# TORSION AND INTEGRABILITY OF SOME CLASSES OF ALMOST KÄHLER MANIFOLDS 

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#### Abstract

We study almost Kähler manifolds whose Riemann curvature tensor is subject to the second or the third curvature condition of Gray ( $\mathcal{A} \mathcal{K}_{2}$ respectively $\mathcal{A K} \mathcal{K}_{3}$ for short). This conditions are interpreted in terms of the torsion of the first canonical Hermitian connection and shown to forces the torsion of the latter to be parallel in directions orthogonal to the Kähler nullity of the almost complex structure. The first main result is that $\mathcal{A} \mathcal{K}_{2}$-manifolds must have parallel intrinsic torsion tensor. In the case of parallel torsion, the Einstein condition and the reducibility of the canonical Hermitian connection is studied. Secondly we show that strictly normal $\mathcal{A K}_{3}$-manifolds are obtained up to local Riemannian products from $\mathcal{A} \mathcal{K}_{2}$-structures and a class of toric Kähler manifolds.


## Contents

1. Introduction ..... 1
2. Almost Kähler geometry ..... 5
2.1. Preliminaries ..... 5
2.2. Intrinsic torsion and the Kähler nullity ..... 7
2.3. Riemannian curvature and integrability ..... 11
3. Gray's curvature conditions ..... 13
3.1. The partial parallelism of the torsion ..... 15
3.2. Examples ..... 16
4. A first decomposition result ..... 19
5. Curvature properties ..... 22
5.1. On parallel torsion ..... 26
6. Local structure of $\mathcal{A} \mathcal{K}_{3}$-manifolds ..... 27
7. Normal structures ..... 32
7.1. General observations ..... 32
7.2. Transverse geometry ..... 34
7.3. The canonical foliation ..... 39
8. Classification results ..... 43
8.1. Integrability ..... 44
References ..... 47

## 1. Introduction

Let $\left(M^{2 m}, \omega\right)$ be a symplectic manifold. An almost complex structure $J$ on $M$ calibrates $\omega$ if and only if $\omega$ is of type $(1,1)$ w.r.t $J$ that is $\omega(J \cdot, J \cdot)=\omega$ and

[^0]$\omega(\cdot, J \cdot)>0$. Every such almost complex structure yields a Riemannian metric on $M$ via $g=\omega(\cdot, J \cdot)$. We shall call such metrics compatible with the symplectic form $\omega$ and the corresponding space will be denoted by $\mathcal{M}(\omega)$. When $M$ is compact several functionals can be used to determine if $\mathcal{M}(\omega)$ contains any distinguished elements. For instance the total scalar curvature $\mathbb{S}: \mathcal{M}(\omega) \rightarrow \mathbb{R}$ functional is given by
$$
\mathbb{S}(g)=\int_{M} s c a l_{g}
$$
for all $g$ in $\mathcal{M}(\omega)$. Its critical points, at fixed volume, are $\omega$-compatible Riemannian metrics $g$ on $M$ satisfying $\operatorname{Ric}_{g}(J \cdot, J \cdot)=\operatorname{Ric}_{g}$. The total energy functional $\mathbb{E}$ : $\mathcal{M}(\omega) \rightarrow \mathbb{R}$ is given by
$$
\mathbb{E}(g)=\int_{M}\left|\nabla^{g} \omega\right|_{g}^{2}
$$
where for all $g$ in $\mathcal{M}(\omega)$ we denote by $\nabla^{g}$ the Levi-Civita connection of the metric $g$. A compatible metric $g$ is critical(again when the volume is fixed) if and only if
$$
\left(\nabla^{g}\right)^{\star} \nabla^{g} \omega \in \lambda^{1,1} M=\left\{\alpha \in \Lambda^{2} M: \alpha(J \cdot, J \cdot)=\alpha\right\}
$$

Such metrics have been studied in the compact setting in [25] and large classes of examples were constructed in dimension 4 in [5, 27, 36].

In this paper we shall study metrics $g$ in $\mathcal{M}(\omega)$ which satisfy additional curvature properties for their Riemannian curvature tensor. They also appear to be critical metrics for both of the functionals introduced above. We consider the real vector bundle

$$
\lambda^{2} M=\left\{\alpha \in \Lambda^{2} M: \alpha(J \cdot, J \cdot)=-\alpha\right\}
$$

with its natural complex structure given by $\mathbb{J} \alpha=\alpha(J \cdot, \cdot)$ for all $\alpha$ in $\lambda^{2} M$. With respect to the splitting $\Lambda^{2} M=\lambda^{1,1} M \oplus \lambda^{2} M$ the Riemann curvature tensor of $g$ can be written in block form

$$
R=\left(\begin{array}{ll}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{array}\right)
$$

Following [21, 33] we shall say that the metric $g$ belongs to the class $\mathcal{A} \mathcal{K}_{2}$ if and only if

$$
\begin{aligned}
& R_{12}=R_{21}=0 \\
& R_{22} \circ \mathbb{J}+\mathbb{J} \circ R_{22}=0
\end{aligned}
$$

The class $\mathcal{A} \mathcal{K}_{3}$ is defined to contain those metrics $g$ which only satisfy

$$
R_{12}=R_{21}=0
$$

Typical examples of non-Kähler $\mathcal{A K}_{2}$-structures are supported by twistor spaces of quaternion-Kähler manifolds of negative scalar curvature [2]. The classes $\mathcal{A} \mathcal{K}_{2}$ and $\mathcal{A} \mathcal{K}_{3}$ have been vigourously studied in dimension 4 by Apostolov et al. $[4,5,7,10,8]$, where full structure results have been obtained.

It is the aim of this work to study the case of higher dimensions. If $\eta$ denotes the intrinsic torsion tensor of some almost-Kähler manifold $(M, g, J)$ and

$$
H=\left\{X \in T M: \eta_{X}=0\right\}
$$

is the Kähler nullity of the structure, the latter is called normal if $\eta_{\mathcal{V}} \mathcal{V} \subseteq H$ where $H=\mathcal{V}^{\perp}$. For local considerations, and in particular when dealing with closed conditions, one can always assume that these distributions are of constant rank, since this happens anyway on connected components of some dense open subset of $M$. Moreover, the almost-Kähler structure is called strict if $\eta$ is non-degenerate as a map from $T M$ into $\Lambda^{2} M$. Our first main result is as follows.

Theorem 1.1. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{3}$. Then over connected component of some dense open subset, the manifold $M$ is locally the Riemannian product of a strict almost Kähler manifold with parallel torsion and a normal $\mathcal{A K}_{3}$-manifold.

Here the torsion and its parallelism are understood w.r.t. canonical Hermitian connection of the almost-Kähler structure. The prove this we first observe that having an almost-Kähler structure in the class $\mathcal{A} \mathcal{K}_{3}$ amounts to require the intrinsic torsion $\eta$ be holomorphic in a suitably extended way; differential consequences include the parallelism (w.r.t. the canonical connection) of $\eta$ along $\mathcal{V}$ (see sections $2 \& 3)$. In particular the integral manifolds of $\mathcal{V}$ are, in the induced structure, almostKähler manifolds with parallel intrinsic torsion. We use a screening procedure to prove first theorem 1.1 along the leaves of $\mathcal{V}$ (see section 4); next the differential constraints on the intrinsic torsion indicate how to perform extension to the whole of $T M$ and also identify the obstruction to the splitting of the metric, which is a sub-space of $\mathcal{V}$. To obtain its vanishing we observe that it is enough to prove integrability for almost-Kähler structures with parallel torsion and subject to an Einstein(-type) condition; this is achieved by the local version in [6] of Sekigawa's formula [35]. This is done in sections $5 \& 6$ of the paper.

Therefore is enough to consider normal $\mathcal{A} \mathcal{K}_{3}$-structures specific examples of which can be constructed(see section 3.2) by re-calibrating product symplectic forms as follows.

Theorem 1.2. Consider the product $\mathbb{R}^{2 p} \times Z$ equipped the symplectic form

$$
\omega=\sum_{i=1}^{p} d x_{i} \wedge d y_{i}+\omega_{h}
$$

where $(Z, h, I)$ is a Kähler manifold and $\left\{x_{i}, y_{i}, 1 \leq i \leq p\right\}$ are co-ordinates on $\mathbb{R}^{2 p}$. The metric

$$
\begin{equation*}
g=\sum_{i j} G_{i j} d x_{i} d y_{j}+h \tag{1.1}
\end{equation*}
$$

defines an $\mathcal{A K}_{3}$ structure compatible with $\omega$ where the matrix $G=(1+w)^{-1}(1-$ $w)$ for some holomorphic map $w$ from $Z$ into symmetric, anti-hermitian $2 p \times 2 p$ matrices, such that $|w|_{\infty}<1$.

The structure above is normal, non-Kähler, unless $w$ is constant, nor in the class $\mathcal{A K}_{2}$, unless ( $h, I$ ) defines a Hermitian symmetric space. The symplectic form $-\sum_{i=1}^{p} d x_{i} \wedge d y_{i}+\omega_{h}$ admits a $g$-orthogonal Kähler structure with toric symmetry; this essentially means that the Kähler nullity of $g$ is parallel w.r.t the Hermitian
connection of the almost-Kähler structure. When $p=1$ and $Z$ is a Riemann surface these examples first appeared, up to a Möbius transformation, in [7, 8]. If Einstein, the metrics above must be Ricci flat; examples have been constructed in dimension $4[30,5,7,10]$. We construct new examples in higher dimensions by suitable deformation of metric cones over Sasaki-Einstein manifolds.

In the rest of the paper we look at the structure of normal $\mathcal{A} \mathcal{K}_{3}$-manifolds. We show in section 7 that every such structure admits a canonical foliation with totally geodesic, holomorphic (in extended sense) leaves. Moreover the foliation is also totally geodesic w.r.t canonical Hermitian connection and in the induced metric the leaves are $\mathcal{A} \mathcal{K}_{3}$ manifolds of null type. For a normal $\mathcal{A} \mathcal{K}_{3}$-structure this is defined by requiring the integral manifolds of $\mathcal{V}$ be flat in the induced metric. For instance the examples in theorem 1.2 are of null type, although this class is presumably larger. We also develop tools to deal with the Einstein equation and an integrability criterion based on a Walczak-type formula (see [38, 37]).

With these in hands we prove in section 8 that
Theorem 1.3. Let $\left(M^{2 m}, g, J\right)$ be a strictly normal $\mathcal{A K}_{3}$-structure. Then it is given by (1.1) where $w$ is (suitably non-degenerate) and immersive.

In fact we show that this holds up to products with four dimensional $\mathcal{A K}_{3}$ manifolds and the conclusion follows by the result in [8].

The elementary observation that normal $\mathcal{A} \mathcal{K}_{2}$-structure are, up to local products with Kähler manifolds, strictly normal combines with thms. 1.3 and 1.1 in
Theorem 1.4. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{2}$. The following hold:
(i) the intrinsic torsion is parallel w.r.t. the canonical connection;
(ii) $(g, J)$ is locally the Riemannian product of a Kähler manifold, a strict almost Kähler manifold with parallel torsion and a 3-symmetric space.

When $m=2$ any almost Kähler structure is normal and the result above has been proved in [7]; note that in dimension 4 there is only one(up to isometry) 3-symmetric space, see [23]. Also note that the 3 -symmetric factor in (ii) above corresponds to the strictly normal piece in $(g, J)$.

From thms. 1.1 and 1.3 and properties of the canonical foliation we also get
Theorem 1.5. Let $(M, g, J)$ belong to the class $\mathcal{A K}_{3}$ and such that $g$ is an Einstein metric. Then:
(i) either $J$ is integrable or $g$ is Ricci flat and of null type, up to local products with Ricci flat Kähler manifolds;
(ii) if $M$ is compact then $J$ is integrable.

Note that (ii) follows from (i) and the fact that almost-Kähler, Einstein, metrics of positive scalar curvature are Kähler [35]; moreover for $\mathcal{A} \mathcal{K}_{2}$ structures we show that the presence of an Einstein metric implies, even locally, integrability. In fact, in dimension 4 the only known examples of non-integrable almost-Kähler, Einstein metrics are Ricci flat $[5,8,30]$. Non-Ricci flat homogeneous examples in dimension $4 m, m \geq 6$ have been first constructed in [1].

There are some direct applications relating to the integrability of orthogonal almost Kähler structures in complex hyperbolic geometry.
Theorem 1.6. Let $\left(M^{2 m}, g, J\right)$ be a Kähler manifold of constant negative holomorphic sectional curvature. If $I$ is an almost complex structure such that $(g, I)$ is almost Kähler and $[I, J]=0$, then $(g, I)$ must be Kähler.

In dimension 4 this has been proved by J. Armstrong [10], as a consequence of his classification of almost Kähler, Einstein, 4-manifolds of class $\mathcal{A K}_{3}$. Note that, by results in [7], there are Hermitian symmetric spaces of non-compact type, admitting a reversing strictly almost Kähler structure. By contrast, positively oriented, orthogonal almost-Kähler structures on Kähler-Einstein surfaces are necessarily integrable [17].

Further progress on the classification problem of $\mathcal{A K}_{3}$-structures relies on the study of the canonical foliation and a first step should therefore comprise the classification of structures of null-type.

## 2. Almost Kähler geometry

2.1. Preliminaries. Let us consider an almost Hermitian manifold ( $\left.M^{2 m}, g, J\right), m \geq$ 2 , that is a Riemannian manifold $\left(M^{2 m}, g\right)$ endowed with a compatible almost complex structure $J: T M \rightarrow T M$ such that $g(J \cdot, J \cdot)=g(\cdot, \cdot)$. The Kähler form $\omega=g(J \cdot, \cdot)$ is nondegenerate and provides a natural orientation on $M$. In what follows we shall recall the definitions and basic properties of some of the $U(\mathrm{~m})$ modules of relevance for the rest of the paper. First we extend the almost complex structure $J: T M \rightarrow T M$ to $\mathcal{J}: \Lambda^{p} M \rightarrow \Lambda^{p} M$ acting on a $p$-form $\alpha$ by

$$
(J \alpha)\left(X_{1}, \ldots, X_{p}\right)=\alpha\left(J X_{1}, \ldots, J X_{p}\right)
$$

whenever $X_{1}, \ldots X_{p}$ belong to $T M$. Clearly $J^{2}=(-1)^{p}$ on $\Lambda^{p} M$ and if $X \mapsto X^{b}$ denotes the isomorphism of $T M$ and $\Lambda^{1} M$ given by the metric $g$ then $J X^{b}=-(J X)^{b}$ for all $X$ in $T M$. Let us consider the operator $\mathcal{J}: \Lambda^{p} M \rightarrow \Lambda^{p} M$ given by

$$
(\mathcal{J} \alpha)\left(X_{1}, \ldots, X_{p}\right)=\sum_{k=1}^{p} \alpha\left(X_{1}, \ldots, J X_{k}, \ldots, X_{p}\right)
$$

for all $X_{1}, \ldots X_{p}$ in $T M . \mathcal{J}$ acts as a derivation on $\Lambda^{\star} M$ and gives the complex bi-grading of the exterior algebra in the following sense. Let $\lambda^{p, q} M$ be given as the $-(p-q)^{2}$-eigenspace of $\mathcal{J}^{2}$. Then

$$
\Lambda^{s} M=\sum_{p+q=s} \lambda^{p, q} M
$$

is an orthogonal, direct sum. Note that $\lambda^{p, q} M=\lambda^{q, p} M$. Of special importance in our discussion are the spaces $\lambda^{p} M=\lambda^{p, 0} M$; forms $\alpha$ in $\lambda^{p} M$ are such that $\left(X_{1}, \ldots, X_{p}\right) \rightarrow \alpha\left(J X_{1}, X_{2}, \ldots, X_{p}\right)$ is still an alternating form which equals $p^{-1} \mathcal{J} \alpha$. Let $\lambda^{p} M \otimes_{1} \lambda^{q} M$ be the space of tensors $Q: \lambda^{p} M \rightarrow \lambda^{q} M$ which satisfy

$$
\left[(\mathbb{J} Q)\left(X_{1}, \ldots, X_{p}\right)\right]\left(Y_{1}, \ldots, Y_{q}\right)=-\left[\mathbb{J}\left(Q\left(X_{1}, \ldots, X_{p}\right)\right)\right]\left(Y_{1}, \ldots, Y_{q}\right)
$$

(here $\mathbb{J}$ as a map of $\lambda^{p} M$ stands in fact for $p^{-1} \mathcal{J}$ ). We also define $\lambda^{p} M \otimes_{2} \lambda^{q} M$ to be the space of tensors $Q: \lambda^{p} M \rightarrow \lambda^{q} M$ such that $Q \mathbb{J}=\mathbb{J} Q$.

We now briefly discuss the various spaces of algebraic curvature tensors of relevance for us. Recall that the Bianchi contraction map $b_{1}: \Lambda^{2} M \otimes \Lambda^{2} M \rightarrow$ $\Lambda^{1} M \otimes \Lambda^{3} M$ is given by

$$
\left(b_{1} Q\right)_{X}=\sum_{i=1}^{2 m} e_{i} \wedge Q\left(e_{i}, X\right)
$$

for all $X$ in $T M$ and whenever $Q$ belongs to $\Lambda^{2} M \otimes \Lambda^{2} M$. Here $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$ is some local orthonormal frame on $M$. The space of algebraic curvature tensors on $M$ is then defined by

$$
\mathcal{K}(\mathfrak{s o}(2 m))=\operatorname{ker} b_{1} \cap\left(\Lambda^{2} M \otimes \Lambda^{2} M\right)=S^{2}\left(\Lambda^{2} M\right) \cap \operatorname{ker}(a) .
$$

Similarly, the space of algebraic Kähler curvature tensors is defined by

$$
\mathcal{K}(\mathfrak{u}(m))=S^{2}\left(\lambda^{1,1} M\right) \cap \operatorname{ker}(a) .
$$

We shall also make use of the space

$$
\mathcal{K}\left(\mathfrak{u}^{\perp}(m)\right)=\mathcal{K}(\mathfrak{s o}(2 m)) \cap S^{2}\left(\lambda^{2} M\right)
$$

In order to outline some its properties we need to recall some facts on the various embeddings of the space $\lambda^{2,2} M$. In fact, given $\Omega$ in $\lambda^{2,2} M$ we consider $\Omega^{-}$in $S^{2}\left(\lambda^{2} M\right)$ defined by

$$
\left.\left.\Omega^{-}(X, Y)=(X\lrcorner Y\right\lrcorner \Omega\right)_{\lambda^{2} M}
$$

for all $X, Y$ in $T M$. Elementary considerations which are left to the reader prove that

Lemma 2.1. The following hold whenever $\Omega$ is in $\lambda^{2,2} M$ :
(i) $\Omega^{-} \in \lambda^{2} M \otimes_{1} \lambda^{2} M$;
(ii) $\left.\left(b_{1} \Omega^{-}\right)_{X}=X\right\lrcorner \Omega$ for all $X$ in $T M$.

In particular the map $\Omega \in \lambda^{2,2} M \mapsto \Omega^{-}$is injective. Let now $a: \Lambda^{p} M \otimes \Lambda^{q} M \rightarrow$ $\Lambda^{p+q} M$ be the alternation map,

$$
a(Q)=\sum_{I=\left(1 \leq i_{1}, \ldots, i_{p} \leq 2 m\right)} e^{I} \wedge Q\left(e_{I}\right)
$$

whenever $Q$ belongs to $\Lambda^{p} M \otimes \lambda^{q} M$ and where $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$ is some local orthonormal basis in TM. A straightforward verification, involving if necessary checking on sample elements yields:
Lemma 2.2. The following hold:
(i) $\mathcal{K}\left(\mathfrak{u}^{\perp}(m)\right) \subseteq\left(\lambda^{2} M \otimes_{2} \lambda^{2} M\right) \cap \operatorname{ker}(a)$;
(ii) the alternation map $a: S^{2}\left(\lambda^{2} M\right) \cap\left(\lambda^{2} M \otimes_{1} \lambda^{2} M\right) \rightarrow \lambda^{2,2} M$ is an isomorphism.

Note that the inclusion in (i) above is strict provided that $m \geq 3$. To end this section if $\Omega$ is in $\lambda^{2,2} M$ we consider $\Omega^{+}$in $S^{2}\left(\lambda^{1,1} M\right)$ given by

$$
\left.\left.\Omega^{+}(X, Y)=(X\lrcorner Y\right\lrcorner \Omega\right)_{\lambda^{1,1} M}
$$

for all $X, Y$ in $T M$. We have

$$
\left.\left(b_{1} \Omega^{+}\right)_{X}=2 X\right\lrcorner \Omega
$$

for all $X$ in $T M$. This renders explicit the decomposition

$$
S^{2}\left(\lambda^{1,1} M\right)=\mathcal{K}(\mathfrak{u}(m)) \oplus \lambda^{2,2} M
$$

2.2. Intrinsic torsion and the Kähler nullity. Let $\left(M^{2 m}, g, J\right)$ be almost Hermitian. We denote by $\nabla$ the Levi-Civita connection of the Riemannian metric $g$. Consider now the tensor $\nabla J$, the first derivative of the almost complex structure, and recall that for all $X$ in $T M$ we have that $\nabla_{X} J$ is a skew-symmetric (with respect to $g$ ) endomorphism of $T M$, which anticommutes with $J$. The tensor $\nabla J$ can be used to distinguish various classes of almost Hermitian manifolds (see [22]). For instance ( $M^{2 m}, g, J$ ) is quasi-Kähler iff

$$
\nabla_{J X} J=-J\left(\nabla_{X} J\right)
$$

for all $X$ in $T M$. If $\omega=g(J \cdot, \cdot)$ denotes the Kähler form of the almost Hermitian structure $(g, J)$, we have an almost Kähler structure iff $d \omega=0$. We also recall the well known fact that almost Kähler manifolds are always quasi-Kähler. The almost complex structure $J$ defines a Hermitian structure if it is integrable, that is the Nijenhuis tensor $N_{J}$ defined by

$$
N_{J}(X, Y)=[X, Y]-[J X, J Y]+J[X, J Y]+J[J X, Y]
$$

for all vector fields $X$ and $Y$ on $M$ vanishes. This is also equivalent to

$$
\nabla_{J X} J=J \nabla_{X} J
$$

whenever $X$ is in $T M$. Therefore, an almost Kähler manifold which is also Hermitian must be Kähler.

In this paper we will deal mainly with almost Kähler ( $\mathcal{A K}$ for short)-manifolds. Let therefore $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}$ and let

$$
\bar{\nabla}_{X} Y=\nabla_{X} Y+\eta_{X} Y
$$

whenever $X, Y$ are vector fields on $M$ be the first canonical Hermitian connection of $(g, J)$. Here we have denoted by $\eta=\frac{1}{2}(\nabla J) J$ the intrinsic torsion tensor of the associated $U(m)$-structure. As recalled above $\eta$ belongs to $\lambda^{1} M \otimes_{2} \lambda^{2} M$, in particular $a(\eta)=0$. The connection $\bar{\nabla}$ is metric and Hermitian, that is $\bar{\nabla} g=0$ and $\bar{\nabla} J=0$. The torsion tensor $T$ of $\bar{\nabla}$ defined by

$$
\begin{aligned}
T_{X} Y & =\bar{\nabla}_{X} Y-\bar{\nabla}_{Y} X-[X, Y] \\
& =\eta_{X} Y-\eta_{Y} X
\end{aligned}
$$

for all vector fields $X, Y$ on $M$ lives in $\lambda^{2} M \otimes \lambda^{1} M$; in addition, it satisfies

$$
\begin{equation*}
\left\langle T_{X} Y, Z\right\rangle=-\left\langle\eta_{Z} X, Y\right\rangle \tag{2.1}
\end{equation*}
$$

for all $X, Y, Z$ in $T M$.
Notation: If $E$ and $F$ and vector sub-bundles of $T M$ and $Q$ is a tensor of type $(2,1)$, we will denote by $Q(E, F)$ (or $Q_{E} F$ ) the sub-bundle of $T M$ generated by elements of the form $Q(u, v)$ where $u$ belongs to $E$ and $v$ is in $F$.

