, approximate the nonlinear system by an appropriate linear system.
- ▶ For initial conditions near the equilibrium, solutions of the nonlinear system stay close to solutions of the approximate linear system, at least for some interval of time.
- ▶ Thus, the type of equilibrium at the origin in the linearised system gives information about the type of the corresponding equilibrium in the nonlinear system.

# Example 1 again

Returning to the original system in Example 1, we now consider the equilibria at $(1, 0)$ and $(-1, 0)$.



dx/dt = y
dy/dt = x − x³ − 0.5 y
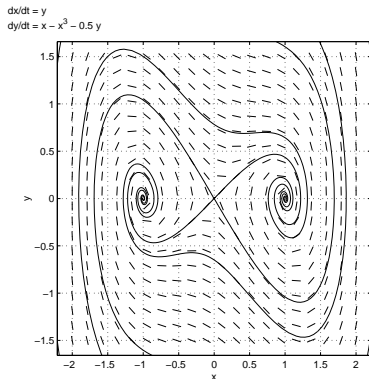
# First shift to the origin

To approximate the behaviour near $(1, 0)$ by a linear system, we must first shift the equilibrium to the origin - because linear systems usually only have an equilibrium at the origin.

We change the coordinates as follows:

▶ Write $u = x - 1$, $v = y$, so the equilibrium $(x, y) = (1, 0)$ is now at $(u, v) = (0, 0)$.

▶ Then the system becomes:

$$\frac{du}{dt} =$$

$$\frac{dv}{dt} =$$

## Ignore the small terms

For $u$ and $v$ small, $-3u^2$ and $u^3$ are very, very small. We ignore these nonlinear terms and approximate the system by:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$
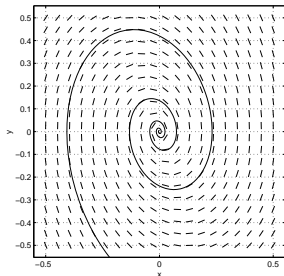
Eigenvalues are $-\dfrac{1}{4} \pm \dfrac{1}{4}\sqrt{31}i$.

So the equilibrium is a spiral sink in the linear approximation.
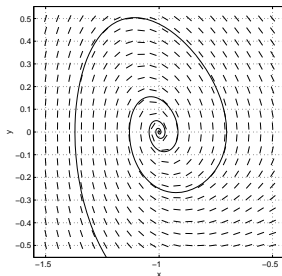
## Comparison

The following pictures illustrate the similarity between the phase portrait for the linearised system (on the left) and the phase portrait near the equilibrium at $(1, 0)$ in the nonlinear system (on the right).



Similar calculations give similar results for the equilibrium at $(-1, 0)$.

## More generally...

If the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

has an equilibrium at $(x_0, y_0)$, we can construct a linear approximation to the system, valid for $x$ and $y$ values near $(x_0, y_0)$, as follows.

► First move the equilibrium to the origin: write $u = x - x_0$ and $v = y - y_0$. The nonlinear equations in the new coordinates are:

$$\frac{du}{dt} = \frac{dx}{dt} = f(x, y) = f(x_0 + u, y_0 + v)$$
$$\frac{dv}{dt} = \frac{dy}{dt} = g(x, y) = g(x_0 + u, y_0 + v)$$

# Taylor expansion...

- ▶ Now use a Taylor expansion to rewrite $f$ and $g$. (This works because $u$ and $v$ can be assumed to be small):

$$f(x_0+u, y_0+v) = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right] u + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right] v + \text{h.o.t}$$

$$g(x_0+u, y_0+v) = g(x_0,y_0) + \left[\frac{\partial g}{\partial x}(x_0,y_0)\right] u + \left[\frac{\partial g}{\partial y}(x_0,y_0)\right] v + \text{h.o.t}$$

- ▶ Recall that $f(x_0,y_0) = g(x_0,y_0) = 0$, so if we ignore the higher order terms then we get an approximate linear system:

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

# The Jacobian matrix

Thus, the behaviour of solutions to the nonlinear system near the equilibrium $(x_0, y_0)$ can be approximated by the behaviour of solutions in the linearised system given above.

The matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\[2ex] \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

is called the Jacobian matrix, evaluated at $(x_0, y_0)$.

# Example 1 again

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - \frac{1}{2}y$$

$J(x, y) =$

# Classifying equilibria in nonlinear systems

- An equilibrium is a (nonlinear) <span style="color:red">sink</span> if all solutions that start close to the equilibrium stay close to the equilibrium for all time and tend to the equilibrium as $t$ increases.

- An equilibrium is a (nonlinear) <span style="color:red">source</span> if all solutions that start close to the equilibrium move away from the equilibrium as $t$ increases.

- An equilibrium is a (nonlinear) <span style="color:red">saddle</span> if there are curves of solutions that tend towards the equilibrium as $t$ increases and curves of solutions that tend towards the equilibrium solution as $t$ decreases. All other solutions started near the equilibrium move away from the equilibrium as $t$ increases and decreases.

These are different definitions than those used for equilibria in linear systems, but are consistent with those.
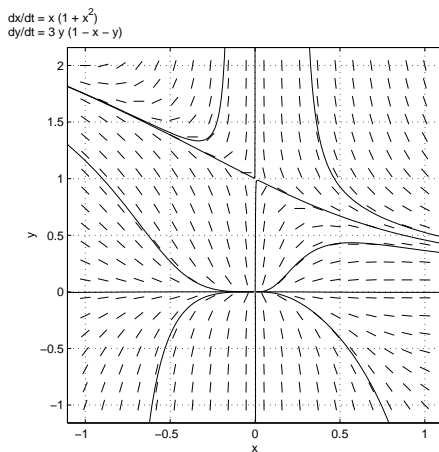
## Example 2

Consider the system

$$\frac{dx}{dt} = x(1 + x^2)$$
$$\frac{dy}{dt} = 3y(1 - y - x)$$

Find the equilibria and determine their types. For each equilibrium, sketch a phase portrait showing the behaviour of solutions in the associated linearised system.

## Example 2

The phase portrait for this system, drawn with *pplane*, is given below.

Note the source at $(0, 0)$ and the saddle at $(0, 1)$ as predicted by our calculations.



dx/dt = x (1 + x²)
dy/dt = 3 y (1 − x − y)

## Example 3

Consider the system

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = x - y^2$$

Find the equilibria and determine their types. For each equilibrium, sketch a phase portrait showing the behaviour of solutions in the associated linearised system.

# Important ideas from today

If a system of nonlinear equations

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

has an equilibrium at $(x_0, y_0)$, then the behaviour of solutions near that equilibrium can be approximated by the behaviour of solutions near the origin for the linearised system

$$\begin{pmatrix} \frac{du}{dt} \\ \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \begin{pmatrix} u \\ \\ v \end{pmatrix}$$

We continue with this idea in the next lecture.