An important object associated with an almost Kähler manifold $\left(M^{2 m}, g, J\right)$ is its Kähler nullity. This is the vector bundle $H$ over $M$ defined at a point $m$ of $M$ by
$H_{m}=\left\{v \in T_{m} M: \nabla_{v} J=0\right\}$. We also define $\mathcal{V}$ to be the orthogonal complement of $H$ in $T M$. Hence, we have an orthogonal, $J$-invariant decomposition

$$
\begin{equation*}
T M=\mathcal{V} \oplus H \tag{2.2}
\end{equation*}
$$

Using the almost Kähler condition under the form (2.1) an orthogonality argument shows that at each point of $M$

$$
\begin{equation*}
\mathcal{V}=T(T M, T M) \tag{2.3}
\end{equation*}
$$

In other words, the torsion of $(g, J)$ is concentrated in $\mathcal{V}$. For further use we observe that (2.3) yields $T(\mathcal{V}, H) \subseteq \mathcal{V}$ hence

$$
\begin{equation*}
\eta_{\mathcal{V}} H \subseteq \mathcal{V} \tag{2.4}
\end{equation*}
$$

Without further assumptions $H$ does not have necessarily constant rank over $M$. However, this is true locally, in the following sense. Call a point $m$ of $M$ regular if the rank of $\eta$ attains a local maximum at $m$. Using standard continuity arguments, it follows that around each regular point, the rank of $\eta$, and hence that of $H$ is constant in some open subset. It is also easy to see that the set of regular points is dense in $M$, provided that the manifod is connected. As we are concerned with the local (in some neighbourhod of a regular point ) structure of certain $\mathcal{A K}$-manifolds we can assume, without loss of generality, that $H$ has constant rank over $M$. This assumption will be made, as applicable, in the whole rest of this paper.

The following two classes of almost-Kähler manifolds will receive special attention in what follows.
Definition 2.1. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}$ such that the Kähler nullity $H$ is of constant rank over M. It is called
(i) normal if $\eta_{\mathcal{V}} \mathcal{V} \subseteq H$;
(ii) strictly normal if $\eta_{\mathcal{V}} \mathcal{V}=H$.

Opposite to the class of normal structures is the following sub-class.
Definition 2.2. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}$. It called strict if and only if $T(T M, T M)=T M$; equivalently the Kähler nullity vanishes at each point of $M$.

We will be particularly interested in normal $\mathcal{A K}$ manifolds such that $\mathcal{V}$ is an integrable distribution, when any of its integrable manifolds is Kähler with respect to the induced structure. Furthermore, in the strictly normal case the definition prevents taking products with Kähler manifolds. Every 4-dimensional $\mathcal{A K}$ manifold is normal in the sense of definition 2.1 on the open set where its Nijenhuis tensor does not vanish. This is a consequence of the fact that the vector bundle $\lambda^{2} M$ has, in this case, real rank 2.

The straightforward fact below will be used in the next sections.
Lemma 2.3. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}$-manifold. Then:
(i) $T(\mathcal{V}, \mathcal{V})=0$;
(ii) $\eta_{\mathcal{V}} H=\mathcal{V}$.

Proof. (i) is clear from the definition. To prove (ii) we consider the orthogonal complement $F$ of $\eta_{\mathcal{V}} H$ in $\mathcal{V}$. Then $\eta_{\mathcal{V}} F=0$ and the vanishing of the torsion on $\mathcal{V}$ implies that $\eta_{F} \mathcal{V}=0$. In other words $\eta_{F} H$ is orthogonal to $\mathcal{V}$, hence it must vanish.

We showed that $F$ is in fact contained in the Kähler nullity of $(g, J)$ hence $F=0$ and our assertion follows.

Resuming our general considerations, we define now $d_{\bar{\nabla}} \eta$ in $\Lambda^{2} M \otimes \lambda^{2} M$ by

$$
d_{\bar{\nabla}} \eta(X, Y)=\left(\bar{\nabla}_{X} \eta\right)_{Y}-\left(\bar{\nabla}_{Y} \eta\right)_{X}
$$

whenever $X, Y$ belong to $T M$. It splits as

$$
d_{\bar{\nabla}} \eta=d_{\bar{\nabla}}^{+} \eta+d_{\bar{\nabla}}^{-} \eta
$$

along $\Lambda^{2} M \otimes \lambda^{2} M=\left(\lambda^{1,1} M \otimes \lambda^{2} M\right) \oplus\left(\lambda^{2} M \otimes \lambda^{2} M\right)$. The fact that $\eta$ belongs to $\left(\lambda^{1} M \otimes_{1} \lambda^{2} M\right) \cap \operatorname{Ker}(a)$ makes that

$$
\begin{equation*}
d_{\bar{\nabla}}^{-} \eta \in\left(\lambda^{2} M \otimes_{2} \lambda^{2} M\right) \cap \operatorname{Ker}(a) \tag{2.5}
\end{equation*}
$$

as an easy verification shows. The exterior derivative of the intrinsic torsion tensor is directly related to the curvature tensor of the connection $\bar{\nabla}$ which is defined by $\bar{R}(X, Y)=-\left[\bar{\nabla}_{X}, \bar{\nabla}_{Y}\right]+\bar{\nabla}_{[X, Y]}$ for all vector fields $X, Y$ on $M$. Note that $\bar{R}$ belongs to $\Lambda^{2} M \otimes \lambda^{1,1} M$, since $\bar{\nabla}$ is a Hermitian connection. It is related to the Riemann curvature tensor $R$ by the comparison formula

$$
\begin{equation*}
\bar{R}(X, Y)=R(X, Y)+\left[\eta_{X}, \eta_{Y}\right]-\eta_{T_{X} Y}-d_{\bar{\nabla}} \eta(X, Y) \tag{2.6}
\end{equation*}
$$

for all vector fields $X, Y$ on $M$. At this stage some information is also required on the quadratic algebraic terms in $\eta$ above, as follows. The tensor $\left[\eta^{2}\right]$ defined by $(X, Y) \mapsto\left[\eta_{X}, \eta_{Y}\right]$ belongs to $\lambda^{1,1} M \otimes \lambda^{1,1} M$ and hence it splits as

$$
\left[\eta^{2}\right]=S_{\eta}+A_{\eta}
$$

where $S_{\eta}$ and $A_{\eta}$ belong to $S^{2}\left(\lambda^{1,1} M\right)$ and $\Lambda^{2}\left(\lambda^{1,1} M\right)$ respectively. Let now $\Omega_{\eta}$ in $\lambda^{2,2} M$ be given by

$$
\Omega_{\eta}=\sum_{i=1}^{2 m} \eta_{e_{i}} \wedge \eta_{e_{i}}
$$

where $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$ is some orthonormal basis in $T M$. We consider the tensor $\left(\eta^{2}\right)$ in $\lambda^{2} M \otimes \lambda^{2} M$ given by

$$
(X, Y) \mapsto \eta_{T_{X} Y}
$$

for all $X, Y$ in $T M$. It belongs to $S^{2}\left(\lambda^{2} M\right)$ as it follows from (2.1) and moreover
Lemma 2.4. The following hold:
(i) $\left(\eta^{2}\right)$ belongs to $\lambda^{2} M \otimes_{1} \lambda^{2} M$;
(ii) $\left(\eta^{2}\right)=-\frac{1}{2} \Omega_{\eta}^{-}$;
(iii) $\left\langle\left(\eta^{2}\right)_{X, Y} Z, U\right\rangle=-\left\langle T_{X} Y, T_{Z} U\right\rangle$ for all $X, Y, Z, U$ in $T M$.

Proof. (i) follows directly from $\eta$ in $\lambda^{1} M \otimes_{2} \lambda^{2} M$.
(ii) We compute in some local orthonormal basis $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$

$$
\begin{aligned}
a\left(\eta^{2}\right) & =\sum_{1 \leq i, j \leq 2 m} e_{i} \wedge e_{j} \wedge \eta_{T_{e_{i}} e_{j}}=\sum_{1 \leq i, j, k \leq 2 m}\left\langle T_{e_{i}} e_{j}, e_{k}\right\rangle e_{i} \wedge e_{j} \wedge \eta_{e_{k}} \\
& =-\sum_{1 \leq i, j, k \leq 2 m}\left\langle\eta_{e_{k}} e_{i}, e_{j}\right\rangle e_{i} \wedge e_{j} \wedge \eta_{e_{k}}=-\sum_{1 \leq i, k \leq 2 m} e_{i} \wedge \eta_{e_{k}} e_{i} \wedge \eta_{e_{k}} \\
& =-2 \Omega_{\eta}
\end{aligned}
$$

By (ii) in lemma (2.1) it follows that the tensor $\left(\eta^{2}\right)+\frac{1}{2} \Omega_{\eta}^{-}$belongs to $S^{2}\left(\lambda^{2} M\right) \cap$ $\left(\lambda^{2} M \otimes_{1} \lambda^{2} M\right) \cap \operatorname{ker}(a)$ and since the latter space vanishes by lemma 2.2 , (ii) the claim is proved.
(iii) follows directly from (2.1).

A computation similar to that in (ii) above yields the equality

$$
a\left(\left[\eta^{2}\right]\right)=-\Omega_{\eta}
$$

a consequence of which is the splitting

$$
S_{\eta}=R_{\eta}+\frac{1}{8} \Omega_{\eta}^{+}
$$

where $R_{\eta}$ belongs to $\mathcal{K}(\mathfrak{u}(m))$. We are now ready to describe the main fact in this section.

Proposition 2.1. Let $\left(M^{2 m}, g, J\right)$ be almost Kähler. The following hold:
(i) $d_{\bar{\nabla}}^{-} \eta$ belongs to $\mathcal{K}\left(\mathfrak{u}^{\perp}(m)\right)$;
(ii) the curvature tensor $\bar{R}$ splits as

$$
\bar{R}=R^{K}+\frac{1}{8} \Omega_{\eta}^{+}+A_{\eta}+\left(d_{\bar{\nabla}}^{+} \eta\right)^{\star}
$$

where $R^{K}$ belongs to $\mathcal{K}(\mathfrak{u}(m))$.
Proof. (i) Since $R$ belongs to $S^{2}\left(\Lambda^{2} M\right)$ it follows from (2.6) that

$$
\begin{align*}
\bar{R}(X, Y, Z, U)-\bar{R}(Z, U, X, Y)= & \left\langle\left[\eta_{X}, \eta_{Y}\right] Z, U\right\rangle-\left\langle\left[\eta_{Z}, \eta_{U}\right] X, Y\right\rangle \\
& -\left\langle\left(d_{\bar{\nabla}} \eta\right)(X, Y) Z, U\right\rangle+\left\langle\left(d_{\bar{\nabla}} \eta\right)(Z, U) X, Y\right\rangle \\
= & 2 A_{\eta}(X, Y, Z, U)  \tag{2.7}\\
& -\left\langle\left(d_{\bar{\nabla}} \eta\right)(X, Y) Z, U\right\rangle+\left\langle\left(d_{\bar{\nabla}} \eta\right)(Z, U) X, Y\right\rangle
\end{align*}
$$

for all $X, Y, Z, U$ in $T M$. Since the tensors $\bar{R}$ and $A_{\eta}$ belong to $\Lambda^{2} M \otimes \lambda^{1,1} M$, after projection on $\lambda^{2} M \otimes \lambda^{2} M$ we find that $d_{\bar{\nabla}}^{-} \eta$ belongs to $S^{2}\left(\lambda^{2} M\right)$. Given that $a\left(d_{\bar{\nabla}}^{-} \eta\right)=0$ by (2.5) the claim is proved.
(ii) we consider the splitting $\bar{R}=R_{1}+R_{2}+R_{3}$ along

$$
\Lambda^{2} M \otimes \lambda^{1,1} M=S^{2}\left(\lambda^{1,1} M\right) \oplus \Lambda^{2}\left(\lambda^{1,1} M\right) \oplus\left(\lambda^{2} M \otimes \lambda^{1,1} M\right)
$$

By projecting (2.7) on $\Lambda^{2}\left(\lambda^{1,1} M\right)$ and $\lambda^{1,1} M \otimes \lambda^{2} M$ respectively we obtain after taking (i) into account that

$$
R_{2}=A_{\eta}, \quad R_{3}=\left(d_{\bar{\nabla}}^{+} \eta\right)^{\star}
$$

Clearly $a\left(R_{2}\right)=a\left(R_{3}\right)=0$ hence $a(\bar{R})=a\left(R_{1}\right)$. On the other hand side, from the comparison formula (2.6) it follows that

$$
a(\bar{R})=a\left(\left[\eta^{2}\right]-\left(\eta^{2}\right)\right)
$$

since $a\left(d_{\bar{\nabla}} \eta\right)=0$. But $a\left[\eta^{2}\right]=-\Omega_{\eta}$ hence by (ii) in lemma 2.4 we get $a\left(R_{1}\right)=\Omega_{\eta}$. In other words $R^{K}=R_{1}-\frac{1}{8} \Omega_{\eta}^{+}$belongs to $\mathcal{K}(\mathfrak{u}(m))$ and the claim is proved.
Corollary 2.1. For a given almost Kähler structure $(g, J)$ on some manifold $M^{2 m}$ its associated canonical Hermitian connection $\bar{\nabla}$ is flat if and only if $(g, J)$ is Kähler.

Proof. Let us assume that $\bar{R}=0$. Using (ii) in proposition 2.1 above we find that $\Omega_{\eta}^{+}=0$. Therefore $\Omega_{\eta}=0$ which leads to $\left(\eta^{2}\right)=0$ and finally to $\eta=0$ by means of (ii) and (iii) in lemma 2.4.

This has been first proved in [16] by using the existence of special orthonormal frames adapted to the connection $\bar{\nabla}$. A useful identity for what follows is contained in the lemma below.

Lemma 2.5. Let $\left(M^{2 m}, g, J\right)$ be almost Kähler. Then

$$
\left(\bar{\nabla}_{J X} \eta\right)_{J Y}+\left(\bar{\nabla}_{X} \eta\right)_{Y}=d_{\bar{\nabla}}^{+} \eta(X, Y)+\mathbb{J} d_{\bar{\nabla}}^{+} \eta(J X, Y)
$$

whenever $X, Y$ belong to $T M$.
Proof. Since $d_{\bar{\nabla}} \eta(X, Y)=\left(\bar{\nabla}_{X} \eta\right)_{Y}-\left(\bar{\nabla}_{Y} \eta\right)_{X}$ it follows that

$$
\begin{aligned}
d_{\bar{\nabla}} \eta(J X, Y)(J Z, U) & =\left\langle\left(\bar{\nabla}_{J X} \eta\right)_{Y} J Z-\left(\bar{\nabla}_{Y} \eta\right)_{J X} J Z, U\right\rangle \\
& =\left\langle\left(\bar{\nabla}_{J X} \eta\right)_{J Y} Z+\left(\bar{\nabla}_{Y} \eta\right)_{X} Z, U\right\rangle
\end{aligned}
$$

for all $X, Y, Z, U$ in $T M$. We deduce that

$$
\mathbb{J} d_{\bar{\nabla}} \eta(J X, Y)=\left(\bar{\nabla}_{J X} \eta\right)_{J Y}+\left(\bar{\nabla}_{X} \eta\right)_{Y}-d_{\bar{\nabla}} \eta(X, Y)
$$

whenever $X, Y$ in $T M$ and the claim follows.
2.3. Riemannian curvature and integrability. In the rest of this section $\left(M^{2 m}, g, J\right), m \geq$ 2 will be an almost Kähler manifold. Let $R$ be the curvature tensor of the metric $g$, with the convention that $R(X, Y)=\nabla_{[X, Y]}-\left[\nabla_{X}, \nabla_{Y}\right]$ for all vector fields $X$ and $Y$ on $M$. With respect to the bi-type splitting $\Lambda^{2} M=\lambda^{1,1} M \oplus \lambda^{2} M$ it can be written in block form as

$$
R=\left(\begin{array}{ll}
R_{11} & R_{12}  \tag{2.8}\\
R_{21} & R_{22}
\end{array}\right)
$$

Given that $R$ is in $S^{2}\left(\Lambda^{2} M\right)$ we have that $R_{21}=R_{12}^{\star}$ and the type of the components is given as follows

$$
\begin{aligned}
& R_{11} \text { in } S^{2}\left(\lambda^{1,1} M\right), \quad R_{12} \text { in } \lambda^{1,1} M \otimes \lambda^{2} M \\
& R_{22} \text { in } S^{2}\left(\lambda^{2} M\right) .
\end{aligned}
$$

The relation with the decomposition of the Hermitian curvature tensor in proposition 2.1 is made through the following.
Proposition 2.2. Let $\left(M^{2 m}, g, J\right)$ be an almost Kähler manifold. We have:
(i) $R_{11}=R^{K}-R_{\eta}$;
(ii) $R_{12}=d_{\bar{\nabla}}^{+} \eta$;
(iii) $R_{22}=d_{\bar{\nabla}}^{-} \eta-\frac{1}{2} \Omega_{\eta}^{-}$.

Proof. All claims follow by making use of proposition 2.1. Indeed, the comparaison formula (2.6) reads

$$
\bar{R}=R+\left[\eta^{2}\right]-\left(\eta^{2}\right)-d_{\bar{\nabla}}^{+} \eta-d_{\bar{\nabla}}^{-} \eta
$$

We recall that $d_{\bar{\nabla}}^{+} \eta$ belongs to $\lambda^{1,1} M \otimes \lambda^{2} M, d_{\bar{\nabla}}^{-} \eta$ belongs to $\mathcal{K}\left(\mathfrak{u}^{\perp}(m)\right)$ and also that the identities

$$
\left(\eta^{2}\right)=-\frac{1}{2} \Omega_{\eta}^{-},\left[\eta^{2}\right]=R_{\eta}+\frac{1}{8} \Omega_{\eta}^{+}+A_{\eta}
$$

hold, where the different components have been introduced in the previous section. It is now enough to combine proposition 2.1 with (2.8) when taking into account the algebraic type of each component.

Note that along $\lambda^{2} M \otimes \lambda^{2} M=\left(\lambda^{2} M \otimes_{1} \lambda^{2} M\right) \oplus\left(\lambda^{2} M \otimes_{2} \lambda^{2} M\right)$ we can split $R_{22}=R_{22}^{\prime}+R_{22}^{\prime \prime}$. Proposition 2.2, (iii) yields then the following explicit formulas

$$
\begin{equation*}
R_{22}^{\prime}=-\frac{1}{2} \Omega_{\eta}^{-}, \quad R_{22}^{\prime \prime}=d_{\bar{\nabla}}^{-} \eta \tag{2.9}
\end{equation*}
$$

the former having been computed first in [21][page 604, Cor.4.3].
We recall now some facts about the various notions of Ricci tensors. Let Ric be the Ricci tensor of the Riemannian metric $g$. We denote by Ric ${ }^{\prime}$ and Ric ${ }^{\prime \prime}$ the $J$-invariant resp. the $J$-anti-invariant part of the tensor Ric. Then the Ricci form is defined by $\rho=\left\langle R i c^{\prime} J \cdot, \cdot\right\rangle$. The $\star$-Ricci form is given by $\rho^{\star}=\frac{1}{2} \sum_{i=1}^{2 m} R\left(e_{i}, J e_{i}\right)$ where $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$ is any local orthonormal basis in $T M$ and satisfies

$$
\begin{equation*}
\rho^{\star}-\rho=\frac{1}{2} \nabla^{\star} \nabla \omega \text {. } \tag{2.10}
\end{equation*}
$$

The proof of this fact (see [3]) consists in using the Weitzenböck formula for the harmonic 2-form $\omega$. Taking the scalar product with $\omega$ we obtain:

$$
s^{\star}-s=\frac{1}{2}|\nabla J|^{2}
$$

where the $\star$-scalar curvature is defined by $s^{\star}=2\langle R(\omega), \omega\rangle$. For further use we introduce the forms $\Psi$ and $\Phi$ in $\Lambda^{1,1} M$ given by

$$
\begin{aligned}
& \Psi(X, Y)=\sum_{i=1}^{2 m}\left\langle\left(\nabla_{e_{i}} J\right) J X,\left(\nabla_{e_{i}} J\right) Y\right\rangle \\
& \Phi(X, Y)=\frac{1}{2} \sum_{i=1}^{2 m}\left\langle\left(\nabla_{J X} J\right) e_{i},\left(\nabla_{Y} J\right) e_{i}\right\rangle
\end{aligned}
$$

for all $X, Y$ in $T M$ and any local orthonormal frame $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$. It then easy to see that

$$
\begin{equation*}
\nabla^{\star} \nabla \omega=\Psi-2 g\left(\left(\delta_{\bar{\nabla}} \eta\right) J \cdot, \cdot\right) \tag{2.11}
\end{equation*}
$$

where $\delta_{\bar{\nabla}}$ denotes co-differentiation w.r.t $\bar{\nabla}$. As a consequence we have

$$
\begin{equation*}
\gamma_{1}=\rho+\frac{1}{2}(\Psi-\Phi)-g\left(\left(\delta_{\bar{\nabla}} \eta\right) J \cdot, \cdot\right) \tag{2.12}
\end{equation*}
$$

where the first Chern form of $(g, J)$, computed w.r.t. the canonical connection is given by $\gamma_{1}=\frac{1}{2} \sum_{i=1}^{2 m} \bar{R}\left(\cdot, \cdot, e_{i}, J e_{i}\right)$. The intrinsic torsion tensor $\eta$ and the Ricci tensor enter the following important formula, which gives an obstruction to the existence of almost Kähler, non-Kähler structures.

Proposition 2.3. [6] Let $\left(M^{2 m}, g, J\right)$ be an almost Kähler manifold. Then the following holds:

$$
\begin{aligned}
\Delta\left(s^{\star}-s\right) & =-4 \delta\left(J \delta\left(J \operatorname{Ric}^{\prime \prime}\right)\right)+8 \delta\left(\left\langle\rho^{\star}, \nabla \cdot \omega\right\rangle\right)+2\left|R i c^{\prime \prime}\right|^{2} \\
& +4\left\langle\rho, \Phi-\nabla^{\star} \nabla \omega\right\rangle-|\Phi|^{2}-\left|\nabla^{\star} \nabla \omega\right|^{2}-8\left|R_{22}^{\prime \prime}\right|^{2} .
\end{aligned}
$$

Here $\delta$ denotes co-differentiation with respect to the Levi-Civita connection $\nabla$, acting on 1-forms and 2-tensors.

Under an integral form the formula above has been proved in [35], where it has been used to show that Einstein, almost Kähler manifolds with positive scalar curvature are, in the compact case, Kähler.

## 3. Gray's curvature conditions

Let $\left(M^{2 m}, g, J\right)$ be almost Kähler with Riemannian curvature tensor to be denoted by $R$. We begin by recalling how one can distinguish several classes of almost Hermitian manifolds by "the degree of ressemblance" of their Riemannian curvature tensor with the curvature tensor of a Kähler manifold [21, 33]:

$$
\begin{aligned}
\left(G_{1}\right): & R(X, Y, J Z, J U)=R(X, Y, Z, U) \\
\left(G_{2}\right): & R(X, Y, Z, U)-R(J X, J Y, Z, U)=R(J X, Y, J Z, U)+R(J X, Y, Z, J U) \\
\left(G_{3}\right): & R(J X, J Y, J Z, J U)=R(X, Y, Z, U)
\end{aligned}
$$

Following [21], the class $\mathcal{A} \mathcal{K}_{i}, 1 \leq i \leq 3$ is defined to contain those almost Kähler manifolds whose curvature tensor satisfies the condition $\left(G_{i}\right)$.

In terms of the block structure of the Riemann curvature operator in (2.8) it is elementary to check that Gray's curvature conditions can be equivalently rephrased as follows.

$$
\begin{align*}
& \left(G_{1}\right): R_{12}=R_{22}=0 \\
& \left(G_{2}\right): R_{12}=0, R_{22}^{\prime \prime}=0  \tag{3.1}\\
& \left(G_{3}\right): R_{12}=0 .
\end{align*}
$$

In particular a Kähler structure satisfies all of the three conditions and also the implications $G_{1} \Rightarrow G_{2} \Rightarrow G_{3}$ hold hence

$$
\mathcal{A K}_{1} \subseteq \mathcal{A K}_{2} \subseteq \mathcal{A K}_{3}
$$

In fact it was shown in [20] that locally $\mathcal{A K}_{1}=\mathcal{K}$, where $\mathcal{K}$ denotes the class of Kähler manifolds. A simple proof of this result (for another proof using special frames see [15]) is given below.

Proposition 3.1. Any almost Kähler manifold $\left(M^{2 m}, g, J\right)$ satisfying condition $\left(G_{1}\right)$ is Kähler.

Proof. Since $R_{22}=0$ it follows from (2.9) that $\Omega_{\eta}^{-}=0$. By using (i) in lemma 2.4 we conclude that $\left(\eta^{2}\right)=0$ hence a positivity argument based on (iii) in the same lemma yields that $T$ vanishes and thus so does $\eta$.

The other inclusions between the previously the classes $\mathcal{A K}_{i}, 1 \leq i \leq 3$ are strict in dimensions $2 m \geq 6$, as showed by the examples in [14], multiplied by Kähler manifolds.

Remark 3.1. In the same spirit as above, the class $\mathcal{A H}_{i}, 1 \leq i \leq 3$ contains those almost Hermitian manifolds whose Riemannian curvature tensor satisfies condition $\left(G_{i}\right)$. As opposed to the almost Kähler case, conditions $\left(G_{2}\right)$ and $\left(G_{3}\right)$ are equivalent in the class of Hermitian manifolds [21] and hence in the class of locally conformally Kähler manifolds [18].

Let us now examine the structure of the intrinsic torsion of an almost Kähler structure satisfying the curvature condition $\left(G_{i}\right), i=2,3$.

Proposition 3.2. Let $\left(M^{2 m}, g, J\right)$ be almost Kähler. The following are equivalent:
(i) $(g, J)$ belongs to $\mathcal{A K}_{3}$;
(ii) $d_{\bar{\nabla}}^{+} \eta=0$;
(iii) $\left(\bar{\nabla}_{J X} \eta\right)(J Y, Z)+\left(\bar{\nabla}_{X} \eta\right)(Y, Z)=0$ for all $X, Y$ in $T M$.

Proof. Since $\left(G_{3}\right)$ is satisfied if and only if $R_{12}=0$ the claim follows from proposition 2.2 , (ii) combined with lemma 2.5 .

Corollary 3.1. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{3}$. The following hold whenever $X, Y, Z, U$ belong to $T M$
(i) $\bar{R}(X, Y, Z, U)-\bar{R}(Z, U, X, Y)=\left\langle\left[\eta_{X}, \eta_{Y}\right] Z, U\right\rangle-\left\langle\left[\eta_{Z}, \eta_{U}\right] X, Y\right\rangle$;
(ii) $\bar{R}(J X, J Y)=\bar{R}(X, Y)$;
(iii) the algebraic Bianchi identity for the connection $\bar{\nabla}$ takes the form

$$
\sigma_{X, Y, Z}\left[\bar{R}(X, Y) Z+T_{T_{X} Y} Z\right]=0
$$

where $\sigma$ denotes the cyclic sum.
Proof. (i) and (ii) follow from the vanishing of $d_{\bar{\nabla}}^{+} \eta$ and (ii) in proposition 2.1. (iii) We compute by using (2.1) and (ii) in proposition 3.2

$$
\begin{aligned}
\left\langle\left(\bar{\nabla}_{X} T\right)(Y, Z), U\right\rangle & =-\left\langle\left(\bar{\nabla}_{X} \eta\right)_{U} Y, Z\right\rangle=-\left(d_{\bar{\nabla}} \eta\right)(X, U)(Y, Z)-\left\langle\left(\bar{\nabla}_{U} \eta\right)_{X} Y, Z\right\rangle \\
& =-\left(d_{\bar{\nabla}}^{-} \eta\right)(X, U)(Y, Z)-\left\langle\left(\bar{\nabla}_{U} \eta\right)_{X} Y, Z\right\rangle
\end{aligned}
$$

for all $X, Y, Z, U$ in $T M$. By proposition 2.1, (i) we know that $d_{\bar{\nabla}}^{-} \eta$ is an algebraic curvature tensor and since $a(\eta)=0$ after taking the cyclic sum on $X, Y, Z$ we obtain that $\sigma_{X, Y, Z}\left\langle\left(\bar{\nabla}_{X} T\right)(Y, Z), U\right\rangle=0$ whenever $X, Y, Z, U$ belong to $T M$. The claim follows now from the algebraic Bianchi identity for the connection $\bar{\nabla}$.

Proposition 3.3. Let $\left(M^{2 m}, g, J\right)$ be almost Kähler. Then $(g, J)$ satisfies condition $\left(G_{2}\right)$ if and only if $d_{\bar{\nabla}} \eta=0$. In particular we have

$$
\begin{equation*}
\left(\bar{\nabla}_{J X} \eta\right)_{J Y}+\left(\bar{\nabla}_{X} \eta\right)_{Y}=0 \tag{3.2}
\end{equation*}
$$

whenever $X, Y$ belong to $T M$, provided $(g, J)$ is of class $\mathcal{A} \mathcal{K}_{2}$.
Proof. Since the curvature condition $\left(G_{2}\right)$ is satisfied if and only if $R_{12}=0$ and $R_{22}^{\prime \prime}=0$ the claim follows from proposition 2.2, (ii) and (2.9).

Finally we record some information on the differential Bianchi identity.

Proposition 3.4. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A} \mathcal{K}_{3}$. We have:

$$
\begin{aligned}
& \left(\bar{\nabla}_{X} \bar{R}\right)(Y, Z)+\left(\bar{\nabla}_{Y} \bar{R}\right)(Z, X)+\left(\bar{\nabla}_{Z} \bar{R}\right)(X, Y)=0 \\
& \bar{R}\left(T_{X} Y, Z\right)+\bar{R}\left(T_{Y} Z, X\right)+\bar{R}\left(T_{Z} X, Y\right)=0
\end{aligned}
$$

whenever $X, Y, Z$ belong to $T M$.
3.1. The partial parallelism of the torsion. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}$. We investigate here the impact of having Gray's curvature conditions satisfied on the intrinsic torsion tensor $\eta$ of $(g, J)$. Notationwise, the action $G \eta$ of an endomorphism $G$ of $T M$ on the tensor $\eta$ is defined by

$$
(G \eta)_{X}=G \eta_{X}-\eta_{X} G-\eta_{G X}
$$

for all $X$ in $T M$. Our main observation in this section is
Proposition 3.5. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{3}$. Then

$$
\bar{\nabla}_{T_{X} Y} \eta=0
$$

for all $X, Y$ in $T M$.
Proof. Differentiating (iii) in proposition 3.2 we obtain

$$
\left(\bar{\nabla}_{X, J Y}^{2} \eta\right)_{J Z}+\left(\bar{\nabla}_{X, Y}^{2} \eta\right)_{Z}=0
$$

and also

$$
-\left(\bar{\nabla}_{J Y, X}^{2} \eta\right)_{J Z}+\left(\bar{\nabla}_{J Y, J X}^{2} \eta\right)_{Z}=0
$$

for all $X, Y, Z$ in $T M$, after performing the variable change $(X, Y) \mapsto(J Y, J X)$. After addition of these two equations and by making use of the Ricci identity w.r.t. the connection $\bar{\nabla}$ we obtain that

$$
\begin{equation*}
(\bar{R}(X, J Y) \eta)_{J Z}+\left(\bar{\nabla}_{T_{X} J Y} \eta\right)_{J Z}=\left(\bar{\nabla}_{X, Y}^{2} \eta\right)_{Z}+\left(\bar{\nabla}_{J Y, J X}^{2} \eta\right)_{Z} \tag{3.3}
\end{equation*}
$$

for all $X, Y, Z$ in $T M$. Now we antisymmetrize (3.3) in $X$ and $Y$ in order to get

$$
\begin{aligned}
& (\bar{R}(X, J Y) \eta+\bar{R}(J X, Y) \eta)_{J Z}+\left(\bar{\nabla}_{T_{X}(J Y)+T_{J X} Y} \eta\right)_{J Z} \\
= & (\bar{R}(J X, J Y) \eta-\bar{R}(X, Y) \eta)_{Z}+\left(\bar{\nabla}_{T_{J X}(J Y)-T_{X} Y} \eta\right)_{Z}
\end{aligned}
$$

whenever $X, Y, Z$ belong to $T M$, where we have used twice the Ricci identity for $\bar{\nabla}$. By corollary 3.1, (ii) the curvature tensor $\bar{R}$ belongs to $\lambda^{1,1} M \otimes \lambda^{1,1} M$ hence $\left(\bar{\nabla}_{T_{X}(J Y)+T_{J X} Y} \eta\right)_{J Z}=\left(\bar{\nabla}_{T_{J X}(J Y)-T_{X} Y} \eta\right)_{Z}$ and further $\left(\bar{\nabla}_{J T_{X} Y} \eta\right)_{J Z}=\left(\bar{\nabla}_{T_{X} Y} \eta\right)_{Z}$ for all $X, Y, Z$ in $T M$ since $\left(M^{2 m}, g, J\right)$ is quasi-Kähler. The claim follows now by using (iii) in proposition 3.2.

As far as the class $\mathcal{A} \mathcal{K}_{2}$ is concerned additional information relating torsion and curvature is available.

Proposition 3.6. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{2}$. Then

$$
\sigma_{X, Y, Z}\left[\bar{R}(X, Y), \eta_{Z}\right]=\sigma_{X, Y, Z} \eta_{\bar{R}(X, Y) Z}
$$

for all $X, Y, Z$ in $T M$.
Proof. Because $R_{12}=0$ and $R_{22}^{\prime \prime}=0$ we have $d_{\bar{\nabla}} \eta=0$. After differentiating we find by using the Ricci identity that $\sigma_{X, Y, Z}(\bar{R}(X, Y) \eta)_{Z}+\left(\bar{\nabla}_{T_{X} Y} \eta\right)_{Z}=0$ for all $X, Y, Z$ in $T M$. The claim follows now by using proposition 3.5.
3.2. Examples. Let us now describe the main class of examples in this paper. On $\mathbb{R}^{2 p}$ with co-ordinates $x_{i}, 1 \leq i \leq 2 p$ we consider the standard flat metric $g_{0}$ and compatible complex structure $J_{0}$ given by $J_{0}\left(\frac{d}{d x_{i}}\right)=\frac{d}{d x_{p+i}}, 1 \leq i \leq p$; moreover let $\omega_{0}=g_{0}\left(J_{0} \cdot, \cdot\right)$ be the induced symplectic form.

Let $(Z, h, I)$ be a Kähler manifold and let

$$
w: Z \rightarrow S^{2,-}\left(\mathbb{R}^{2 p}\right)=\left\{S \in S^{2}\left(\mathbb{R}^{2 p}\right): S J_{0}+J_{0} S=0\right\}
$$

be such that $|w|_{\infty}<1$. We consider the product $M=\mathbb{R}^{2 p} \times Z$ where the $\mathbb{R}^{2 p}$ is equipped with the metric and almost complex structure given by

$$
\begin{aligned}
& g_{w}=g_{0}\left((1+w)^{-1}(1-w) \cdot, \cdot\right) \\
& J_{w}=(1-w)^{-1} J_{0}(1-w)
\end{aligned}
$$

in the basis $\left\{\frac{d}{d x_{i}}, 1 \leq i \leq 2 p\right\}$. These objects extend to a Riemannian metric and two almost complex structures on $M$ given in block form by

$$
g=\left(\begin{array}{ll}
g_{w} & 0 \\
0 & h
\end{array}\right), J=\left(\begin{array}{ll}
J_{w} & 0 \\
0 & I
\end{array}\right), \tilde{J}=\left(\begin{array}{ll}
-J_{w} & 0 \\
0 & I
\end{array}\right)
$$

If $\omega^{Z}=h(I \cdot, \cdot)$ denotes the Kähler form of $(h, I)$ an important feature of this construction is to have the Kähler forms of $(g, J)$ respectively $(g, \tilde{J})$ equal to $\omega_{0}+\omega^{Z}$ and $\omega_{0}-\omega^{Z}$ respectively. Therefore both of the structures $(g, J)$ and $(g, \tilde{J})$ are automatically almost-Kähler.

Of special significance to us is the case when $w: Z \rightarrow S^{2,-}\left(\mathbb{R}^{2 p}\right)$ is anti-holomorphic in the sense that $I d w=-(d w) J_{0}$, where in right hand side matrix multiplication is meant. Using appropriate complex notation this is easily seen to be equivalent to the usual notion.

Let now the distribution $\mathcal{V}$ be spanned by $\left\{\frac{d}{d x_{i}}, 1 \leq i \leq 2 p\right\}$ and let $H$ be its orthogonal complement in $T M$, w..r.t., say, $g$.

Proposition 3.7. If $w: Z \rightarrow S^{2,-}\left(\mathbb{R}^{2 p}\right)$ is anti-holomorphic, non-constant and moreover $|w|_{\infty}<1$ holds we have:
(i) $(M, g, \tilde{J})$ is Kähler;
(ii) $(g, J)$ is a normal almost Kähler, non-Kähler structure with Kähler nullity containing $H$;moreover the canonical connection of $(g, J)$ leaves the splitting $T M=\mathcal{V} \oplus H$ invariant;
(iii) $(M, g, J)$ belongs to the class $\mathcal{A K}_{3}$;
(iv) the metric $g$ is Einstein if and only if the Kähler structures $(h, I)$ has Ricci form $\rho^{h}=2(d I d) \ln \operatorname{det}(1-w)$.

Proof. (i) Let $V_{i}=\frac{d}{d x_{i}}, 1 \leq i \leq 2 p$ such that in matrix terms we have $-\tilde{J} V_{i}=$ $\sum_{j=1}^{2 p} J_{w}^{i j} V_{j}$ for $1 \leq i \leq 2 p$. The Nijenhuis tensor $N_{\tilde{J}}$ vanishes on $\Lambda^{2} \mathcal{V} \oplus \Lambda^{2} H$ by the construction of $\tilde{J}$ and the fact that $I$ is integrable. Unless otherwise specified in what follows we will work with with invariant vector fields $X$ in $T Z$, that is
$\left[V_{i}, X\right]=0,1 \leq i \leq 2 p$. A straightforward computation shows that

$$
\begin{aligned}
N_{\tilde{J}}\left(V_{i}, X\right) & =\left[V_{i}, X\right]+\left[J_{w} V_{i}, I X\right]+\tilde{J}\left[V_{i}, I X\right]-\tilde{J}\left[J_{w} V_{i}, X\right] \\
& =\left[J_{w} V_{i}, I X\right]-\tilde{J}\left[J_{w} V_{i}, X\right] \\
& =\sum_{j=1}^{2 p}\left(L_{I X} J_{w}^{i j}\right) V_{j}-\sum_{j=1}^{2 p}\left(L_{X} J_{w}^{i j}\right) \tilde{J} V_{j}
\end{aligned}
$$

for all $1 \leq i \leq 2 p$ and for all invariant $X$ in $T Z$. Hence $\tilde{J}$ is integrable if and only if

$$
0=L_{I X} J_{w}+\left(L_{X} J_{w}\right) J_{w}=(1-w)^{-1}\left(\left(L_{I X} w\right) J_{0}-L_{X} w\right)\left(J_{0} J_{w}-1\right)
$$

holds for any invariant $X$ in $T Z$; the claim follows since the matrix $J_{0} J_{w}-1$ is invertible.
(ii) Let us compute the intrinsic torsion tensor $\eta$ of the almost Kähler structure $(g, J)$. As in (i) the Nijenhuis tensor $N_{J}$ vanishes on $\Lambda^{2} \mathcal{V} \oplus \Lambda^{2} H$. Moreover by a computation similar to that in (i) taking into account that $w$ is anti-holomorphic yields

$$
N_{J}\left(V_{i}, X\right)=2 \sum_{j=1}^{2 p}\left(L_{X} J_{w}^{i j}\right) J_{w} V_{j}
$$

for all $1 \leq i \leq 2 p$ and for all invariant $X$ in $T Z$. Since $(g, J)$ is almost Kähler we have $g\left(N_{J}\left(U_{1}, U_{2}\right), U_{3}\right)=4 g\left(\eta_{U_{3}} U_{1}, U_{2}\right)$ whenever $U_{i}, 1 \leq i \leq 3$ belong to $T M$. It follows that

$$
\eta_{H}=0, \eta_{\mathcal{V}} \mathcal{V} \subseteq H
$$

and moreover, when writing

$$
\begin{equation*}
2 \eta_{V_{i}} X=\sum_{j=1}^{2 p} \alpha_{i j}(X) V_{j} \tag{3.4}
\end{equation*}
$$

for all invariant $X$ in $H$ and $1 \leq i \leq 2 p$ the matrix of 1-forms $\alpha=\left(\alpha_{i j}\right)_{1 \leq i, j \leq 2 p}$ is given by

$$
\begin{equation*}
\alpha(X)=-\left(L_{X} G_{w}\right) G_{w}^{-1} \tag{3.5}
\end{equation*}
$$

Here we have used the notation $G_{w}=(1+w)^{-1}(1-w)$ and that $J_{0} G_{w}^{-1}=J_{w}$. The canonical Hermitian connection of $(g, J)$ leaves the splitting $T M=\mathcal{V} \oplus H$ invariant since $(g, \tilde{J})$ is Kähler and $[J, \tilde{J}]=0$. The structure is non-Kähler because by (3.5) $\eta$ vanishes if and only if $w$ is constant.
(iii) Since the metric $g_{w}$ does not depend on $\mathbb{R}^{2 p}$ the vector fields $V_{i}, 1 \leq i \leq 2 p$ are Killing. By projection of the Killing equation on $\mathcal{V} \times H$ it is easy to obtain that

$$
\begin{equation*}
\bar{\nabla}_{X} V_{i}=-\eta_{V_{i}} X \tag{3.6}
\end{equation*}
$$

for all $X$ in $H$ and whenever $1 \leq i \leq 2 p$. Again because $g_{w}$ does not depend on $\mathbb{R}^{2 p}$ and because the vector fields $V_{i}$ mutually commute the Koszul formula and (3.5) lead directly to

$$
\bar{\nabla}_{V} V_{i}=0
$$

for all $1 \leq i \leq 2 p$ and whenever $V$ is in $\mathcal{V}$. To verify that $(g, J)$ belongs to the class $\mathcal{A} \mathcal{K}_{3}$ we use the criterion in (iii) of proposition 3.2, as follows. From (3.4) it is clear
that $\bar{\nabla}_{V} \eta=0$ for all $V$ in $\mathcal{V}$ and since $\mathcal{V}$ is parallel w.r.t $\bar{\nabla}$ and $\eta_{H}=0$ it is enough to check that

$$
\left(\bar{\nabla}_{J X} \eta\right)_{J V} Y+\left(\bar{\nabla}_{X} \eta\right)_{V} Y=0
$$

for all $X, Y$ in $H$ and $V$ in $\mathcal{V}$. We fix now $X, Y$ in $H$. After differentiation in (3.4) we find by using (3.6) that

$$
\left(\bar{\nabla}_{X} \eta\right)_{V_{i}} Y=\eta_{\eta_{V_{i}} X} Y-\eta_{\eta_{V_{i}} Y} X+\frac{1}{2} \sum_{j=1}^{2 p}\left(\bar{\nabla}_{X} \alpha_{i j}\right) Y V_{j}
$$

for all $1 \leq i \leq 2 p$. The quantity to examine is then

$$
\begin{aligned}
\left(\bar{\nabla}_{J X} \eta\right)_{J V_{i}} Y+\left(\bar{\nabla}_{X} \eta\right)_{V_{i}} Y= & \left(\bar{\nabla}_{J X} \eta\right)_{V_{i}} J Y+\left(\bar{\nabla}_{X} \eta\right)_{V_{i}} Y \\
= & 2\left(\eta_{\eta_{V_{i}} X} Y-\eta_{\eta_{V_{i}} Y} X\right) \\
& +\frac{1}{2} \sum_{j=1}^{2 p}\left(\left(\bar{\nabla}_{I X} \alpha_{i j}\right) I Y+\left(\bar{\nabla}_{X} \alpha_{i j}\right) Y\right) V_{j} .
\end{aligned}
$$

Now $\left(\bar{\nabla}_{I X} \alpha_{i j}\right) I Y+\left(\bar{\nabla}_{X} \alpha_{i j}\right) Y=d \alpha_{i j}(X, Y)+d\left(I \alpha_{i j}\right)(I X, Y)$; from (3.5) it follows that

$$
d \alpha(X, Y)=-[\alpha(X), \alpha(Y)]
$$

for all $X, Y$ in $H$ as well as $I \alpha=-2 d\left((1+w)^{-1} J_{0}\right)$, in particular

$$
d(I \alpha)=0
$$

on $H$. Therefore

$$
\begin{aligned}
& \sum_{j=1}^{2 p}\left(\left(\bar{\nabla}_{I X} \alpha_{i j}\right) I Y+\left(\bar{\nabla}_{X} \alpha_{i j}\right) Y\right) V_{j}=-\sum_{j=1}^{2 p}[\alpha(X), \alpha(Y)]_{i j} V_{j} \\
& =\sum_{j, k=1}^{2 p}\left(\alpha_{i k}(Y) \alpha_{k j}(X)-\alpha_{i k}(X) \alpha_{k j}(Y)\right) V_{j}=2 \sum_{k=1}^{2 p} \alpha_{i k}(Y) \eta_{V_{k}} X-2 \sum_{k=1}^{2 p} \alpha_{i k}(X) \eta_{V_{k}} Y \\
& =4 \eta_{\eta_{V_{i}} Y} X-4 \eta_{\eta_{V_{i}} X} Y .
\end{aligned}
$$

It follows that $\left(\bar{\nabla}_{J X} \eta\right)_{J V_{i}} Y+\left(\bar{\nabla}_{X} \eta\right)_{V_{i}} Y=0$ for all $1 \leq i \leq 2 p$ and the claim is proved.
(iv) follows by a standard computation using the explicit form of the metric $g$.

We note, as in the proof above, that $V_{i}=\frac{d}{d x_{i}}, 1 \leq i \leq 2 p$ are Killing vector fields for $g_{w}$, holomorphic w.r.t. $\tilde{J}$. This aspect will be discussed further in section 8. We will call $w$ non-degenerate if

$$
\left\{\lambda=\left(\begin{array}{c}
\lambda_{1} \\
\vdots \\
\lambda_{2 p}
\end{array}\right) \in \mathbb{R}^{2 p}:(d w) \lambda=0\right\}
$$

vanishes at each point of $Z$. In this case the Kähler nullity of $(g, J)$ coincides with $H$. For example, under the identification $S^{2,-}\left(\mathbb{R}^{2 p}\right)=\left\{A \in M_{p}(\mathbb{C}): A^{T}=A\right\}$ one can take

$$
w=\operatorname{Diag}\left(w_{1}, \ldots, w_{p}\right)
$$

where $w_{i}: Z \rightarrow \mathbb{C}$ are holomorphic, $1 \leq i \leq p$; then $w$ is non-degenerate on the open set where none of $w_{i}, 1 \leq i \leq p$ is constant. Moreover an $\mathcal{A} \mathcal{K}_{3}$ structure given by a diagonal $w$ as above is strictly normal if and only if $\left(w_{1}, \ldots, w_{p}\right): Z \rightarrow \mathbb{C}^{p}$ is an immersion.

When $p=1$ and $Z$ is a Riemann surface the construction above first appeared in [8], where it has been shown to exaust, locally, the class of four dimensional $\mathcal{A K}_{3}$ manifolds.

In section 8.1 we will show that $(g, J)$ belongs to the class $\mathcal{A} \mathcal{K}_{2}$ if and only if it is a 3 -symmetric space and in particular $(h, I)$ is a Hermitian symmetric space.

Non-product examples of Kähler manifolds ( $Z, h, I$ ) leading to Einstein(in fact Ricci flat) almost-Kähler metrics $g$ as above can be constructed as follows; when $Z$ is a Riemann surface the metric $\left(1-|w|^{2}\right) h$ is flat. In higher dimensions we have the following.

Example 3.1. Let $(S, k)$ be a Sasakian manifold with Reeb vector field $\zeta$ and contact distribution $\mathcal{D}$; we denote by $\theta=k(\zeta, \cdot)$ the contact and by $\tilde{I}$ the transversal complex structure.

Let $(Z, \tilde{h})=\left((0, \infty) \times S, d r^{2}+r k\right)$ be the metric cone over $S$; it is a Kähler manifold with Kähler form $d r \wedge \theta+r d \theta$. We assume now that $k$ is an Einstein metric, such that $\tilde{h}$ is Ricci flat.

Pick now $w=w_{1}+i w_{2}: Z \rightarrow \mathbb{C}$ with $|w|_{\infty}<1$ and let the symmetric endomorphism of $T Z$ be given by $\left(\begin{array}{cc}w_{1} & w_{2} \\ w_{2} & -w_{1}\end{array}\right)$, in the basis $\left\{r^{-1} d r, \theta\right\}$ on $\operatorname{span}\left\{r \frac{d}{d r}, \zeta\right\}$ and $S=0$ on $\mathcal{D}$.

We define a metric and compatible complex structure on $Z$ by

$$
h=\tilde{h}\left((1+S)(1-S)^{-1} \cdot, \cdot\right), I=(1-S) \tilde{I}(1-S)^{-1}
$$

Assume now that $L_{\frac{d}{d r}} w=L_{\zeta} w=0$ and $\tilde{I} d w=i d w$; that is $w$ is holomorphic on the local Kähler quotient $S /\{\zeta\}$. Then $(h, I)$ is Kähler and moreover

$$
\rho^{h}=2(d I d) \ln \left(1-|w|^{2}\right)
$$

since $\tilde{h}$ is Ricci flat (see [12]) for details. Now since $w:(Z, I) \rightarrow \mathbb{C}$ is holomorphic, it enters the construction in proposition 3.7 to yield a Ricci-flat almost Kähler metric on $\mathbb{C} \times Z$. Note that these metrics have in fact $T^{3}$-symmetry, because of the invariance properties we have imposed on $w$.

A more general situation when examples as above occur is looked at in [12].

## 4. A first decomposition result

In this section we start to analyse geometric consequences of the proposition 3.5 established in the last section. Our main object of study will be an almost Kähler manifold $\left(M^{2 m}, g, J\right)$ in the class $\mathcal{A} \mathcal{K}_{3}$ with Kähler nullity $H$. Our analysis is built around the partial parallelism of the torsion, that is

$$
\begin{equation*}
\bar{\nabla}_{V} \eta=0 \tag{4.1}
\end{equation*}
$$

for all $V$ in $\mathcal{V}=H^{\perp}$, which follows immediately from proposition 3.5 and (2.3). Let us examine now some elementary properties of the decomposition $T M=\mathcal{V} \oplus H$.

Proposition 4.1. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{3}$. We have:
(i) $\bar{\nabla}_{V} X$ belongs to $H$ for all $V$ in $\mathcal{V}$ and $X$ in $H$;
(ii) $\bar{\nabla}_{V} W$ belongs to $\mathcal{V}$ if $V, W$ are in $\mathcal{V}$;
(iii) $\bar{R}(V, W) \eta=0$ whenever $V, W$ belong to $\mathcal{V}$;
(iv) the distribution $\mathcal{V}$ is integrable.

Proof. (i) We know from (4.1) that $\left(\bar{\nabla}_{V} \eta\right)(X, U)=0$ for all $U$ in $T M$. Since $\eta_{X}=0$ this gives $\eta_{\bar{\nabla}_{V} X} U=0$ and the proof is finished. Now, (ii) follows from (i) since $\bar{\nabla}$ is a metric connection and $\mathcal{V}$ and $H$ are orthogonal.
(iii) follows by differentiation of (4.1) in directions coming from $\mathcal{V}$ together with $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$.
(iv) is implied by (ii) and the fact that $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$.

We shall now work to obtain a first decomposition of the vector bundle $\mathcal{V}$ having good algebraic properties with respect to the torsion tensor $T$. Define the subbundle $\mathcal{V}_{0}$ of $\mathcal{V}$ by setting

$$
\mathcal{V}_{0}=T(\mathcal{V}, \mathcal{V})
$$

and let $\mathcal{V}_{1}$ be its orthogonal complement in $\mathcal{V}$. In fact, $\mathcal{V}_{1}$ can be seen as the Kähler nullity of the foliation induced by $\mathcal{V}$ with respect to the induced almost Kähler structure. More precisely, we define the tensor

$$
\hat{\eta}: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \text { by } \hat{\eta}_{V} W=\left(\eta_{V} W\right)_{\mathcal{V}}
$$

where the subscript indicates orthogonal projection. Then we have

$$
\begin{equation*}
\mathcal{V}_{1}=\left\{V \in \mathcal{V}: \hat{\eta}_{V} \mathcal{V}=0\right\} \tag{4.2}
\end{equation*}
$$

In other words $\eta_{\mathcal{V}_{1}} \mathcal{V} \subseteq H$ holds whence $T\left(\mathcal{V}_{1}, \mathcal{V}_{1}\right)=0$. By (2.1) we have that $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$ hence the tensor $\hat{\eta}$ completely determines the torsion over $\mathcal{V}$, that is

$$
\hat{\eta}_{V} W-\hat{\eta}_{W} V=T(V, W)
$$

for all $V, W$ in $\mathcal{V}$.
In the subsequent, we will assume that the sub-bundle $\mathcal{V}_{0}$ has constant rank over $M$. As our study is purely local and we have already assumed that $H$ has constant rank over $M$, there is no loss of generality since $\mathcal{V}_{0}$ has constant rank over each connected component of some open dense subset of $M$.

To summarise, any integral manifold of the distribution $\mathcal{V}$ carries, w.r.t the induced metric and almost complex structure an almost Kähler structure with Kähler nullity $\mathcal{V}_{1}$ and intrinsic torsion tensor $\hat{\eta}$. Moreover, w.r.t the induced structure the intrinsic torsion tensor is parallel by (4.1), since the first canonical Hermitian connection of such an integral manifold, coincides with the restriction of $\bar{\nabla}$ to $\mathcal{V}$.

We now look at the Kähler nullity of integral manifolds, first in the induced structure only; later on in section 6 we will show how these considerations can be extended over the whole of $M$.
Lemma 4.1. The orthogonal decomposition $\mathcal{V}=\mathcal{V}_{0} \oplus \mathcal{V}_{1}$ is $J$-invariant and $\bar{\nabla}$ parallel inside $\mathcal{V}$.
Proof. From (4.1) we get that $\bar{\nabla}_{V} T=0$ for all $V$ in $\mathcal{V}$. Then proposition 4.1, (ii) and the definition of $\mathcal{V}_{0}$ yields the parallelism of $\mathcal{V}_{0}$ and hence that of $\mathcal{V}_{1}$ inside $\mathcal{V}$.

In the following lemma we show that the existence of a decomposition as that of $\mathcal{V}$ above, generates strong algebraic restrictions involving the tensors $T$ and $\hat{\eta}$.
Lemma 4.2. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A} \mathcal{K}_{3}$. If $\mathcal{V}=D_{1} \oplus D_{2}$ is an orthogonal and J-invariant decomposition such that the distributions $D_{1}$ and $D_{2}$ are $\bar{\nabla}$-parallel inside $\mathcal{V}$ then:
(i) $\bar{R}\left(V_{1}, W_{1}, W, W_{2}\right)=-\left\langle T_{W_{1}} W, \hat{\eta}_{W_{2}} V_{1}\right\rangle-\left\langle T_{W} V_{1}, \hat{\eta}_{W_{2}} W_{1}\right\rangle$ for all $V_{1}, W_{1}$ in $D_{1}$, $W$ in $\mathcal{V}$ and $W_{2}$ in $D_{2}$;
(ii) $\hat{\eta}_{D_{2}} T\left(D_{1}, D_{1}\right)=0$;
(iii) $\bar{R}\left(V, W, V_{1}, W_{1}\right)=0$ for all $V, W$ in $\mathcal{V}_{0}$ and $V_{1}, W_{1}$ in $\mathcal{V}_{1}$;
(iv) for all $V_{i}, 1 \leq i \leq 4$ in $\mathcal{V}$ we have

$$
\bar{R}\left(V_{1}, V_{2}, V_{3}, V_{4}\right)-\bar{R}\left(V_{3}, V_{4}, V_{1}, V_{2}\right)=\left\langle\left[\hat{\eta}_{V_{V}}, \hat{\eta}_{V_{2}}\right] V_{3}, V_{4}\right\rangle-\left\langle\left[\hat{\eta}_{V_{3}}, \hat{\eta}_{V_{4}}\right] V_{1}, V_{2}\right\rangle .
$$

Proof. We prove (i) and (ii) at the same time. Using the first Bianchi identity for the connection $\bar{\nabla}$ in corollary 3.1, (iii) we get:

$$
\begin{aligned}
& \bar{R}\left(V_{1}, W_{1}, W, W_{2}\right)+\bar{R}\left(W_{1}, W, V_{1}, W_{2}\right)+\bar{R}\left(W, V_{1}, W_{1}, W_{2}\right) \\
& +\left\langle T_{V_{1}} W_{1}, \eta_{W_{2}} W\right\rangle+\left\langle T_{W_{1}} W, \eta_{W_{2}} V_{1}\right\rangle+\left\langle T_{W} V_{1}, \eta_{W_{2}} W_{1}\right\rangle=0 .
\end{aligned}
$$

Now the second and the third term below vanish since $D_{1}, D_{2}$ are $\bar{\nabla}$-parallel inside the integrable distribution $\mathcal{V}$. Using that $\bar{R}\left(J V_{1}, J W_{1}, W, W_{2}\right)=\bar{R}\left(V_{1}, W_{1}, W, W_{2}\right)$ and the $J$-invariance properties of the tensor $T$ it follows that $\left\langle T_{V_{1}} W_{1}, \eta_{W_{2}} W\right\rangle=0$. By (2.3) we further get that $\left\langle T_{V_{1}} W_{1}, \hat{\eta}_{W_{2}} W\right\rangle=0$ and since $W$ in $\mathcal{V}$ was chosen arbitrarily the claim in (ii) follows. The proof of (i) is now obtained by updating the Bianchi identity above.

To prove (iii) we take $D_{1}=\mathcal{V}_{0}, D_{2}=\mathcal{V}_{1}$ in (i) and use that $\mathcal{V}_{1}$ is the Kähler nullity of $\hat{\eta}$ as indicated in (4.2).

The proof of (iv) follows by an easy computation from corollary 3.1, (i) and the symmetry of the tensor $\eta^{H}$ defined by $\eta_{V}^{H} W=\left(\eta_{V} W\right)_{H}$ for all $V, W$ in $\mathcal{V}$ (which is a consequence of having the torsion concentrated in $\mathcal{V}$ ).

Let us define $\mathcal{W}_{1}=T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right)$ and let $\mathcal{W}_{2}$ be the orthogonal complement of $\mathcal{W}_{1}$ in $\mathcal{V}_{0}$. The splitting

$$
\mathcal{V}_{0}=\mathcal{W}_{1} \oplus \mathcal{W}_{2}
$$

is clearly $J$-invariant and $\bar{\nabla}$-parallel inside $\mathcal{V}$. We are now able to prove our first decomposition result as follows.
Proposition 4.2. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{3}$. The following hold:
(i) $\mathcal{W}_{1}=T\left(\mathcal{W}_{1}, \mathcal{W}_{1}\right)$;
(ii) $\hat{\eta}_{\mathcal{W}_{1}} \mathcal{W}_{2}=\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{1}=0$;
(iii) $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{2} \subseteq \mathcal{V}_{1}$;
(iv) $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}=\mathcal{W}_{2}$.

Proof. We consider the $\bar{\nabla}$-parallel decomposition (inside $\mathcal{V}$ )

$$
\mathcal{V}=\mathcal{W}_{1} \oplus\left(\mathcal{W}_{2} \oplus \mathcal{V}_{1}\right)
$$

which is also orthogonal and $J$-invariant. Using lemma 4.2,(ii) for $D_{1}=\mathcal{W}_{1}, D_{2}=$ $\mathcal{W}_{2} \oplus \mathcal{V}_{1}$ we get $\hat{\eta}_{\mathcal{W}_{1}} T\left(\mathcal{W}_{2}, \mathcal{V}_{1}\right)=0$ hence

$$
\begin{equation*}
\hat{\eta}_{\mathcal{W}_{1}} \hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}=0 \tag{4.3}
\end{equation*}
$$

since $\hat{\eta}_{\mathcal{V}_{1}} \mathcal{V}=0$. It follows that

$$
\begin{equation*}
\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1} \perp \hat{\eta}_{\mathcal{W}_{1}} \mathcal{W}_{2} \tag{4.4}
\end{equation*}
$$

The definition of $\mathcal{W}_{2}$ implies that $\mathcal{W}_{2} \perp T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right)$ hence the use of the almost Kähler condition (2.1) gives

$$
\begin{equation*}
\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{0} \subseteq \mathcal{V}_{1} \tag{4.5}
\end{equation*}
$$

in particular $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{1} \subseteq \mathcal{V}_{1}$. Then $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1} \perp \hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{1}$ since the vanishing of the torsion on $\mathcal{V}_{1}$ implies that $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}$ is orthogonal to $\mathcal{V}_{1}$. Because of (4.4) we arrive at $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}, T\left(\mathcal{W}_{1}, \mathcal{W}_{2}\right)$, in other words $\hat{\eta}_{\mathcal{W}_{2}} T\left(\mathcal{W}_{1}, \mathcal{W}_{2}\right)$ is orthogonal to $\mathcal{V}_{1}$. But $T\left(\mathcal{W}_{1}, \mathcal{W}_{2}\right) \subseteq \mathcal{W}_{1}$ by the definition of $W_{1}$ and we saw that $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{1} \subseteq \mathcal{V}_{1}$. Therefore we are lead to

$$
\begin{equation*}
\hat{\eta}_{\mathcal{W}_{2}} T\left(\mathcal{W}_{1}, \mathcal{W}_{2}\right)=0 \tag{4.6}
\end{equation*}
$$

Consider now the orthogonal, $J$-invariant and $\bar{\nabla}$-parallel (inside $\mathcal{V}$ ) decomposition $\mathcal{V}=\mathcal{W}_{2} \oplus\left(\mathcal{W}_{1} \oplus \mathcal{V}_{1}\right)$. Using again lemma 4.2, (ii) we get

$$
\begin{equation*}
\hat{\eta}_{\mathcal{W}_{2}} T\left(\mathcal{W}_{1}, \mathcal{W}_{1}\right)=0 \tag{4.7}
\end{equation*}
$$

Now, $T\left(\mathcal{W}_{2}, \mathcal{W}_{2}\right)=0$ (this follows immediately from (4.5) hence by the definition of $\mathcal{W}_{1}$ we have that $\mathcal{W}_{1}$ is generated by $T\left(\mathcal{W}_{1}, \mathcal{W}_{1}\right)$ and $T\left(\mathcal{W}_{2}, \mathcal{W}_{1}\right)$. By means of (4.6) and (4.7) we obtain

$$
\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{1}=0
$$

and the second half of the claim in (ii) is now proved.
Now, $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}$ is orthogonal to $\mathcal{W}_{1}$, but we also know that it is orthogonal to $\mathcal{V}_{1}$ as $T\left(\mathcal{V}_{1}, \mathcal{V}_{1}\right)=0$. It follows that $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1} \subseteq \mathcal{W}_{2}$. Let us define $E=\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}$ and let $F$ be the orthogonal complement of $E$ in $\mathcal{W}_{2}$. Then $\hat{\eta}_{\mathcal{W}_{2}} F$ is orthogonal to $\mathcal{V}_{1}$ and by means (4.5) we obtain that $\hat{\eta}_{\mathcal{W}_{2}} F=0$. Since $T\left(\mathcal{W}_{2}, \mathcal{W}_{2}\right)=0$ we also have $\hat{\eta}_{F} \mathcal{W}_{2}=0$. But $\hat{\eta}_{F} \mathcal{V}_{1} \subseteq E \subseteq \mathcal{W}_{2}$ and then $\hat{\eta}_{F} \mathcal{V}_{1}=0$. Or $F$ is contained in $W_{2}$ hence $\hat{\eta}_{F} \mathcal{W}_{1}=0$. We showed that $\hat{\eta}_{F} \mathcal{V}=0$ and since $F$ is contained in $\mathcal{V}_{0}$ it has to vanish. That is, $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}=\mathcal{W}_{2}$ proving the claim in (iv).

Now (4.3) leads to $\hat{\eta}_{\mathcal{W}_{1}} \mathcal{W}_{2}=0$ which proves completely the claim in (ii). Then $T\left(\mathcal{W}_{1}, \mathcal{W}_{2}\right)=0$ making that $\mathcal{W}_{1}=T\left(\mathcal{W}_{1}, \mathcal{W}_{1}\right)$ where we have taken again into account the vanishing of $T$ on $\mathcal{W}_{2}$. Thus (i) is also proved, and (iii) is an easy consequence of (ii) and of the vanishing of the torsion on $\mathcal{W}_{2}$.

In other words we have proved
Corollary 4.1. Let $(M, g, J)$ belong to the class $\mathcal{A K}_{3}$. With respect to the induced structure, every integral manifold of the distribution $\mathcal{V}$ is locally the Riemannian product of a strict $\mathcal{A K}$ manifold and a normal $\mathcal{A K}$ manifold, both having parallel torsion.

## 5. Curvature properties

In this section we examine some of the curvature properties of a local $\mathcal{A K}_{3}{ }^{-}$ manifold. In fact we will show how transverse to $\mathcal{V}$, this time, curvature behaviour entails the vanishing of the factor $\mathcal{W}_{2}$ in proposition 4.2 ; in other words we will show that integral manifolds of $\mathcal{V}$ are Riemannian products, in the induced metric, of strict $\mathcal{A} \mathcal{K}$-structures with parallel torsion and Kähler manifolds.

Unless otherwise stated, throughout this section $\left(M^{2 m}, g, J\right), m \geq 2$ will be an almost Kähler manifold in the class $\mathcal{A K}_{3}$. All the notations in the previous section will be used without further comment.

Lemma 5.1. Let $V, V_{i}, 1 \leq i \leq 3$ be in $\mathcal{V}$ and $X$ in $H$. We have:
(i) $\bar{R}\left(V_{1}, V_{2}, V_{3}, X\right)=0$;
(ii) $\bar{R}\left(X, V_{1}, V_{2}, V_{3}\right)=-\left\langle\left[\eta_{V_{2}}, \eta_{V_{3}}\right] X, V_{1}\right\rangle$;
(iii) $\left(\bar{\nabla}_{V} \bar{R}\right)\left(X, V_{1}, V_{2}, V_{3}\right)=0$.

Proof. (i) follows from proposition 4.1, (ii) and the integrability of $\mathcal{V}$. To obtain (ii) one uses (i) and the symmetry property of corollary 3.1, (i). Finally, (iii) follows by differentiation from (ii) when taking into account (4.1) as well as proposition (4.1), (i).

We will now use the second Bianchi identity for the canonical Hermitian connection in order to get more information about the algebraic properties of $\eta$ with respect to the decomposition (2.2).

Proposition 5.1. Let $X, V_{i}, 1 \leq i \leq 4$ be vector fields in $H$ and $\mathcal{V}$ respectively. We have
(i) $\bar{R}\left(\eta_{V_{2}} X, V_{1}, V_{3}, V_{4}\right)-\bar{R}\left(\eta_{V_{1}} X, V_{2}, V_{3}, V_{4}\right)=-\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] X, T_{V_{1}} V_{2}\right\rangle$
(ii) $\left(\bar{\nabla}_{X} \bar{R}\right)\left(V_{1}, V_{2}, V_{3}, V_{4}\right)=0$.

Proof. Using the second Bianchi identity we obtain

$$
\begin{aligned}
& \left(\bar{\nabla}_{X} \bar{R}\right)\left(V_{1}, V_{2}, V_{3}, V_{4}\right)+\left(\bar{\nabla}_{V_{1}} \bar{R}\right)\left(V_{2}, X, V_{3}, V_{4}\right)+\left(\bar{\nabla}_{V_{2}} \bar{R}\right)\left(X, V_{1}, V_{3}, V_{4}\right) \\
& +\bar{R}\left(T_{X} V_{1}, V_{2}, V_{3}, V_{4}\right)+\bar{R}\left(T_{V_{1}} V_{2}, X, V_{3}, V_{4}\right)+\bar{R}\left(T_{V_{2}} X, V_{1}, V_{3}, V_{4}\right)=0
\end{aligned}
$$

Now, the second and the third terms of this equation vanish by lemma 5.1,(iii). But the first term is $J$-invariant in $V_{1}$ and $V_{2}$ by corollary 3.1 , (ii) and that all the remaining terms are $J$-anti-invariant in $V_{1}$ and $V_{2}$ since $\eta$ belongs to $\lambda^{1} M \otimes_{2} \lambda^{2} M$. Therefore, (ii) follows and moreover we obtain

$$
\bar{R}\left(T_{X} V_{1}, V_{2}, V_{3}, V_{4}\right)+\bar{R}\left(T_{V_{1}} V_{2}, X, V_{3}, V_{4}\right)+\bar{R}\left(T_{V_{2}} X, V_{1}, V_{3}, V_{4}\right)=0
$$

Since $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$ it suffices to use lemma 5.1, (ii) in order to conclude.
An important consequence of proposition 5.1, (i) is:
Corollary 5.1. We have that $\eta_{\mathcal{V}_{1}} \hat{\eta}_{\mathcal{V}_{0}} \mathcal{V}_{0}=0$.
Proof. Take $V_{3}$ in $\mathcal{V}_{0}, V_{4}$ in $\mathcal{V}_{1}$ and $V_{1}, V_{2}$ in $\mathcal{V}$ in (i) of proposition 5.1. Since $\mathcal{V}_{i}, i=0,1$ are orthogonal and $\bar{\nabla}$-parallel inside $\mathcal{V}$ we have that

$$
\left.\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] X, T_{V_{1}} V_{2}\right)\right\rangle=0
$$

for all $X$ in $H$. By definition $T(\mathcal{V}, \mathcal{V})=\mathcal{V}_{0}$ hence it follows that $\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] X, U\right\rangle=0$ for all $U$ in $\mathcal{V}_{0}$. Because $\eta_{\mathcal{V}_{0}} H \subseteq \mathcal{V}$ by (2.4) and $\eta_{\mathcal{V}_{1}} \mathcal{V} \subseteq H$ by (4.2), we have that $\eta_{V_{4}} \eta_{V_{3}} X$ belongs to $H$, in particular $\left\langle\eta_{V_{4}} \eta_{V_{3}} X, U\right\rangle=0$ for all $U$ in $\mathcal{V}_{0}$. We are left with $0=\left\langle\eta_{V_{3}} \eta_{V_{4}} X, U\right\rangle=-\left\langle\eta_{V_{4}} X, \hat{\eta}_{V_{3}} U\right\rangle=\left\langle X, \eta_{V_{4}} \hat{\eta}_{V_{3}} U\right\rangle$ for all $U$ in $\mathcal{V}_{0}$. In other words $\eta_{\mathcal{V}_{1}} \hat{\eta}_{\nu_{0}} \mathcal{V}_{0}$ is orthogonal to $H$. It is also contained in $\eta_{\mathcal{V}_{1}} \mathcal{V} \subseteq H$ and the claim follows.

With these in hands we set out to show that the space $\mathcal{W}_{2}$ in proposition 4.2 must vanish. We consider the decomposition

$$
\mathcal{V}_{1}=E^{\prime} \oplus E
$$

where $E^{\prime}=\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{2} \subseteq \mathcal{V}_{1}$ and $E$ denotes the orthogonal complement of $E^{\prime}$ in $\mathcal{V}_{1}$. This splitting is $J$-invariant and $\bar{\nabla}$-parallel inside $\mathcal{V}$ and moreover

Lemma 5.2. The distribution $E^{\prime}$ satisfies $\eta_{E^{\prime}} H \subseteq \mathcal{W}_{2}$.
Proof. By corollary 5.1, the fact that $T\left(\mathcal{W}_{1}, \mathcal{W}_{1}\right)=\mathcal{W}_{1}$ (see proposition 4.2, (i)) and the definition of $E^{\prime}$ we obtain easily that

$$
\begin{equation*}
\eta_{\nu_{1}} \mathcal{W}_{1}=0 \tag{5.1}
\end{equation*}
$$

and

$$
\eta_{\nu_{1}} E^{\prime}=0
$$

The second equation gives us $\eta_{E} E^{\prime}=\eta_{E^{\prime}} E^{\prime}=0$. Therefore $\eta_{E^{\prime}} E$ is contained in $T\left(E, E^{\prime}\right)=0$ (since $T\left(\mathcal{V}_{1}, \mathcal{V}_{1}\right)=0$ ) and thus it vanishes, showing that

$$
\begin{equation*}
\eta_{E^{\prime}} \mathcal{V}_{1}=0 \tag{5.2}
\end{equation*}
$$

Hence $\eta_{E^{\prime}} H$ is orthogonal to $\mathcal{V}_{1}$ and by the first equation it is also orthogonal to $\mathcal{W}_{1}$ and the claim follows.

Let the symmetric and $J$-invariant tensor $\hat{r}: \mathcal{V}_{0} \rightarrow \mathcal{V}_{0}$ be defined by

$$
\left\langle\hat{r} V_{0}, W_{0}\right\rangle=\sum_{v_{k} \in \mathcal{V}_{1}}\left\langle\hat{\eta}_{V_{0}} v_{k}, \hat{\eta}_{W_{0}} v_{k}\right\rangle
$$

for all $V_{0}, W_{0}$ in $\mathcal{V}_{0}$, where $\left\{v_{k}\right\}$ is an arbitrary local orthonormal basis in $\mathcal{V}_{1}$.
Lemma 5.3. We have
(i) $\bar{R}\left(V_{1}, W_{1}, V_{0}, W_{0}\right)=-\left\langle\left[\hat{\eta}_{V_{0}}, \hat{\eta}_{W_{0}}\right] V_{1}, W_{1}\right\rangle$ for all $V_{1}, W_{1}$ in $\mathcal{V}_{1}$ and $V_{0}, W_{0}$ in $\mathcal{V}_{0}$;
(ii) $\hat{r}$ preserves $\mathcal{W}_{2}$ and the restriction of $\hat{r}$ to $\mathcal{W}_{2}$ has no kernel;
(iii) let $U$ be in $\mathcal{V}$ such that

$$
\bar{R}\left(U, V_{1}, V_{2}, V_{3}\right)=0
$$

for all $V_{1}$ in $E^{\prime}$ and $V_{2}, V_{3}$ in $\mathcal{V}_{1}$. Then $U \perp E^{\prime}$.
Proof. (i) It suffices to apply lemma 4.2 ,(iii) and (iv).
(ii) An elementary computation based on (i) leads to

$$
\begin{equation*}
\sum_{v_{k} \in \mathcal{V}_{1}} \bar{R}\left(v_{k}, J v_{k}\right) V=-2(\hat{r} J) V \tag{5.3}
\end{equation*}
$$

whenever $V$ belongs to $\mathcal{V}_{0}$. Because $\mathcal{W}_{2}$ is $\bar{\nabla}$-parallel inside $\mathcal{V}$ it must therefore be preserved by $\hat{r}$.

Let $V$ in $\mathcal{W}_{2}$ be such that $\hat{r} V=0$. From the definition of $\hat{r}$ we get by a positivity argument that $\hat{\eta}_{V} \mathcal{V}_{1}=0$. Hence the space $\hat{\eta}_{V} \mathcal{W}_{2}$ is orthogonal to $\mathcal{V}_{1}$ and we know that it is also contained in $\mathcal{V}_{1}$ by proposition 4.2 , (iii). Therefore $\hat{\eta}_{V} \mathcal{W}_{2}=0$, and again the fact that $V$ belongs to $\mathcal{W}_{2}$ yields $\hat{\eta}_{V} \mathcal{W}_{1}=0$ by using proposition 4.2 , (ii). Altogether $\hat{\eta}_{V} \mathcal{V}=0$, in other words $V$ must belong to $\mathcal{V}_{1}$ by (4.2). At the same
time $V$ is in $\mathcal{W}_{2} \subseteq \mathcal{V}_{0} \perp \mathcal{V}_{1}$ and so it must vanish.
(iii) Using the symmetry formula in lemma 4.2, (iv) we obtain that

$$
\bar{R}\left(V_{2}, V_{3}, V_{1}, U\right)=0
$$

for all $V_{2}, V_{3}$ in $\mathcal{V}_{1}$ and $V_{1}$ in $E^{\prime}$. Let now $V_{0}, W_{0}$ be in $\mathcal{W}_{2}$. If $\left\{v_{k}\right\}$ is an orthonormal basis in $\mathcal{V}_{1}$ then by proposition 4.1, (iii) we have that:

$$
\bar{R}\left(v_{k}, J v_{k}\right) \eta_{V_{0}} W_{0}=\eta_{\bar{R}\left(v_{k}, J v_{k}\right) V_{0}} W_{0}+\eta_{V_{0}}\left(\bar{R}\left(v_{k}, J v_{k}\right) W_{0}\right)
$$

We now project on $\mathcal{V}$ and sum over $k$ to find by means of (5.3) that

$$
\sum_{v_{k} \in \mathcal{V}_{1}} \bar{R}\left(v_{k}, J v_{k}\right) \hat{\eta}_{V_{0}} W_{0}=2 J\left[\hat{\eta}_{\hat{r} V_{0}} W_{0}+\hat{\eta}_{V_{0}}\left(\hat{r} W_{0}\right)\right] .
$$

By definition, $E^{\prime}=\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{2}$ hence taking the scalar product with $U$ above yields $\left\langle\hat{\eta}_{\hat{r} V_{0}} W_{0}+\hat{\eta}_{V_{0}}\left(\hat{r} W_{0}\right), J U\right\rangle=0$ for all $V_{0}, W_{0}$ in $\mathcal{W}_{2}$. Because $\hat{r}$ is positive definite on $\mathcal{W}_{2}$, by considering its spectral decomposition we deduce that $J U$ (and thus $U$ ) is orthogonal to $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{2}=E^{\prime}$ and the proof is finished.

These preparations allow to show that
Proposition 5.2. The orthogonal, $J$-invariant and $\bar{\nabla}$-parallel (inside $\mathcal{V}$ ) decomposition $\mathcal{V}=\mathcal{V}_{0} \oplus \mathcal{V}_{1}$ satisfies:
(i) $\mathcal{V}_{0}=T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right)$;
(ii) $\eta_{\mathcal{V}_{1}} \mathcal{V}_{0}=0$.

Proof. We first show that $E^{\prime}=0$. Indeed, let us consider $V_{0}$ in $\mathcal{W}_{2}, W_{0}$ in $E^{\prime}$ and $V_{1}, V_{2}$ in $\mathcal{V}_{1}$ as well as $X$ in $H$. Using proposition 5.1, (i) we obtain that

$$
\bar{R}\left(\eta_{V_{0}} X, W_{0}, V_{1}, V_{2}\right)-\bar{R}\left(\eta_{W_{0}} X, V_{0}, V_{1}, V_{2}\right)=-\left\langle\left[\eta_{V_{1}}, \eta_{V_{2}}\right] X, T_{W_{0}} V_{0}\right\rangle
$$

But $\eta_{W_{0}} X$ belongs to $\mathcal{W}_{2} \subseteq \mathcal{V}_{0}$ by lemma 5.2 , hence the second term of the left hand side vanishes by lemma 4.2 , (iii). Because $T_{W_{0}} V_{0}$ is in $\mathcal{V}$ whilst $\left[\eta_{V_{1}}, \eta_{V_{2}}\right] X$ belongs to $\eta_{\mathcal{V}_{1}}\left(\eta_{\mathcal{V}_{1}} H\right) \subseteq \eta_{\mathcal{V}_{1}} \mathcal{V} \subseteq H$ it follows that the right hand side of the equation above vanishes as well. Hence

$$
\bar{R}\left(\eta_{V_{0}} X, W_{0}, V_{1}, V_{2}\right)=0
$$

for all $W_{0}$ in $E^{\prime}$ and whenever $V_{1}, V_{2}$ belong to $\mathcal{V}_{1}$. Applying lemma 5.3, (iii) we obtain that $\eta_{\mathcal{W}_{2}} H$ is orthogonal to $E^{\prime}$ thus $\eta_{\mathcal{W}_{2}} E^{\prime} \subseteq \mathcal{V}=\mathcal{W}_{1} \oplus \mathcal{W}_{2} \oplus \mathcal{V}_{1}$. But $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{1}=0, \hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}=\mathcal{W}_{2}$ by proposition 4.2 , (ii) and (iv) hence $\eta_{\mathcal{W}_{2}} E^{\prime} \subseteq \mathcal{W}_{2}$. Since $T\left(E^{\prime}, \mathcal{W}_{2}\right) \subseteq \mathcal{V}$ we deduce that $\eta_{E^{\prime}} \mathcal{W}_{2} \subseteq \mathcal{V}$. At the same time, $\eta_{E^{\prime}} \mathcal{W}_{2} \subseteq \eta_{\nu_{1}} \mathcal{V}_{0} \subseteq H$ results in $\eta_{E^{\prime}} \mathcal{W}_{2}=0$. As a consequence lemma 5.2 yields $\eta_{E^{\prime}} H=0$. But using (5.1) we get that $\eta_{E^{\prime}} \mathcal{W}_{1} \subseteq \eta_{\mathcal{V}_{1}} \mathcal{W}_{1}=0$ whilst (5.2) makes that $\eta_{E^{\prime}} \mathcal{V}_{1}=0$.

Collecting the facts above we see that $E^{\prime}$ is contained in $H$, the Kähler nullity of $(g, J)$ which leads to $E^{\prime}=0$.

From the vanishing of $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{W}_{2}$ it follows that $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}$ is orthogonal to $\mathcal{W}_{2}$. However $\hat{\eta}_{\mathcal{W}_{2}} \mathcal{V}_{1}=\mathcal{W}_{2}$ by proposition 4.2 , (iv) hence necessarily $\mathcal{W}_{2}=0$. It follows that $\mathcal{W}_{1}=\mathcal{V}_{0}$ hence (i) holds by proposition 4.2, (i) whilst (ii) follows from (5.1).
5.1. On parallel torsion. We pause now to gather additional information related to the class of $\mathcal{A K}$ manifolds with parallel torsion. We will prove, in this situation, that the Einstein condition and some of its generalisations force integrability; along the way we also obtain some information about the holonomy of the canonical Hermitian connection. These results will be used in the next section to extract the remaining information from proposition 5.2.

Let $\left(M^{2 m}, g, J\right)$ be almost Kähler and have parallel torsion w.r.t. the first canonical connection or equivalently $\bar{\nabla} \eta=0$. By proposition 3.3 it follows that $\left(M^{2 m}, g, J\right)$ belongs to the class $\mathcal{A} \mathcal{K}_{2}$. Define the symmetric and $J$-invariant tensor $r: T M \rightarrow$ $T M$ by

$$
\langle r X, Y\rangle=\sum_{i=1}^{2 m}\left\langle\left(\nabla_{e_{i}} J\right) X,\left(\nabla_{e_{i}} J\right) Y\right\rangle
$$

for all $X$ and $Y$ in $T M$, where $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$ is an arbitrary local orthonormal basis of $T M$. Then $\Psi=\langle r J \cdot$,$\rangle belongs to \lambda^{1,1} M$. We also let $\bar{\rho}$ in $\Lambda^{2} M$ be defined by

$$
\bar{\rho}=\sum_{i=1}^{2 m} \bar{R}\left(e_{i}, J e_{i}\right)
$$

where $\left\{e_{i}, 1 \leq i \leq 2 m\right\}$ is an arbitrary local orthonormal basis in $T M$.
Lemma 5.4. For any $\mathcal{A} \mathcal{K}$-manifold $\left(M^{2 m}, g, J\right)$ with parallel intrinsic torsion tensor we have:
(i) $\nabla^{\star} \nabla \omega=\Psi$;
(ii) $\bar{\rho}=2 \rho+\frac{1}{2} \Psi$;
(iii) $4\langle\rho, \Phi-\Psi\rangle=|\Phi|^{2}+|\Psi|^{2}$;
(iv) $4\langle\rho, \Phi+\Psi\rangle+\langle\Psi, \Phi+\Psi\rangle=0$.

Proof. (i) we have $\left(\nabla_{X} \omega\right)(Y, Z)=-2\left\langle\eta_{X}(J Y), Z\right\rangle$ for all $X, Y, Z$ in $T M$. By using that $\bar{\nabla} \eta=0$ a simple algebraic computation which is left to the reader yields the desired result.
(ii) from (2.6) we get that $\bar{\rho}=2 \rho^{\star}-\frac{1}{2} \Psi$ and the claim follows by using (2.10) and (i).
(iii) we update the integrability condition in proposition 2.3 by the additional information available in the parallel torsion case. We have $R_{22}^{\prime \prime}=0$ by (2.9). Moreover the function $s^{\star}-s=\frac{1}{2}|\nabla J|^{2}$ is constant and Ric ${ }^{\prime \prime}$ vanishes since $(g, J)$ satisfies condition $\left(G_{3}\right)$. The claim follows now by using (i) and proposition 2.3.
(iv) Since the torsion is parallel, we have that $\bar{R}(X, Y) \eta=0$ for all $X, Y$ in $T M$. It follows that $\bar{\rho} \eta=0$ and taking the scalar product with $J \eta$ we obtain after an easy computation that $\langle\bar{\rho}, \Phi+\Psi\rangle=0$. To conclude it suffices now to use (ii).
Theorem 5.1. Let $\left(M^{2 m}, g, J\right)$ be almost Kähler with parallel torsion. The following hold:
(i) If $g$ is Einstein then $J$ is integrable;
(ii) If $J$ is not integrable the connection $\bar{\nabla}$ has complex reducible holonomy.

Proof. We prove both assertions at the same time. Let us suppose that we have

$$
\begin{equation*}
2\langle\rho, \Phi\rangle=\langle\rho, \Psi\rangle \tag{5.4}
\end{equation*}
$$

and prove that $J$ is integrable. Using (5.4), the relations in (iii) and (iv) of lemma 5.4 become

$$
\begin{aligned}
& -2\langle\rho, \Psi\rangle=|\Phi|^{2}+|\Psi|^{2} \\
& 6\langle\rho, \Psi\rangle+|\Psi|^{2}+\langle\Psi, \Phi\rangle=0 .
\end{aligned}
$$

We deduce that $3|\Phi|^{2}+2|\Psi|^{2}=\langle\Phi, \Psi\rangle$. Since $\langle\Phi, \Psi\rangle \leq|\Phi||\Psi|$ we have clearly that $\Psi=\Phi=0$, that is $(g, J)$ is a Kähler structure.

Now, if the manifold is Einstein, (5.4) is clearly satisfied, hence (i) is proven. To prove (ii), suppose that $\bar{\nabla}$ has complex irreducible holonomy. Then the $J$ invariant, $\bar{\nabla}$-parallel forms $\Phi, \Psi$ must be multiples of $\omega$ hence $2 \Phi=\Psi$ so (5.4) is again satisfied.

Note that for geometries with parallel, totally skew-symmetric torsion, in particular in the nearly Kähler case, both instances of real irreducible and complex irreducible holonomy for $\bar{\nabla}$ can occur (see [13, 28]). In the same vein, one can also have integrability results in terms of the Hermitian Ricci tensor $\bar{\rho}$.
Proposition 5.3. Let $\left(M^{2 m}, g, J\right)$ be almost Kähler with parallel torsion. If $\bar{\rho}=0$ then $(g, J)$ is a Kähler structure.

Proof. Because $\bar{\rho}$ vanishes we have that $\rho=-\frac{1}{4} \Psi$ by using lemma 5.4, (ii). Then (iii) in the same lemma leads to $|\Phi|^{2}+\langle\Psi, \Phi\rangle=0$. But $\langle\Psi, \Phi\rangle \geq 0$ because both of $\Psi, \Phi$ are positive semidefinite and the claim follows.

We will rely on the observation above to complete, in the next section, the last part of the argument of the proof of theorem 1.1.

## 6. Local structure of $\mathcal{A K}_{3}$-MANifolds

We shall investigate now some of the geometric implications of the decomposition in proposition 5.2. Let $\tilde{\nabla}$ be the linear connection in $T M_{\tilde{\nabla}}$ obtained by orthogonal projection of $\bar{\nabla}$ onto the splitting $T M=\mathcal{V} \oplus H$. Clearly $\tilde{\nabla} g=0$ and $\tilde{\nabla} J=0$ that is the connection is metric and Hermitian. If $\tilde{\eta}$ is the difference tensor between the connections $\bar{\nabla}$ and $\tilde{\nabla}$, that is $\bar{\nabla}=\tilde{\nabla}+\tilde{\eta}$ it follows that $\tilde{\eta}$ belongs to $\Lambda^{1} M \otimes \lambda^{1,1} M$.

By construction $\tilde{\nabla}$ leaves the distributions $\mathcal{V}$ and $H$ invariant and $\tilde{\eta}_{H} H \subseteq \mathcal{V}$ whilst $\tilde{\eta}_{H} \mathcal{V} \subseteq H$. As a consequence of (i) and (ii) in proposition 4.1 we have that

$$
\tilde{\eta}_{\mathcal{V}}=0 .
$$

Proposition 6.1. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A} \mathcal{K}_{3}$. Then the distribution $\mathcal{V}_{0}$ is $\bar{\nabla}$-parallel.

Proof. Let us denote by $\tilde{\eta}^{+}$and $\tilde{\eta}^{-}$the symmetric resp. the skew-symmetric component of the tensor $\eta$. If $V, W$ and $X, Y$ are in $\mathcal{V}$ and $H$ respectively we must have by proposition 3.2, (ii) that

$$
\left(d_{\bar{\nabla}} \eta\right)(X, V)(W, Y)=\left(d_{\bar{\nabla}} \eta\right)(W, Y)(X, V)
$$

Taking into account the parallelism of $\eta$ over $\mathcal{V}$ in (4.1) it follows that

$$
\left\langle\left(\bar{\nabla}_{X} \eta\right)_{V} W, Y\right\rangle=-\left\langle\left(\bar{\nabla}_{Y} \eta\right)_{W} X, V\right\rangle=\left\langle\left(\bar{\nabla}_{Y} \eta\right)_{W} V, X\right\rangle
$$

We antisymmetrise in $V$ and $W$ and we arrive at

$$
\left\langle\left(\bar{\nabla}_{X} T\right)(V, W), Y\right\rangle+\left\langle\left(\bar{\nabla}_{Y} T\right)(V, W), X\right\rangle=0
$$

Since $T(T M, T M) \subseteq \mathcal{V}$ this is equivalent with $\left\langle T_{V} W, \tilde{\eta}_{X}^{+} Y\right\rangle=0$ so as to obtain that $\tilde{\eta}_{X}^{+} Y$ is in $\mathcal{V}_{1}$. At the same time, by (ii) in proposition 3.2 we have that

$$
\left(d_{\bar{\nabla}} \eta\right)(X, Y)(V, W)=\left(d_{\bar{\nabla}} \eta\right)(V, W)(X, Y)
$$

and once again the parallelism of $\eta$ over $\mathcal{V}$ leads to $\left\langle\left(\bar{\nabla}_{X} \eta\right)_{Y} V-\left(\bar{\nabla}_{Y} \eta\right)_{X} V, W\right\rangle=0$. Since $\eta_{H}=0$ this is equivalent to $\left\langle\eta_{\tilde{\eta}_{X} Y} V, W\right\rangle=0$ and the almost Kähler condition (2.1) ensures that $\left\langle\tilde{\eta}_{X}^{-} Y, T_{V} W\right\rangle=0$ hence $\tilde{\eta}_{X}^{-} Y$ belongs to $\mathcal{V}_{1}$. To summarise we have showed that $\tilde{\eta}_{X} Y$ belongs to $\mathcal{V}_{1}$ therefore $\tilde{\eta}_{H} \mathcal{V}_{0}=0$. In other words

$$
\begin{equation*}
\bar{\nabla}_{X} V_{0} \text { is in } \mathcal{V} \tag{6.1}
\end{equation*}
$$

for all $X$ in $H$ and $V_{0}$ in $\mathcal{V}_{0}$. As a consequence $\left(\bar{\nabla}_{X} T\right)\left(V_{0}, W_{0}\right)$ belongs to $\mathcal{V}$ for all $V_{0}, W_{0}$ in $\mathcal{V}_{0}$, but since $\left\langle\left(\bar{\nabla}_{X} \eta\right)_{V} W, U\right\rangle=0$ for all $V, W, U$ in $\mathcal{V}$ (again by using that $d_{\bar{\nabla}} \eta$ belongs to $S^{2}\left(\lambda^{2} M\right)$ and (4.1)) we see that $\left(\bar{\nabla}_{X} T\right)\left(V_{0}, W_{0}\right)$ belongs to $H$, and hence it must vanish. Explicitly

$$
\bar{\nabla}_{X}\left(T_{V_{0}} W_{0}\right)=T_{\bar{\nabla}_{X} V_{0}} W_{0}+T_{V_{0}} \bar{\nabla}_{X} V_{0}
$$

whenever $V_{0}, W_{0}$ belong to $\mathcal{V}_{0}$. The right hand side above belongs to $T\left(\mathcal{V}, \mathcal{V}_{0}\right) \subseteq \mathcal{V}_{0}$ by (6.1) and the parallelism of $\mathcal{V}_{0}$ w.r.t. $\bar{\nabla}$ follows now from the equality $T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right)=$ $\mathcal{V}_{0}$ in proposition 5.2.

The following lemma is a global version of (i) and (ii) in lemma 4.2. The proof is omitted since entirely analogous.

Lemma 6.1. Let $\left(M^{2 m}, g, J\right)$ be an $\mathcal{A}_{3}$-manifold admitting an orthogonal and $J$ invariant decomposition $T M=D_{1} \oplus D_{2}$ which is also $\bar{\nabla}$-parallel. The following hold:
(i) $\eta_{D_{1}} T\left(D_{2}, D_{2}\right)=0$;
(ii) $\bar{R}(V, W, X, Y)=-\left\langle T_{T_{W} X} V+T_{T_{X} V} W, Y\right\rangle$ whenever $V, W$ are in $D_{1}$ and $X, Y$ in $D_{2}$ respectively.

Corollary 6.1. Let $\left(M^{2 m}, g, J\right)$ belong to the class $\mathcal{A K}_{3}$. Then:
(i) $\eta_{\mathcal{V}_{1}} H=\mathcal{V}_{1}$;
(ii) $\eta_{\mathcal{V}_{0}} \mathcal{V}_{1}=0$.

Proof. (i) Since $\eta_{\mathcal{V}_{1}} \mathcal{V}_{0}=0$ by proposition 5.2 , (ii) we have that $\eta_{\mathcal{V}_{1}} H \subseteq \mathcal{V}_{1}$. Consider the decomposition $\mathcal{V}_{1}=E \oplus F$ with $\eta_{\mathcal{V}_{1}} H=E$ and $F$ the orthogonal complement of $E$ in $\mathcal{V}_{1}$. From the definition of $F$ it follows that $\eta_{\mathcal{V}_{1}} F$ is orthogonal to $H$ and hence it vanishes given that $\eta_{\mathcal{V}_{1}} \mathcal{V}_{1} \subseteq H$. Since $T\left(\mathcal{V}_{1}, \mathcal{V}_{1}\right)=0$ it also follows that $\eta_{F} \mathcal{V}_{1}=0$. It implies that $\eta_{F} H$, which a subspace of $\mathcal{V}_{1}$, is orthogonal to $\mathcal{V}_{1}$, and hence $\eta_{F} H=0$. Finally since $\eta_{F} \mathcal{V}_{0} \subseteq \eta_{\mathcal{V}_{0}} \mathcal{V}_{1}=0$ we conclude that $F$ is contained in the Kähler nullity of $(g, J)$ and then, of course, $F=0$.
(ii) Because $\bar{\nabla}$ is metric it follows from proposition 6.1 that $\mathcal{V}_{1} \oplus H$ is a $\bar{\nabla}$-parallel distribution. Using lemma 6.1, (i) for the $\bar{\nabla}$-parallel decomposition $T M=\mathcal{V}_{0} \oplus$ $\left(\mathcal{V}_{1} \oplus H\right)$ we find that

$$
\eta_{\mathcal{V}_{0}} T\left(\mathcal{V}_{1}, H\right)=\eta_{\mathcal{V}_{0}} \eta_{\mathcal{V}_{1}} H=0 .
$$

Combining this with $\eta_{\nu_{1}} H=\mathcal{V}_{1}$ in (i) finishes the proof of the lemma.

At this stage further screening of the distribution $\mathcal{V}_{0}$ is necessary. We consider the orthogonal decomposition

$$
\mathcal{V}_{0}=E_{1} \oplus E_{2}
$$

where $E_{2}=\eta_{\mathcal{V}_{0}} H$ and $E_{1}$ is the orthogonal complement of $E_{2}$ in $\mathcal{V}_{0}$. It is clear that this is a $J$-invariant decomposition.

Lemma 6.2. The following hold:
(i) $\eta_{E_{2}} E_{1}=\eta_{E_{1}} E_{2}=0$;
(ii) $T\left(E_{i}, E_{i}\right)=E_{i}, i=1,2$.

Proof. From the definition of $E_{1}$ we get that $\eta_{\mathcal{V}_{0}} E_{1} \perp H$ and since $T\left(E_{1}, \mathcal{V}_{0}\right) \subseteq \mathcal{V}_{0}$ it follows that $\eta_{E_{1}} \mathcal{V}_{0}$ is orthogonal to $H$. Equivalently $\eta_{E_{1}} H$ is orthogonal to $\mathcal{V}_{0}$ hence contained in $\mathcal{V}_{1}$ and using that $\eta_{\mathcal{V}_{0}} \mathcal{V}_{1}=0$ from (ii) in corollary 6.1 we arrive at

$$
\begin{equation*}
\eta_{E_{1}} H=0 . \tag{6.2}
\end{equation*}
$$

Consequently, the definition of $E_{2}$ yields

$$
\begin{equation*}
\eta_{E_{2}} H=E_{2} \tag{6.3}
\end{equation*}
$$

From their definition and the parallelism of $\eta$ in vertical directions it is easy to see that $E_{1}$ and $E_{2}$ are $\bar{\nabla}$-parallel inside $\mathcal{V}$. Therefore, taking $V_{3}$ in $E_{1}$ and $V_{4}$ in $E_{2}$ in (i) of proposition 5.1 we find that

$$
\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] X, T_{V_{1}} V_{2}\right\rangle=0
$$

for all $V_{i}$ in $\mathcal{V}, i=1,2$ and $X$ in $H$. Taking into account that $T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right)=\mathcal{V}_{0}$ as of proposition 5.2 , (i) and (6.2) it follows that $\left\langle\eta_{V_{3}} \eta_{V_{4}} X, U\right\rangle=0$ for all $U$ in $\mathcal{V}_{0}$. Now $\eta_{E_{2}} H=E_{2}$ implies that $\eta_{E_{1}} E_{2} \perp \mathcal{V}_{0}$. But $\eta_{E_{1}} E_{2}$ is orthogonal to $H$ by (6.2) and also to $\mathcal{V}_{1}$ by corollary 6.1, (ii). We arrive at

$$
\begin{equation*}
\eta_{E_{1}} E_{2}=0 \tag{6.4}
\end{equation*}
$$

Using (6.4) and the almost Kähler condition we get $T\left(E_{2}, E_{2}\right) \perp E_{1}$ ensuring that $T\left(E_{2}, E_{2}\right) \subseteq E_{2}$. Again by (6.4) $\eta_{E_{1}} E_{1}$ is orthogonal to $E_{2}$ and after antisymmetrisation we arrive at $T\left(E_{1}, E_{1}\right) \subseteq E_{1}$. Thus $T\left(E_{1}, E_{1}\right) \perp E_{2}$ and through the almost Kähler condition (2.1) the spaces $\eta_{E_{2}} E_{1}$ and $E_{1}$ must be orthogonal, in particular $T\left(E_{1}, E_{2}\right)=\eta_{E_{2}} E_{1} \subseteq E_{2}$. Now $\mathcal{V}_{0}=T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right)$, in other words

$$
E_{1} \oplus E_{2}=T\left(E_{1}, E_{1}\right)+T\left(E_{1}, E_{2}\right)+T\left(E_{2}, E_{2}\right)
$$

The above inclusions, followed by a dimension argument yield $T\left(E_{1}, E_{1}\right)=E_{1}$. Using lemma 4.2, (ii) applied to the distributions $D_{1}=E_{1} \oplus \mathcal{V}_{1}$ and $D_{2}=E_{2}$ leads to $\hat{\eta}_{E_{2}} T\left(E_{1}, E_{1}\right)=0$ henceforth $\hat{\eta}_{E_{2}} E_{1}=0$. But $\eta_{E_{2}} E_{1}=T\left(E_{1}, E_{2}\right)$ is contained in $\mathcal{V}_{0}$ by using the definition of the latter, thus $\eta_{E_{2}} E_{1}=0$. The proof of the lemma is now finished.

The last ingredient we need before giving the proof of the main splitting result in this section is related to the behaviour of the curvature tensor $\bar{R}$ on $E_{2}$.
Lemma 6.3. We have $\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{1}, V_{2}, V_{3}, V_{4}\right)=0$ for all $U$ in $\mathcal{V}$ and $V_{i}, 1 \leq i \leq 4$ in $E_{2}$.

Proof. By differentiation of (i) in proposition 5.1 in the direction of $U$ and taking into account the symmetry property of $\bar{R}$ in corollary 3.1 , (i) together with the parallelism of the torsion over $\mathcal{V}$ we find that:

$$
\begin{equation*}
\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}, \eta_{V_{2}} X, V_{1}\right)=\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}, \eta_{V_{1}} X, V_{2}\right) \tag{6.5}
\end{equation*}
$$

for all $X$ in $H$ and $V_{i}$ in $\mathcal{V}, 1 \leq i \leq 4$. Now, by proposition 4.1, (iii), we know that

$$
\bar{R}\left(V_{3}, V_{4}\right)\left(\eta_{V_{2}} X\right)=\eta_{V_{2}} \bar{R}\left(V_{3}, V_{4}\right) X+\eta_{\bar{R}\left(V_{3}, V_{4}\right) V_{2}} X
$$

whenever $V_{i}, 1 \leq i \leq 3$ belong to $\mathcal{V}$ and $X$ is in $H$. Recall that $\mathcal{V}_{0}$ is $\bar{\nabla}$-parallel, and $T\left(\mathcal{V}_{0}, H\right) \subseteq \mathcal{V}_{0}, T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right) \subseteq \mathcal{V}_{0}$; after applying lemma 6.1, (ii) to $D_{1}=\mathcal{V}_{0}$ and $D_{2}=\mathcal{V}_{1} \oplus H$ we find that

$$
\begin{equation*}
\bar{R}\left(V_{3}, V_{4}\right) X=0 \tag{6.6}
\end{equation*}
$$

for all $V_{3}, V_{4}$ in $\mathcal{V}_{0}$ and for all $X$ in $H$. Therefore

$$
\bar{R}\left(V_{3}, V_{4}\right)\left(\eta_{V_{2}} X\right)=\eta_{\bar{R}\left(V_{3}, V_{4}\right) V_{2}} X
$$

for all $V_{2}, V_{3}, V_{4}$ in $\mathcal{V}_{0}$ and whenever $X$ belongs to $H$. Differentiating the last equation in the direction of $U$ and invoking again the parallelism of $\eta$ in the $\mathcal{V}$ direction we compute:

$$
\begin{aligned}
\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}, \eta_{V_{2}} X, V_{1}\right) & =\left\langle\eta_{\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}\right) V_{2}} X, V_{1}\right\rangle=-\left\langle X, \eta_{\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}\right) V_{2}} V_{1}\right\rangle \\
& =-\left\langle X, \eta_{V_{1}}\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}\right) V_{2}\right\rangle=\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}, V_{2}, \eta_{V_{1}} X\right) \\
& =-\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}, \eta_{V_{1}} X, V_{2}\right) .
\end{aligned}
$$

To obtain the third equality above we have used that $\left(\bar{\nabla}_{U} \bar{R}\right)\left(V_{3}, V_{4}\right) V_{2}$ belongs to $\mathcal{V}$ as it follows by lemma 5.1, (i) and also that $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$.

By comparison with (6.5) and since $\eta_{E_{2}} H=E_{2}$ by (6.3), it is now easy to conclude.

The proof of theorem 1.1 in the introduction is now at hand. The argument relies on the properties of the decomposition $T M=\left(\left(E_{0} \oplus E_{1}\right) \oplus \mathcal{V}_{1}\right) \oplus H$ we have previously established; moreover the transverse (to $\mathcal{V}$ ) curvature information which is available turns out to force the vanishing of the Hermitian Ricci form of $E_{2}$, in the induced metric; this makes it possible to use the integrability results in section 5.1.

## Proof of Theorem 1.1:

We are going to show that $E_{2}=0$, in other words $\eta_{\mathcal{V}_{0}} H=0$. To this extent we consider the partial Hermitian Ricci curvature tensor $\bar{\rho}_{E_{2}}: \mathcal{V}_{0} \rightarrow \mathcal{V}_{0}$ defined by

$$
\bar{\rho}_{E_{2}}=\sum_{v_{k} \in E_{2}} \bar{R}\left(v_{k}, J v_{k}\right)
$$

where $\left\{v_{k}\right\}$ is an arbitrary orthonormal basis in $E_{2}$. By taking the appropriate contraction in proposition 4.1, (iii) we get after also using (6.6) that

$$
\bar{\rho}_{E_{2}}\left(\eta_{V} X\right)=\eta_{\bar{\rho}_{E_{2}}}(V) X
$$

for all $V$ in $\mathcal{V}_{0}$ and $X$ in $H$. Writing $\bar{\rho}_{E_{2}}=S J$ where $S$ is a symmetric, $J$-invariant endomorphism of $\mathcal{V}_{0}$ the previous relation reads

$$
\begin{equation*}
S\left(\eta_{V} X\right)=-\eta_{S V} X \tag{6.7}
\end{equation*}
$$

whenever $V$ is in $\mathcal{V}_{0}$ and $X$ in $H$. The tensor $\bar{\rho}_{E_{2}}$ preserves the distribution $E_{2}$, since the latter is $\bar{\nabla}$-parallel inside $\mathcal{V}$. Moreover, lemma 6.3 ensures that the restriction of $S$ to $E_{2}$ is $\bar{\nabla}$-parallell inside $\mathcal{V}$.

We are now going to show that $S=0$. By contradiction, let us assume that the restriction of $S$ to $E_{2}$ has a non-zero eigenfunction $\lambda$, and let us denote the corresponding eigendistribution by $\mathcal{D}_{1}$. We also define $\mathcal{D}_{2}=\eta_{\mathcal{D}_{1}} H$ and note that by (6.7) we have that $\mathcal{D}_{2} \subseteq \operatorname{ker}(S+\lambda)$. In particular $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are orthogonal, and again by (6.7) it follows that $\eta_{\mathcal{D}_{2}} H \subseteq \mathcal{D}_{1}$. Let $\mathcal{D}$ be the orthogonal complement of $\mathcal{D}_{1} \oplus \mathcal{D}_{2}$ in $E_{2}$. Then by (6.7) we get that $\eta_{\mathcal{D}} H \subseteq \mathcal{D}$, since $S \pm \lambda: \mathcal{D} \rightarrow \mathcal{D}$ is injective. Now $\eta_{E_{2}} H=E_{2}$, hence a dimension argument shows that we must have

$$
\begin{equation*}
\eta_{\mathcal{D}_{2}} H=\mathcal{D}_{1}, \quad \eta_{\mathcal{D}} H=\mathcal{D} . \tag{6.8}
\end{equation*}
$$

From the $\bar{\nabla}$-parallelism of $S$ when directions from $\mathcal{V}$ are taken it follows immediately that the distributions $\mathcal{D}_{i}, i=1,2$ and $\mathcal{D}$ are equally $\bar{\nabla}$-parallel inside $\mathcal{V}$. With this fact in mind we will now make use of proposition 5.1,(i). By taking $V_{3}$ in $\mathcal{D}$ and $V_{4}$ in $\mathcal{D}_{1}$ we obtain

$$
\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] X, T\left(V_{1}, V_{2}\right)\right\rangle=0
$$

for all $V_{1}, V_{2}$ in $\mathcal{V}$ and $X$ in $H$. Since $T\left(\mathcal{V}_{0}, \mathcal{V}_{0}\right)=\mathcal{V}_{0}$ it follows further that $\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] X, U\right\rangle=0$ for all $U$ in $\mathcal{V}_{0}$. Now a $J$-invariance argument in the variables $V_{3}, X$ yields $\left\langle\eta_{V_{3}} \eta_{V_{4}} X, U\right\rangle=\left\langle\eta_{V_{4}} \eta_{V_{3}} X, U\right\rangle=0$. Using (6.8), we get that $\eta_{\mathcal{D}} \mathcal{D}_{1}$ and $\eta_{\mathcal{D}_{1}} \mathcal{D}$ are orthogonal to $\mathcal{V}_{0}$. But these spaces are orthogonal to $\mathcal{V}_{1}$ by corollary 6.1, (ii) and also to $H$ by (6.8). Therefore

$$
\begin{equation*}
\eta_{\mathcal{D}} \mathcal{D}_{1}=\eta_{\mathcal{D}_{1}} \mathcal{D}=0 \tag{6.9}
\end{equation*}
$$

and in a completely analogous manner, using the pairs of distributions $\mathcal{D}_{1}, \mathcal{D}_{2}$ and $\mathcal{D}, \mathcal{D}_{2}$ respectively, one arrives to

$$
\begin{equation*}
\eta_{\mathcal{D}_{2}} \mathcal{D}_{1}=\eta_{\mathcal{D}_{1}} \mathcal{D}_{2}=\eta_{\mathcal{D}} \mathcal{D}_{2}=\eta_{\mathcal{D}_{2}} \mathcal{D}=0 \tag{6.10}
\end{equation*}
$$

Using the almost Kähler condition (2.1) it is easy to derive from these properties that

$$
\begin{aligned}
& T(\mathcal{D}, \mathcal{D}) \subseteq \mathcal{D} \\
& T\left(\mathcal{D}_{i}, \mathcal{D}_{i}\right) \subseteq \mathcal{D}_{i}, i=1,2 \\
& T\left(\mathcal{D}, \mathcal{D}_{i}\right)=0, i=1,2
\end{aligned}
$$

But $T\left(E_{2}, E_{2}\right)=E_{2}$ and $E_{2}=\mathcal{W}_{1} \oplus \mathcal{D}_{2} \oplus \mathcal{D}$ hence a dimension argument shows that $T\left(\mathcal{D}_{i}, \mathcal{D}_{i}\right)=\mathcal{D}_{i}, i=1,2$ and $T(\mathcal{D}, \mathcal{D}) \subseteq \mathcal{D}$. In particular:

$$
\begin{equation*}
T\left(\mathcal{D}_{1}, \mathcal{D}_{1}\right)=\mathcal{D}_{1} \tag{6.11}
\end{equation*}
$$

Now, the space $\eta_{\mathcal{D}_{1}} \mathcal{D}_{1}$ is orthogonal to $\mathcal{D}_{2} \oplus \mathcal{D}$ by (6.9) and (6.10), to $E_{1}$ by lemma 6.2 ,(i) to $\mathcal{V}_{1}$ by corollary 6.1 , (ii) and finally to $H$ by (6.8). We conclude that

$$
\begin{equation*}
\eta_{\mathcal{D}_{1}} \mathcal{D}_{1} \subseteq \mathcal{D}_{1} \tag{6.12}
\end{equation*}
$$

Using again (iii) in proposition 4.1 we find that

$$
\bar{\rho}_{E_{2}}\left(\eta_{V} W\right)=\eta_{\bar{\rho}_{E_{2}} V} W+\eta_{V} \bar{\rho}_{E_{2}} W
$$

for all $V, W$ in $\mathcal{D}_{1}$. In view of (6.12) and of the fact that $\bar{\rho}_{E_{2}}$ acts on $\mathcal{D}_{1}$ as $\lambda J$ we obtain that $\eta_{\mathcal{D}_{1}} \mathcal{D}_{1}$ vanishes and then so does $\mathcal{D}_{1}$ by (6.11).

We have obtained a contradiction leading to the fact that $\bar{\rho}_{E_{2}}=0$ on $E_{2}$.
Now the distribution $E_{2} \subseteq \mathcal{V}$ is totally geodesic w.r.t. to $\bar{\nabla}$ and satisfies $T\left(E_{2}, E_{2}\right)=$ $E_{2}$ in particular it is integrable. The canonical connection of the induced structure is obtained by restricting $\bar{\nabla}$ to $E_{2}$. Therefore, with respect to the induced structure the integral manifolds of $E_{2}$ are almost Kähler manifold with parallel torsion and vanishing Hermitian Ricci tensor. By proposition 5.3 such a structure has to be Kähler; the intrinsic torsion of the induced structure is obtained by projecting $\eta$ on $E_{2}$, however because $T\left(E_{2}, E_{2}\right)=E_{2}$ the torsion of the induced structure is given by the restriction of $T$ to $E_{2}$.

It follows that $T$ vanishes on $E_{2}$ hence $E_{2}=0$; then $\eta_{\mathcal{V}_{0}} H=0$ whence $\eta_{\mathcal{V}_{0}} \mathcal{V}_{0} \subseteq \mathcal{V}_{0}$. Since $\mathcal{V}_{0}$ is a $\bar{\nabla}$-parallel distribution, the last conditions yield that $\mathcal{V}_{0}$ is $\nabla$-parallel and it is now straightforward to conclude by using the de Rham splitting theorem.

Under the assumption of parallel torsion it follows that
Theorem 6.1. Let $\left(M^{2 m}, g, J\right)$ be almost Kähler such that $\bar{\nabla} \eta=0$. Then $M$ is locally the Riemannian product of a strict almost Kähler manifold with parallel torsion and a normal almost Kähler manifold with parallel torsion.

In section 7 we will show that normal almost Kähler manifolds with parallel torsion are in fact 3 -symmetric spaces.

## 7. Normal structures

7.1. General observations. Let us first describe a few facts peculiar to the class of normal $\mathcal{A K}_{2}$-structures. All notations from the previous sections will be tacitly used. The connection $\tilde{\nabla}$ introduced in section 6 preserves the metric $g$ and the almost complex structure $J$ as well as the splitting $T M=\mathcal{V} \oplus H$. In proposition 4.1, (iv) it has been showed that $\mathcal{V}$ is an integrable distribution and we will investigate up to what extent this holds for the distribution $H$. Let $\tilde{J}$ be the $g$-orthogonal almost complex structure on $M$ given by

$$
\tilde{J}=-J \text { on } \mathcal{V}, \tilde{J}=J \text { on } H
$$

Proposition 7.1. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}_{2}$-manifold. We have:
(i) $H$ is an integrable distribution;
(ii) $(g, \tilde{J})$ is an almost Kähler structure;
(iii) $(\nabla \tilde{J}) \tilde{J}=-\tilde{\eta}$ in other words $\tilde{\nabla}+\left(\frac{1}{2} \tilde{\eta}-\eta\right)$ is the canonical Hermitian connection of the almost Kähler structure $(g, \tilde{J})$;
(iv) $(g, \tilde{J})$ is a Kähler structure if and only if $\tilde{\eta}=0$.

Proof. (i) By proposition 3.3 we have $\left(\bar{\nabla}_{X} \eta\right)_{Y} U=\left(\bar{\nabla}_{Y} \eta\right)_{X} U$ for all $X, Y$ in $H$ and $U$ in $T M$. Since $\eta_{H}=0$, this is equivalent to $\eta_{[X, Y]} U=0$ and the integrability of $H$ is proved.
(ii) Let us split $\omega=\omega^{\mathcal{V}}+\omega^{H}$ along $T M=\mathcal{V} \oplus H$. Since the Kähler form of $(g, \tilde{J})$ equals $\omega^{\mathcal{\nu}}-\omega^{H}$ and $d \omega=0$ it suffices to show that $d \omega^{\mathcal{V}}=0$. Because $H$ is integrable and $\omega^{\mathcal{V}}$ vanishes on $H$ the component of $d \omega^{\mathcal{V}}$ in $\Lambda^{3} H$ vanishes. If $V$ and $X, Y$ are in $\mathcal{V}$ and $H$ respectively we get by the definition the exterior derivative coupled with the vanishing of $\omega^{\mathcal{L}}$ on $H$ yields $d \omega^{\mathcal{L}}(V, X, Y)=\omega^{\mathcal{L}}(V,[X, Y])=0$ since $H$ is integrable. Now the integrability of $\mathcal{V}$ and $\eta_{H}=0$ yield $d \omega^{\mathcal{V}}=0$ in $\Lambda^{3} \mathcal{V} \oplus\left(\Lambda^{2} \mathcal{V} \oplus \Lambda^{1} H\right)$ and the claim is proved.
(iii) follows by a direct computation involving only the definition of the connection $\tilde{\nabla}$.
(iv) clearly follows from (iii).

Note that (i) above holds in the more general setting of quasi-Kähler $\mathcal{A H}_{2}$ manifolds [21]. To measure how far a normal $\mathcal{A K}$ manifold $\left(M^{2 m}, g, J\right)$ is from being strictly normal we introduce the distribution

$$
H_{0}=\eta_{\mathcal{V}} \mathcal{V} \subseteq H
$$

It is clearly $J$-invariant and its orthogonal complement $H_{1}$ in $H$ is subject to

$$
\begin{equation*}
\eta_{\mathcal{V}} H_{1}=0 \tag{7.1}
\end{equation*}
$$

Clearly $H_{0}$ and $H_{1}$ are $\bar{\nabla}$-parallel along $\mathcal{V}$.
Within the class of $\mathcal{A} \mathcal{K}_{2}$-manifolds the study of normal structures reduces to that of strictly normal ones as the following shows.

Proposition 7.2. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}_{2}$-manifold. Then:
(i) $\tilde{\nabla} \eta=0$;
(ii) $\left[\tilde{\eta}_{U_{1}}, \eta_{U_{2}}\right]=0$ for all $U_{1}, U_{2}$ in $T M$;
(iii) $\left(M^{2 m}, g, J\right)$ is locally the Riemannian product of a Kähler manifold and a strictly normal $\mathcal{A}_{2}$-manifold.

Proof. We prove both (i) \& (ii) simultaneously. From proposition 3.3 we know that $d_{\bar{\nabla}} \eta=0$ in particular $\left(\bar{\nabla}_{X} \eta\right)_{V} Y=0$ for all $X, Y$ in $H$ and for all $V$ in $\mathcal{V}$, where we have also used the parallelism of $\eta$ in the direction of $\mathcal{V}$. It follows that

$$
\left(\tilde{\nabla}_{X} \eta\right)_{V} Y+\left[\tilde{\eta}_{X}, \eta_{V}\right] Y=0
$$

By using the algebraic properties of $\eta$ and $\tilde{\eta}$ together with the invariance of $\mathcal{V}$ under $\tilde{\nabla}$ the first term above belongs to $\mathcal{V}$ whilst the second is in $H$. Therefore $\left(\tilde{\nabla}_{X} \eta\right)_{V} Y=\left[\tilde{\eta}_{X}, \eta_{V}\right] Y=0$. Through similar arguments, from $\left(\bar{\nabla}_{X} \eta\right)_{V} W=0$ for all $X$ in $H$ and $V, W$ in $\mathcal{V}$ we obtain that $\left(\tilde{\nabla}_{X} \eta\right)_{V} W=\left[\tilde{\eta}_{X}, \eta_{V}\right] W=0$. In particular $\left[\tilde{\eta}_{X}, \eta_{V}\right]=0$ hence (ii) is proved since $\eta_{H}$ and $\tilde{\eta}_{\nu}$ both vanish.

Because $\left(\tilde{\nabla}_{X} \eta\right)_{V}=0$ in order to prove (i) it suffices to examine the remaining components of the tensor $\tilde{\nabla} \eta$. Indeed, the compnent on $H \otimes H$ vanishes since $H$ is preserved by $\tilde{\nabla}$ and $\eta_{H}=0$. Finally $\tilde{\nabla}_{V} \eta=0$ for all $V$ in $\mathcal{V}$ since $\eta$ is parallel in the direction of $\mathcal{V}$ and $\tilde{\eta}_{\mathcal{V}}=0$.
(iii) By (i) it follows that $H_{0}$ is parallel w.r.t $\tilde{\nabla}$, hence so is $H_{1}$ since $\tilde{\nabla}$ is metric and preserves $H$. From (7.1) and the commutation formula in (ii) we get $\tilde{\eta}_{T M} H_{1} \subseteq H_{1}$. In other words $H_{1}$ is parallel w.r.t $\bar{\nabla}$ and finally w.r.t the Levi-Civita connection of $g$ because $\eta_{H_{1}}=0$. We conclude by using the de Rham splitting theorem.

The situation is different in the case of $\mathcal{A} \mathcal{K}_{3}$-structures where we have no a priori argument to ensure the integrability of $H$ nor the coincidence, up to products with Kähler manifolds, of normal and strictly normal structures. In the next section we provide more details in this direction.
7.2. Transverse geometry. We begin to study in detail the properties of the reversing almost complex structure $\tilde{J}$. As in section 6 we split the intrinsic torsion tensor $\tilde{\eta}=\tilde{\eta}^{+}+\tilde{\eta}^{-}$along $\otimes^{2} T M \otimes T M=\left(\odot^{2} M \otimes T M\right) \oplus\left(\Lambda^{2} M \otimes T M\right)$. Let $r: T M \rightarrow T M$ be given by

$$
r=-\sum_{i=1}^{2 m} \eta_{e_{i}}^{2}
$$

for some local orthonormal basis in $T M$. We examine below what interaction there is between the intrinsic torsions $\eta$ and $\tilde{\eta}$.

Lemma 7.1. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}_{3}$-manifold. The following hold whenever $V, W$ are in $\mathcal{V}$ and $X, Y, Z$ belong to $H$ :
(i) $\tilde{\eta}_{X}\left(\eta_{V} W\right)=\eta_{V}\left(\tilde{\eta}_{X} W\right)$;
(ii) $\left\langle\tilde{\eta}_{X}\left(\eta_{V} Y\right)-\eta_{V}\left(\tilde{\eta}_{X} Y\right), Z\right\rangle=-2\left\langle\eta_{\tilde{\eta}_{Y}^{-}} X, V\right\rangle$;
(iii) $\tilde{\eta}_{J X} Y=\tilde{\eta}_{X}(J Y)$;
(iv) $\left\langle\tilde{\eta}_{X} Y, \eta_{V} Z\right\rangle=\left\langle\tilde{\eta}_{Z} Y, \eta_{V} X\right\rangle$;
(v) $\tilde{\eta}_{X}(r Y)=\tilde{\eta}_{Y}(r X)$.

Proof. (i) Since $d_{\bar{\nabla}} \eta$ belongs to $S^{2}\left(\lambda^{2} M\right)$ according to proposition 3.2, (ii) the partial parallelism of $\eta$ in (4.1) leads to $\left\langle\left(\bar{\nabla}_{X} \eta\right)(V, W), U\right\rangle=0$ whenever $U$ is in $\mathcal{V}$. The claim is now proved by expanding $\bar{\nabla}=\tilde{\nabla}+\tilde{\eta}$ while keeping in mind that $\tilde{\nabla}$ preserves $\mathcal{V}$ and $H, \eta_{\mathcal{V}} H \subseteq \mathcal{V}$ and that $(g, J)$ is normal, i.e. $\eta_{\mathcal{V}} \mathcal{V} \subseteq H$. (ii) again by the symmetry of $d_{\bar{\nabla}} \eta$ and (4.1) we get

$$
\left\langle\left(\bar{\nabla}_{X} \eta\right)_{V} Y, Z\right\rangle=\left\langle\left(\bar{\nabla}_{Y} \eta\right)_{Z} X-\left(\bar{\nabla}_{Z} \eta\right)_{Y} X, V\right\rangle
$$

The claim follows again by using $\bar{\nabla}=\tilde{\nabla}+\tilde{\eta}$ and projecting along $T M=\mathcal{V} \oplus H$. (iii) by proposition 3.2 , (iii) with $X, Y, Z$ in $H$ we obtain $\eta_{\tilde{\eta}_{J X} J Y+\tilde{\eta}_{X} Y}=0$ hence $\tilde{\eta}_{J X} J Y+\tilde{\eta}_{X} Y$ belongs to $H$; since it is also in $\mathcal{V}$ (by the definition of $\tilde{\eta}$ ) it has to vanish.
(iv) we first note that by the definition of $\tilde{\eta}$ together with the symmetry of $\eta$ on $\mathcal{V}$ in the left resp. the right hand side of (ii) one obtains

$$
\begin{equation*}
\left\langle\tilde{\eta}_{X} Y, \eta_{V} Z\right\rangle-\left\langle\tilde{\eta}_{X} Z, \eta_{V} Y\right\rangle=-2\left\langle\tilde{\eta}_{Y}^{-} Z, \eta_{V} X\right\rangle . \tag{7.2}
\end{equation*}
$$

After taking the symmetric sum on $X, Y, Z$ in (7.2) we arrive at $\sigma_{X, Y, Z}\left\langle\tilde{\eta}_{X}^{-} Y, \eta_{V} Z\right\rangle=$ 0 . Swapping $X, Z$ in (7.2), and substracting the result from the same equation we get $\left\langle\tilde{\eta}_{X} Y, \eta_{V} Z\right\rangle-\left\langle\tilde{\eta}_{Z} Y, \eta_{V} X\right\rangle=-2 \sigma_{X, Y, Z}\left\langle\tilde{\eta}_{X}^{-} Y, \eta_{V} Z\right\rangle=0$.
(v) by (i) we have for any $U$ in $\mathcal{V}$

$$
\left\langle\tilde{\eta}_{X} \eta_{V} W, \eta_{U} Y\right\rangle=\left\langle\eta_{V}\left(\tilde{\eta}_{X} W\right), \eta_{U} Y\right\rangle=-\left\langle\tilde{\eta}_{X} W, \eta_{V} \eta_{U} Y\right\rangle .
$$

Now (iv) ensures the symmetry in $X$ and $Y$ of the left hand side term hence taking the trace over $\mathcal{V}$ in $V$ and $U$ yields $\tilde{\eta}_{X}(r Y)=\tilde{\eta}_{Y}(r X)$.

To supply additional algebraic information on the tensor $\tilde{\eta}$ we need the following.

Lemma 7.2. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A}_{3}$-structure. The following hold whenever $X, Y$ belong to $H$ and $V, W$ are in $\mathcal{V}$ :
(i) $\bar{R}(V, X, W, Y)=\left\langle\left(\bar{\nabla}_{V} \tilde{\eta}\right)_{X} Y, W\right\rangle-\left\langle\tilde{\eta}_{\tilde{\eta}_{X} V} W, Y\right\rangle$;
(ii) $\left\langle\left(\bar{\nabla}_{V} \tilde{\eta}^{+}\right)_{X} Y, W\right\rangle=\left\langle\left(\bar{\nabla}_{W} \tilde{\eta}^{+}\right)_{X} Y, V\right\rangle$;
(iii) $\left\langle\tilde{\eta}_{\tilde{\eta}_{X} V} W, Y\right\rangle=\left\langle\tilde{\eta}_{\tilde{\eta}_{Y} W} V, X\right\rangle$;
(iv) $\bar{R}(V, W, X, Y)=\left\langle\tilde{\eta}_{\tilde{\eta}_{X} W} V, Y\right\rangle-\left\langle\tilde{\eta}_{\tilde{\eta}_{X} V} W, Y\right\rangle$;
(v) $\bar{\nabla}_{V} \tilde{\eta}^{-}=0$.

Proof. (i) is an immediate consequence of the definition of the curvature tensor taking into account only proposition 4.1, (i) and (ii). It will be therefore left to the reader.
(ii) \& (iii) will be proved at the same time. From corollary 3.1, (i) and the properties of $\eta$ w.r.t to the splitting $T M=\mathcal{V} \oplus H$ we get $\bar{R}(V, X, W, Y)=\bar{R}(W, Y, V, X)$. Therefore (i) leads to:

$$
\left\langle\left(\bar{\nabla}_{V} \tilde{\eta}\right)_{X} Y, W\right\rangle-\left\langle\tilde{\eta}_{\tilde{\eta}_{X} V} W, Y\right\rangle=\left\langle\left(\bar{\nabla}_{W} \tilde{\eta}\right)_{Y} X, V\right\rangle-\left\langle\tilde{\eta}_{\tilde{\eta}_{Y} W} V, X\right\rangle .
$$

But due to lemma 7.1, (iii) the first terms of each side of the previous equality are $J$-anti-invariant in $X, Y$ whilst the second ones are $J$-invariant. We conclude that

$$
\begin{equation*}
\left\langle\left(\bar{\nabla}_{V} \tilde{\eta}\right)_{X} Y, W\right\rangle=\left\langle\left(\bar{\nabla}_{W} \tilde{\eta}\right)_{Y} X, V\right\rangle \tag{7.3}
\end{equation*}
$$

and

$$
\left\langle\tilde{\eta}_{\tilde{\eta}_{X} V} W, Y\right\rangle=\left\langle\tilde{\eta}_{\tilde{\eta}_{Y} W} V, X\right\rangle .
$$

This proves (iii) and (ii) follows from (7.3) by symmetrisation.
(iv) \& (v) the first Bianchi identity for $\bar{\nabla}$ in corollary 3.1, (iii) and the vanishing of the torsion on $\mathcal{V}$ yield

$$
\bar{R}(V, W, X, Y)=\bar{R}(V, X, W, Y)-\bar{R}(W, X, V, Y)
$$

Using now (i) when recording that the anti-symmetric part of (7.3) leads to

$$
\left\langle\left(\bar{\nabla}_{V} \tilde{\eta}^{-}\right)_{X} Y, W\right\rangle+\left\langle\left(\bar{\nabla}_{W} \tilde{\eta}^{-}\right)_{X} Y, V\right\rangle=0
$$

and further to

$$
\bar{R}(V, W, X, Y)=2\left\langle\left(\bar{\nabla}_{V} \tilde{\eta}^{-}\right)_{X} Y, W\right\rangle+\left\langle\tilde{\eta}_{\tilde{\eta}_{X} W} V, Y\right\rangle-\left\langle\tilde{\eta}_{\tilde{\eta}_{X} V} W, Y\right\rangle
$$

To conclude it is enough to observe that the first term in the right hand side above is $J$-anti-invariant in $(X, Y)$ whilst all the remaining terms of the equation are $J$-invariant in $(X, Y)$.

We can provide now additional information on the type of the almost-Hermitian structure $(g, \tilde{J})$.

Proposition 7.3. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}_{3}$-structure. We have:
(i) $\tilde{\eta}_{H_{1}} H=0$;
(ii) $\tilde{\eta}_{H_{0}}^{-} H_{0}=0$, in other words the restriction of $\tilde{\eta}$ to $H_{0}$ is symmetric.

Proof. (i) we take $Z$ in $H_{1}$ in lemma 7.1, (iv), use (7.1) and recall that $\eta_{\mathcal{V}} H=\mathcal{V}$.
(ii) Let $V, W$ be in $\mathcal{V}$ and $X, Y, Z, Z^{\prime}$ belong to $H$. By (iii) in lemma 7.2 we obtain

$$
\left\langle\tilde{\eta}_{\tilde{\eta}_{X}\left(\eta_{V} Z\right)} W, Y\right\rangle=\left\langle\tilde{\eta}_{\tilde{\eta}_{Y} W} \eta_{V} Z, X\right\rangle=\left\langle\tilde{\eta}_{Z} \eta_{V} \tilde{\eta}_{Y} W, X\right\rangle
$$

where for the last step we used lemma 7.1, (iv). In particular, the transformation $W \mapsto \eta_{W} Z^{\prime}$ yields

$$
\left\langle\tilde{\eta}_{\tilde{\eta}_{X}\left(\eta_{V} Z\right)} \eta_{W} Z^{\prime}, Y\right\rangle=\left\langle\tilde{\eta}_{Z} \eta_{V} \tilde{\eta}_{Y} \eta_{W} Z^{\prime}, X\right\rangle .
$$

By using again (iv) of lemma 7.1 in the left hand side above we get

$$
\left\langle\tilde{\eta}_{Z^{\prime}} \eta_{W} \tilde{\eta}_{X}\left(\eta_{V} Z\right), Y\right\rangle=\left\langle\tilde{\eta}_{Z} \eta_{V} \tilde{\eta}_{Y} \eta_{W} Z^{\prime}, X\right\rangle
$$

Now $\eta_{W} \tilde{\eta}_{X}\left(\eta_{V} Z\right)=\eta_{W} \tilde{\eta}_{Z}\left(\eta_{V} X\right)=\tilde{\eta}_{Z}\left(\eta_{W} \eta_{V} X\right)$ by applying succesively (iv) and (i) in lemma 7.1, (iv) and (i); similarly $\eta_{V} \tilde{\eta}_{Y} \eta_{W} Z^{\prime}=\tilde{\eta}_{Z^{\prime}}\left(\eta_{V} \eta_{W} Y\right)$. Therefore

$$
\left\langle\tilde{\eta}_{Z^{\prime}} \tilde{\eta}_{Z}\left(\eta_{W} \eta_{V} X\right), Y\right\rangle=\left\langle\tilde{\eta}_{Z} \tilde{\eta}_{Z^{\prime}}\left(\eta_{V} \eta_{W} Y\right), X\right\rangle
$$

After taking the trace in $V$ and $W$ we get $\left\langle\tilde{\eta}_{Z^{\prime}} \tilde{\eta}_{Z}(r X), Y\right\rangle=\left\langle\tilde{\eta}_{Z} \tilde{\eta}_{Z^{\prime}}(r Y), X\right\rangle$, hence $\left\langle\tilde{\eta}_{Z}(r X), \tilde{\eta}_{Z^{\prime}} Y\right\rangle=\left\langle\tilde{\eta}_{Z^{\prime}}(r Y), \tilde{\eta}_{Z} X\right\rangle$. But the right hand side in the previous equation is symmetric in $\left(Z^{\prime}, Y\right)$ by lemma 7.1 , (v) showing that $\left\langle\tilde{\eta}_{Z}(r X), \tilde{\eta}_{Z^{\prime}}^{-} Y\right\rangle=0$. The definition of $H_{0}$ ensures that $\operatorname{Im}\left(r_{\mid H}\right)=H_{0}$ hence $\left\langle\tilde{\eta}_{Z} X, \tilde{\eta}_{Z^{\prime}}^{-} Y\right\rangle=0$ whenever $X$ belongs to $H_{0}$ and $Z, Z^{\prime}, Y$ are in $H$. The vanishing of $\tilde{\eta}^{-}$on $H_{0}$ follows by antisymetrisation in $(Z, X)$ and a positivity argument.

In particular, if $(g, J)$ is strictly normal we have $H_{0}=H$ which is therefore an integrable distribution by the proposition above.

At this stage we need to supply more information on the curvature tensor $\bar{R}$ as follows.

Lemma 7.3. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}_{3}$ manifold. We have:

$$
2 \bar{R}\left(V_{3}, V_{4}, V_{2}, \eta_{V_{1}} X\right)=\bar{R}\left(V_{3}, V_{4}, X, \eta_{V_{1}} V_{2}\right)
$$

for all $V_{i}, 1 \leq i \leq 4$ in $\mathcal{V}$ and whenever $X$ is in $H$.
Proof. By proposition 5.1, (i)

$$
\bar{R}\left(\eta_{V_{2}} X, V_{1}, V_{3}, V_{4}\right)-\bar{R}\left(\eta_{V_{1}} X, V_{2}, V_{3}, V_{4}\right)=-\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] X, T_{V_{1}} V_{2}\right\rangle
$$

hence $\bar{R}\left(\eta_{V_{2}} X, V_{1}, V_{3}, V_{4}\right)=\bar{R}\left(\eta_{V_{1}} X, V_{2}, V_{3}, V_{4}\right)$ since $T(\mathcal{V}, \mathcal{V})=0$. Now, using the symmetry property of $\bar{R}$ in corollary 3.1, (i) we obtain:

$$
\begin{aligned}
& \bar{R}\left(V_{3}, V_{4}, \eta_{V_{2}} X, V_{1}\right)+\left\langle\left[\eta_{\eta_{V_{2}} X}, \eta_{V_{1}}\right] V_{3}, V_{4}\right\rangle-\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] \eta_{V_{2}} X, V_{1}\right\rangle \\
= & \bar{R}\left(V_{3}, V_{4}, \eta_{V_{1}} X, V_{2}\right)+\left\langle\left[\eta_{\eta_{V_{1}} X}, \eta_{V_{2}}\right] V_{3}, V_{4}\right\rangle-\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] \eta_{V_{1}} X, V_{2}\right\rangle .
\end{aligned}
$$

Since the restriction of $\eta$ to $\mathcal{V}$ is symmetric a standard computation yields

$$
\left\langle\left[\eta_{\eta_{V_{2}} X}, \eta_{V_{1}}\right] V_{3}, V_{4}\right\rangle-\left\langle\left[\eta_{V_{3}}, \eta_{V_{4}}\right] \eta_{V_{2}} X, V_{1}\right\rangle=0
$$

thus

$$
\begin{equation*}
\bar{R}\left(V_{3}, V_{4}, \eta_{V_{2}} X, V_{1}\right)=\bar{R}\left(V_{3}, V_{4}, \eta_{V_{1}} X, V_{2}\right) \tag{7.4}
\end{equation*}
$$

But $\left(\bar{R}\left(V_{3}, V_{4}\right) \eta\right)\left(V_{2}, X\right)=0$ by proposition 4.1, (iii) hence the previous equation is updated to

$$
\left\langle\eta_{\bar{R}\left(V_{3}, V_{4}\right) V_{2}} X, V_{1}\right\rangle+\left\langle\eta_{V_{2}} \bar{R}\left(V_{3}, V_{4}\right) X, V_{1}\right\rangle=\bar{R}\left(V_{3}, V_{4}, \eta_{V_{1}} X, V_{2}\right)
$$

By proposition 4.1, (ii), the operator $\bar{R}\left(V_{3}, V_{4}\right)$ preserves $\mathcal{V}$ and $H$. Again by using the symmetry of $\eta$ on $\mathcal{V}$ we compute
$\left\langle\eta_{\bar{R}\left(V_{3}, V_{4}\right) V_{2}} X, V_{1}\right\rangle=-\left\langle X, \eta_{\bar{R}\left(V_{3}, V_{4}\right) V_{2}} V_{1}\right\rangle=-\left\langle X, \eta_{V_{1}} \bar{R}\left(V_{3}, V_{4}\right) V_{2}\right\rangle=\bar{R}\left(V_{3}, V_{4}, V_{2}, \eta_{V_{1}} X\right)$
and it is now straightforward to conclude.
The relation in lemma 7.3 shows that the restriction of $\bar{R}$ to $\mathcal{V}$ is completely determined by the component of $\bar{R}$ on $\Lambda^{2} \mathcal{V} \otimes \Lambda^{2} H$ which, in turn, depends only on the intrinsic torsions $\eta$ and $\tilde{\eta}$. We compute now some curvature contractions of relevance, including the restriction of the Ricci tensor to $\mathcal{V}$. Define the symmetric and $J$-invariant tensors $r_{1}: \mathcal{V} \rightarrow \mathcal{V}$ and $r_{2}: H \rightarrow H$ by:

$$
\begin{equation*}
\left\langle r_{1} V, W\right\rangle=\sum_{e_{i} \in H}\left\langle\left(\tilde{\eta}_{e_{i}} V\right)_{H_{0}},\left(\tilde{\eta}_{e_{i}} W\right)_{H_{0}}\right\rangle,\left\langle r_{2} X, Y\right\rangle=\sum_{v_{k} \in \mathcal{V}}\left\langle\left(\tilde{\eta}_{X} v_{k}\right)_{H_{0}},\left(\tilde{\eta}_{Y} v_{k}\right)_{H_{0}}\right\rangle \tag{7.5}
\end{equation*}
$$

for all $X, Y, V, W$ in $H$ and $\mathcal{V}$ respectively, where $\left\{v_{k}\right\},\left\{e_{i}\right\}$ are arbitrary orthonormal basis in $\mathcal{V}$ and $H$ respectively and the subscript indicates orthogonal projection.
Proposition 7.4. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}_{3}$-manifold. We have:
(i) $\sum_{v_{k} \in \mathcal{V}} \bar{R}\left(V, W, v_{k}, J v_{k}\right)=\left\langle r_{1}(J V), W\right\rangle$ for all $V, W$ in $\mathcal{V}$;
(ii) $\sum_{v_{k} \in \mathcal{V}} \bar{R}\left(v_{k}, J v_{k}, X, Y\right)=-2\left\langle r_{2} J X, Y\right\rangle$ whenever $X, Y$ belong to $H$;
(iii) $\rho(V, W)=-\frac{1}{2}\left\langle r_{1} J V, W\right\rangle$ for all $V, W$ in $\mathcal{V}$
where $\left\{v_{k}\right\}$ is some orthonormal basis in $\mathcal{V}$.
Proof. First we record that by (iv) in lemma 7.2 and (i) in proposition 7.3 we get

$$
\begin{equation*}
\bar{R}(\mathcal{V}, \mathcal{V}) H_{1}=0 \tag{7.6}
\end{equation*}
$$

and moreover by also using the symmetry of $\tilde{\eta}$ on $H_{0}$

$$
\begin{equation*}
\bar{R}(V, W, X, Y)=\left\langle\left(\tilde{\eta}_{X} W\right)_{H_{0}},\left(\tilde{\eta}_{Y} V\right)_{H_{0}}\right\rangle-\left\langle\left(\tilde{\eta}_{X} V\right)_{H_{0}},\left(\tilde{\eta}_{Y} W\right)_{H_{0}}\right\rangle \tag{7.7}
\end{equation*}
$$

for all $V, W$ in $\mathcal{V}$ and whenever $X, Y$ belong to $H_{0}$.
(i) we will use repeatedly, as indicated, that $\tilde{\eta}$ is symmetric on $H_{0}$, according to (ii) in proposition 7.3. From lemma (7.3)

$$
2 \sum_{v_{k} \in \mathcal{V}} \bar{R}\left(v_{k}, J v_{k}, V, \eta_{W} X\right)=\sum_{v_{k} \in \mathcal{V}} \bar{R}\left(v_{k}, J v_{k}, X, \eta_{V} W\right)
$$

whenever $X$ is in $H$; in fact we will assume in what follows that $X$ belongs to $H_{0}$ since $\eta_{\mathcal{V}} H_{1}=0$. Now

$$
\bar{R}\left(v_{k}, J v_{k}, X, \eta_{V} W\right)=-2\left\langle\left(\tilde{\eta}_{X} v_{k}\right)_{H_{0}},\left(\tilde{\eta}_{\eta_{V} W} J v_{k}\right)_{H_{0}}\right\rangle
$$

by (7.7) and since $[\tilde{\eta}, J]=0$. After summation,
$\sum_{v_{k} \in \mathcal{V}} \bar{R}\left(v_{k}, J v_{k}, V, \eta_{W} X\right)=-\sum_{v_{k} \in \mathcal{V}, e_{i} \in H_{0}}\left\langle\tilde{\eta}_{X} v_{k}, e_{i}\right\rangle\left\langle\tilde{\eta}_{\eta_{V} W} J v_{k}, e_{i}\right\rangle=\sum_{e_{i} \in H_{0}}\left\langle\tilde{\eta}_{e_{i}} X, J \tilde{\eta}_{e_{i}} \eta_{V} W\right\rangle$
because $\tilde{\eta}$ is symmetric on $H_{0}$ and $\eta_{V} W$ belongs to $H_{0}$. By lemma 7.1, (i)

$$
\begin{aligned}
\left\langle\tilde{\eta}_{e_{i}} X, J \tilde{\eta}_{e_{i}} \eta_{V} W\right\rangle & =\left\langle\tilde{\eta}_{e_{i}} X, J \eta_{V} \tilde{\eta}_{e_{i}} W\right\rangle=\left\langle\eta_{V} \tilde{\eta}_{e_{i}} J X, \tilde{\eta}_{e_{i}} W\right\rangle \\
& =\left\langle\tilde{\eta}_{e_{i}} \eta_{V} J X, \tilde{\eta}_{e_{i}} W\right\rangle+2\left\langle\eta_{\tilde{\eta}_{J X}^{-}\left(\tilde{\eta}_{e_{i}} W\right)} e_{i}, V\right\rangle
\end{aligned}
$$

where in the last equality we have taken into account lemma 7.1, (ii). However, $X$ is in $H_{0}$ and $\tilde{\eta}_{H_{1}} H_{0}=0$ hence $\tilde{\eta}_{J X}^{-}\left(\tilde{\eta}_{e_{i}} W\right)=\frac{1}{2} \tilde{\eta}_{J X}\left(\tilde{\eta}_{e_{i}} W\right)_{H_{1}}$. Therefore

$$
2\left\langle\eta_{\tilde{\eta}_{J X}^{-}\left(\tilde{\eta}_{e_{i}} W\right)} e_{i}, V\right\rangle=\left\langle\eta_{V} e_{i}, \tilde{\eta}_{J X}\left(\tilde{\eta}_{e_{i}} W\right)_{H_{1}}\right\rangle=\left\langle\eta_{V} J X, \tilde{\eta}_{e_{i}}\left(\tilde{\eta}_{e_{i}} W\right)_{H_{1}}\right\rangle
$$

because $\eta$ is symmetric on $\mathcal{V}$ and by also using lemma 7.1, (iv). Summarising, we have proved that

$$
\begin{aligned}
\sum_{v_{k} \in \mathcal{V}} \bar{R}\left(v_{k}, J v_{k}, V, \eta_{W} X\right) & =\sum_{e_{i} \in H_{0}}\left\langle\tilde{\eta}_{e_{i}} \eta_{V} J X, \tilde{\eta}_{e_{i}} W\right\rangle-\sum_{e_{i} \in H_{0}}\left\langle\tilde{\eta}_{e_{i}} \eta_{V} J X,\left(\tilde{\eta}_{e_{i}} W\right)_{H_{1}}\right\rangle \\
& =-\left\langle r_{1}\left(J \eta_{V} X\right), W\right\rangle
\end{aligned}
$$

Since the r.h.s. above is symmetric in $V, W$ by lemma 7.3 and $\eta_{\mathcal{V}} H_{0}=\mathcal{V}$ the claim follows.
(ii) is an imediate consequence of (7.7) and (7.6).
(iii) from (i) and again (7.7) we get for the first Chern form of $(g, J)$

$$
\gamma_{1}(V, W)=\frac{1}{2}\left\langle r_{1} J V, W\right\rangle+\frac{1}{2} \sum_{e_{i} \in H} \bar{R}\left(V, W, e_{i}, J e_{i}\right)=-\frac{1}{2}\left\langle r_{1} J V, W\right\rangle
$$

Because $(g, J)$ belongs to the class $\mathcal{A K}_{3}$ we have $\delta_{\bar{\nabla}} \eta=0$ hence $\gamma_{1}=\rho+\frac{1}{2}(\Psi-\Phi)$ by (2.12). Furthermore, because the structure is normal an easy computation yields that $\Psi(V, W)=\Phi(V, W)$ for all $V, W$ in $\mathcal{V}$ and the claim follows.

A subclass of interest for what follows is introduced below.
Definition 7.1. $A n \mathcal{A K}_{3}$ structure $\left(M^{2 m}, g, J\right)$ is called of null type if it is normal and $\bar{R}(\mathcal{V}, \mathcal{V})=0$, where $\mathcal{V}$ is the orthogonal to the Kähler nullity.

The examples in section 3.2 are all of null type, however not necesarily strictly normal. Being of null type is equivalent by proposition 7.4 with

$$
\tilde{\eta}_{H_{0}}=0
$$

Therefore, if a structure of null type has integrable Kähler nullity, $\tilde{\eta}$ vanishes, that is $(g, \tilde{J})$ is Kähler.

To finish this section we will obtain sufficient conditions to ensure that a normal $\mathcal{A} \mathcal{K}_{3}$ manifold $\left(M^{2 m}, g, J\right)$ is null or to have the reversing almost Hermitian structure $(g, \tilde{J})$ Kähler. This mainly takes into account that the Ricci tensor of $g$ is negative over $\mathcal{V}$. In the compact case a similar argument has been used in [29].

Proposition 7.5. Let $\left(M^{2 m}, g, J\right), m \geq 2$ be a normal $\mathcal{A K}_{3}$-manifold. Then:
(i) we have $\Delta_{\mathcal{V}} \operatorname{Tr}\left(r_{1}\right)=-\left(\left|r_{1}\right|^{2}+4\left|r_{2}\right|^{2}\right)-2\left|\bar{\nabla}_{\mathcal{V}} \tilde{\eta}\right|^{2}$. Here, $\bar{\nabla}_{\mathcal{V}}$ denotes the restriction of $\bar{\nabla}$ to $\mathcal{V}$ and $\Delta_{\mathcal{V}}$ is the corresponding partial Laplacian, acting on functions;
(ii) if $\operatorname{Tr}\left(r_{1}\right)$ is constant along $\mathcal{V}$ then $(g, J)$ is of null type;
(iii) if $\operatorname{Tr}\left(r_{1}\right)$ is constant along $\mathcal{V}$ and $H$ is integrable then $(g, \tilde{J})$ is a Kähler structure.

Proof. (i) From lemma 7.2, (ii) we deduce that

$$
\begin{equation*}
\left(\bar{\nabla}_{J V} \tilde{\eta}\right)_{J X} Y=\left(\bar{\nabla}_{V} \tilde{\eta}\right)_{X} Y \tag{7.8}
\end{equation*}
$$

for all $V$ in $\mathcal{V}$ and $X, Y$ in $H$ respectively. We consider now the partial Bochner Laplacian $D^{\mathcal{V}}$, acting on $\tilde{\eta}$ by $\left(D^{\mathcal{V}} \tilde{\eta}\right)(X, Y)=-\sum_{v_{k} \in \mathcal{V}}\left(\bar{\nabla}_{v_{k}, v_{k}}^{2} \tilde{\eta}\right)(X, Y)$ for all $X, Y$ in
$H$, where $\left\{v_{k}\right\}$ is an arbitrary local orthonormal basis of $\mathcal{V}$. Differentiating (7.8) when taking into account that $H_{0}$ is $\bar{\nabla}$-parallel along $\mathcal{V}$ and proposition 7.4 we get

$$
\left(D^{\mathcal{V}} \tilde{\eta}\right)(X, Y)=\frac{1}{2} J \sum_{v_{k} \in \mathcal{V}}\left(\bar{R}\left(v_{k}, J v_{k}\right) \tilde{\eta}\right)(X, Y)=-\frac{1}{2}\left(r_{1} \tilde{\eta}_{X} Y+2 \tilde{\eta}_{r_{2} X} Y+2 \tilde{\eta}_{X} r_{2} Y\right)
$$

for all $X, Y$ in $H_{0}$. Taking the scalar product with $\tilde{\eta}$ above yields $\left\langle D^{\mathcal{V}} \tilde{\eta}, \tilde{\eta}\right\rangle=$ $-\frac{1}{2}\left(\left|r_{1}\right|^{2}+4\left|r_{2}\right|^{2}\right)$, when keeping in mind that the restriction of $\tilde{\eta}$ to $H_{0}$ is symmetric; we conclude by means of the standard Weitzenböck formula $\frac{1}{2} \Delta_{\mathcal{V}}|\tilde{\eta}|_{H_{0}}^{2}=\left\langle D^{\mathcal{V}} \tilde{\eta}, \tilde{\eta}\right\rangle-$ $|\bar{\nabla} \mathcal{V} \tilde{\eta}|^{2}$ where $|\tilde{\eta}|_{H_{0}}^{2}=\sum_{e_{i}, e_{j} \in H_{0}}\left|\tilde{\eta}_{e_{i}} e_{j}\right|^{2}=\operatorname{Tr}\left(r_{1}\right)$.
(ii) from (i) we get after a positivity argument that $r_{1}=0$ hence the restriction of $\tilde{\eta}$ to $H_{0}$ vanishes, that is $(g, J)$ has null type. (iii) follows from (ii) and proposition 7.1, (iv).

Remark 7.1. (i) In fact proposition 7.5 (i) is a particular case of the Walczak formula $[38,37]$; in the $\mathcal{A} \mathcal{K}_{2}$ case it can be also recovered from proposition 2.3 applied to the almost Kähler structure $(g, \tilde{J})$. For self-containdeness reasons we have adopted here the direct approach.
(ii) if a normal almost-Kähler manifold $\left(M^{2 m}, g, J\right)$ in the class $\mathcal{A K}_{3}$ has the property that $(g, \tilde{J})$ belongs to the class $\mathcal{A} \mathcal{K}_{2}$ then the function $|\nabla \tilde{J}|^{2}$ is constant by proposition 7.2, (i) hence (iii) in proposition 7.5 ensures that $(g, \tilde{J})$ is in fact a Kähler structure.
In this paper we will mainly use the criterion above in two situations: for Einstein, $\mathcal{A} \mathcal{K}_{3}$ structures, below, and in the next section for strictly normal $\mathcal{A} \mathcal{K}_{3}$-structures.
Theorem 7.1. An Einstein, $\mathcal{A K}_{3}$ structure is either
(i) Kähler
or
(ii) Ricci flat and of null type, up to local Riemannian products with Ricci flat Kähler manifolds.
Proof. By thms. 1.1 and 5.1 an Einstein, $\mathcal{A K}_{3}$ structure is locally the Riemannian product of a Kähler-Einstein manifold and an Einstein, normal $\mathcal{A} \mathcal{K}_{3}$ manifold. In the latter situation proposition 7.4, (iii) yields $\rho(V, W)=-\frac{1}{2}\left\langle r_{1}(J V), W\right\rangle$ for all $V, W$ in $\mathcal{V}$; it follows that $\operatorname{Tr}\left(r_{1}\right)$ is constant hence by proposition 7.5 the structure is of null type, in particular Ricci flat. It is now straightforward to conclude.
7.3. The canonical foliation. While keeping all notation from the previous sections we will identify here the main obstruction to the integrability of the Kähler nullity $H$. This leads to the construction of a canonical foliation on any normal $\mathcal{A K}_{3}$ manifold with specific properties. It also helps to gather additional information on the Einstein case. We consider the nullity

$$
\mathcal{N}_{0}=\{V \in \mathcal{V}: \bar{R}(\mathcal{V}, \mathcal{V}) V=0\}
$$

which, as we will see later on, corresponds to a flat factor in the local deRham decomposition of integral manifolds of $\mathcal{V}$, w.r.t. the induced structure. Denote by $\mathcal{N}_{1}$ its orthogonal complement in $\mathcal{V}$ and let

$$
E=\{X \in H: \bar{R}(\mathcal{V}, \mathcal{V}) X=0\}
$$

with orthogonal complement in $H$ to be denoted by $F_{1}$. Because $H_{1} \subseteq E$ by (7.6) we have a further $J$-invariant splitting

$$
E=F_{0} \oplus H_{1}
$$

where $F_{0} \subseteq H_{0}$ is the orthogonal complement of $H_{1}$ in $E$.
Therefore the nullity $\mathcal{N}_{0}$ is maximal if and only if the structure is of null type in the sense of definition 7.1. We now examine the action of the intrinsic torsion $\eta$ on the distributions

$$
\begin{aligned}
& \mathcal{E}_{0}=\mathcal{N}_{0} \oplus F_{0} \oplus H_{1} \\
& \mathcal{E}_{1}=\mathcal{N}_{1} \oplus F_{1} .
\end{aligned}
$$

Lemma 7.4. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A K}_{3}$-structure. We have:
(i) $\eta_{\mathcal{N}_{0}} \mathcal{N}_{0}=F_{0}, \eta_{\mathcal{N}_{1}} \mathcal{N}_{1}=F_{1}$ and $\eta_{\mathcal{N}_{0}} \mathcal{N}_{1}=0$;
(ii) $\eta_{\mathcal{N}_{0}} F_{0}=\mathcal{N}_{0}, \eta_{\mathcal{N}_{1}} F_{1}=\mathcal{N}_{1}$ and $\eta_{\mathcal{N}_{0}} F_{1}=\eta_{\mathcal{N}_{1}} F_{0}=0$.

Proof. (i) From lemma 7.3 combined with the definitions of the spaces $\mathcal{N}_{0}$ and $F_{0}$ it follows that

$$
\begin{equation*}
\eta_{\mathcal{V}} \mathcal{N}_{0} \subseteq F_{0} \tag{7.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta_{\mathcal{V}} F_{0} \subseteq \mathcal{N}_{0} \tag{7.10}
\end{equation*}
$$

By taking orthogonal complements (7.10) yields $\eta_{\mathcal{V}} \mathcal{N}_{1} \subseteq F_{1}$ hence $\eta_{\mathcal{N}_{1}} \mathcal{N}_{1} \subseteq F_{1}$ and $\eta_{\mathcal{N}_{0}} \mathcal{N}_{1} \subseteq F_{0}$. The last inclusion combined with (7.9) leads to

$$
\eta_{\mathcal{N}_{0}} \mathcal{N}_{1}=0 .
$$

It follows that $H_{0}=\eta_{\mathcal{V}} \mathcal{V}=\eta_{\mathcal{N}_{0}} \mathcal{N}_{0}+\eta_{\mathcal{N}_{1}} \mathcal{N}_{1} \subseteq F_{0} \oplus F_{1}$ as shown above, therefore $\eta_{\mathcal{N}_{0}} \mathcal{N}_{0}=F_{0}$ and $\eta_{\mathcal{N}_{1}} \mathcal{N}_{1}=F_{1}$ by a dimension argument.
(ii) From (7.10) we get $\eta_{\mathcal{N}_{0}} F_{0} \subseteq \mathcal{N}_{0}$ and also $\eta_{\mathcal{N}_{1}} F_{0} \subseteq \mathcal{N}_{1}$. Combined with the vanishing of $\eta_{\mathcal{N}_{1}} \mathcal{N}_{2}$ the last inclusion yields $\eta_{\mathcal{N}_{1}} F_{0}=0$. Through similar arguments one arrives at $\eta_{\mathcal{N}_{1}} F_{1} \subseteq \mathcal{N}_{1}$ and $\eta_{\mathcal{N}_{0}} F_{1}=0$. The proof of the claim is now completed by a dimension argument based on $\eta_{\mathcal{V}} H=\mathcal{V}$.

In particular we have

$$
\eta_{\mathcal{E}_{k}} \mathcal{E}_{k} \subseteq \mathcal{E}_{k}, k=0,1 \text { and } \eta_{\mathcal{E}_{0}} \mathcal{E}_{1}=\eta_{\mathcal{E}_{1}} \mathcal{E}_{0}=0
$$

We prove now that the tensor $\tilde{\eta}$ has in fact analogous properties. This will be done by relating first the distributions $\mathcal{W}_{0}$ and $E$ to the nullity of $\tilde{\eta}$ on $\mathcal{V}$ and $H$ respectively.
Lemma 7.5. Let $\left(M^{2 m} g, J\right)$ be a normal $\mathcal{A K}_{3}$-structure. We have that $\tilde{\eta}_{\mathcal{E}_{k}} \mathcal{E}_{k} \subseteq$ $\mathcal{E}_{k}, k=0,1$ and $\tilde{\eta}_{\mathcal{E}_{0}} \mathcal{E}_{1}=\tilde{\eta}_{\mathcal{E}_{1}} \mathcal{E}_{0}=0$.
Proof. Let $X$ belong to $F_{0}$. Then by using (7.7) we obtain $\left\langle\left(\tilde{\eta}_{X} W\right)_{H_{0}},\left(\tilde{\eta}_{Y} V\right)_{H_{0}}\right\rangle-$ $\left\langle\left(\tilde{\eta}_{X} V\right)_{H_{0}},\left(\tilde{\eta}_{Y} W\right)_{H_{0}}\right\rangle=0$ for all $V, W$ in $\mathcal{V}$ and whenever $Y$ in $H$. Since the first summand is $J$-invariant in $(Y, V)$ whilst the second is $J$-anti-invariant it follows easily after a positivity argument that $\left(\tilde{\eta}_{X} \mathcal{V}\right)_{H_{0}}=0$. By orthogonality we have showed that

$$
\begin{equation*}
F_{0}=\left\{X \in H_{0}: \tilde{\eta}_{X} H_{0}=0\right\} . \tag{7.11}
\end{equation*}
$$

In particular $\left(\tilde{\eta}_{H_{0}} \mathcal{V}\right)_{H_{0}} \subseteq F_{1}$. Now if $Y$ in $F_{1}$ is orthogonal to $\left(\tilde{\eta}_{H_{0}} \mathcal{V}\right)_{H_{0}}$ then clearly $\tilde{\eta}_{Y} H_{0}=0$, hence $Y$ must belong to $F_{0} \perp F_{1}$ whence $Y=0$. It folows that

$$
\begin{equation*}
F_{1}=\left(\tilde{\eta}_{H_{0}} \mathcal{V}\right)_{H_{0}} \tag{7.12}
\end{equation*}
$$

Since $\tilde{\eta}_{H_{1}}=0$ when taking $X$ in $H_{1}$ in (iii) of lemma 7.2 we obtain:

$$
0=\left\langle X, \tilde{\eta}_{\tilde{\eta}_{Y} W} V\right\rangle=\left\langle X, \tilde{\eta}_{\left(\tilde{\eta}_{Y} W\right)_{H_{0}}} V\right\rangle
$$

for all $Y$ in $H_{0}$ and $V, W$ in $\mathcal{V}$. Then $H_{1} \perp \tilde{\eta}_{F_{1}} \mathcal{V}$ by (7.12) or equivalently

$$
\begin{equation*}
\tilde{\eta}_{F_{1}} H_{1}=0 . \tag{7.13}
\end{equation*}
$$

It follows that lemma 7.1, (iv) applied to $X$ in $F_{0}, Y$ in $H_{1}$ and $Z$ in $F_{1}$ yields $\tilde{\eta}_{F_{0}} H_{1} \perp \eta_{\nu} F_{1}=\mathcal{N}_{1}$ hence

$$
\begin{equation*}
\tilde{\eta}_{F_{0}} H_{1} \subseteq \mathcal{N}_{0} \tag{7.14}
\end{equation*}
$$

Therefore

$$
\begin{aligned}
\tilde{\eta}_{\mathcal{E}_{0}} \mathcal{E}_{0} & =\tilde{\eta}_{F_{0}} \mathcal{E}_{0}+\tilde{\eta}_{H_{1}} \mathcal{E}_{0}=\tilde{\eta}_{F_{0}} \mathcal{N}_{0}+\tilde{\eta}_{F_{0}} F_{0}+\tilde{\eta}_{F_{0}} H_{1} \\
& =\tilde{\eta}_{F_{0}} \mathcal{N}_{0}+\tilde{\eta}_{F_{0}} H_{1} \subseteq H_{1} \oplus \mathcal{N}_{0} \subseteq \mathcal{E}_{0}
\end{aligned}
$$

by succesive use of the facts above.
Because $\bar{R}\left(\mathcal{N}_{0}, \mathcal{V}\right)=0$ we have

$$
\mathcal{N}_{0} \subseteq\left\{V \in \mathcal{V}: \tilde{\eta}_{H_{0}} V \subseteq H_{1}\right\}
$$

by proposition 7.4, (i) followed by a standard positivity argument. Conversely, if $V$ in $\mathcal{V}$ is such that $\tilde{\eta}_{H_{0}} V \subseteq H_{1}$ then $\eta_{\mathcal{V}} \tilde{\eta}_{H_{0}} V=0$. From (i) in lemma 7.1 it follows that $\tilde{\eta}_{H_{0}}\left(\eta_{\nu} V\right)=0$ therefore $\eta_{\nu} V \subseteq F_{0}$. From lemma 7.3 it is easy to conclude that $V$ belongs to $\mathcal{N}_{0}$ thus

$$
\begin{equation*}
\mathcal{N}_{0}=\left\{V \in \mathcal{V}: \tilde{\eta}_{H_{0}} V \subseteq H_{1}\right\} \tag{7.15}
\end{equation*}
$$

Now $\tilde{\eta}_{F_{1}} \mathcal{N}_{1} \perp H_{1} \oplus F_{0}$ by (7.13), (7.11) and the symmetry of $\tilde{\eta}$ on $H_{0}$ hence

$$
\tilde{\eta}_{F_{1}} \mathcal{N}_{1} \subseteq F_{1}
$$

Furthermore, $\tilde{\eta}_{F_{1}} F_{1} \perp \mathcal{N}_{0}$ by (7.15) making that

$$
\tilde{\eta}_{F_{1}} F_{1} \subseteq \mathcal{N}_{1}
$$

Altogether,

$$
\tilde{\eta}_{\mathcal{E}_{1}} \mathcal{E}_{1}=\tilde{\eta}_{F_{1}} \mathcal{E}_{1}=\tilde{\eta}_{F_{1}} \mathcal{N}_{1}+\tilde{\eta}_{F_{1}} F_{1} \subseteq F_{1} \oplus \mathcal{N}_{1}=\mathcal{E}_{1}
$$

We observe now that

$$
\begin{equation*}
\tilde{\eta}_{F_{1}} \mathcal{N}_{0}=0 \tag{7.16}
\end{equation*}
$$

by using (7.15) and (7.13). We conclude that

$$
\tilde{\eta}_{\mathcal{E}_{1}} \mathcal{E}_{0}=\tilde{\eta}_{F_{1}} \mathcal{E}_{0}=\tilde{\eta}_{F_{1}} \mathcal{N}_{0}+\tilde{\eta}_{F_{1}} F_{0}+\tilde{\eta}_{F_{1}} H_{1}=0
$$

by making use of (7.16), (7.11) and (7.13). The observation that $\tilde{\eta}_{F_{0}} \mathcal{N}_{1} \perp H_{1}$ by (7.14) whilst $\tilde{\eta}_{F_{0}} \mathcal{N}_{1} \subseteq H_{1}$ by (7.11) lead to

$$
\begin{equation*}
\tilde{\eta}_{F_{0}} \mathcal{N}_{1}=0 \tag{7.17}
\end{equation*}
$$

Finally,

$$
\tilde{\eta}_{\mathcal{E}_{0}} \mathcal{E}_{1}=\tilde{\eta}_{F_{0}} \mathcal{E}_{1}+\tilde{\eta}_{H_{1}} \mathcal{E}_{1}=\tilde{\eta}_{F_{0}} \mathcal{N}_{1}+\tilde{\eta}_{F_{0}} F_{1}=0
$$

by (7.17) and (7.11) and the lemma is completely proved.

The obstructive role of the nullity space $\mathcal{N}_{0}$ is now apparent for $\mathcal{N}_{0}=0$ implies $\tilde{\eta}_{F_{0}} H_{1}=0$ by (7.14) and further $\tilde{\eta}_{H_{0}} H_{1}=0$ by also using that $\tilde{\eta}_{\mathcal{E}_{1}} H_{1}=0$, in other words the Kähler nullity $H$ must be integrable.

Summarising the considerations in this section we have
Theorem 7.2. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A} \mathcal{K}_{3}$-manifold. The distribution $\mathcal{E}_{0}$ is
(i) totally geodesic;
(ii) holomorphic in the sense that $\left(L_{X} J\right) T M \subseteq \mathcal{E}_{0}$ whenever $X$ is in $\mathcal{E}_{0}$.

Moreover, in the induced structure every integral manifold of $\mathcal{E}_{0}$ is of null type.
Proof. (i) We will first show that $\mathcal{N}_{0}$ is parallel w.r.t. $\tilde{\nabla}$. That $\mathcal{N}_{0}$ is invariant under $\tilde{\nabla}_{X}$ with $X$ in $H$ follows from proposition 5.1, (ii). If $V$ is in $\mathcal{V}$ and $w_{0}$ and $X_{0}$ are in $\mathcal{N}_{0}$ respectively $F_{0}$ we have

$$
\bar{\nabla}_{V}\left(\eta_{w_{0}} X_{0}\right)=\eta_{w_{0}} \bar{\nabla}_{V} X_{0}+\eta_{\bar{\nabla}_{V} w_{0}} X_{0} \in \eta_{\mathcal{N}_{0}} H+\eta_{\mathcal{V}} F_{0}
$$

and we conclude by lemma 7.4 that $\mathcal{N}_{0}$ is invariant under $\bar{\nabla}_{V}$. By orthogonality $\mathcal{N}_{1}$ is $\tilde{\nabla}$-parallel as well.

Now $\bar{R}(\mathcal{V}, \mathcal{V}) \mathcal{E}_{0}=0$, hence after differentiation we get

$$
\begin{aligned}
-\left(\bar{\nabla}_{U_{1}} \bar{R}\right)\left(V_{1}, V_{2}, U, U_{2}\right)= & \bar{R}\left((\eta+\tilde{\eta})_{U_{1}} V_{1}, V_{2}, U, U_{2}\right)+\bar{R}\left(V_{1},(\eta+\tilde{\eta})_{U_{1}} V_{2}, U, U_{2}\right) \\
& +\bar{R}\left(V_{1}, V_{2}, \bar{\nabla}_{U_{1}} U, U_{2}\right)
\end{aligned}
$$

whenever $U_{1}, U_{2}$ are in $T M$ and $V_{1}, V_{2}, U$ belong to $\mathcal{V}$ and $\mathcal{E}_{0}$ respectively. When taking $U$ in $H_{1}$, the differential Bianchi identity for $\bar{\nabla}$ in proposition 3.4, (i) together with $\bar{R}\left(\mathcal{V}, H_{1}\right)=0$ (an easy consequence of (ii) in the same proposition) yield the vanishing of the l.h.s., since all the distributions under consideration are invariant under $\bar{\nabla}_{V}, V$ in $\mathcal{V}$. It follows that $\bar{R}(\mathcal{V}, \mathcal{V}) \bar{\nabla}_{U_{1}} U=0$ i.e.

$$
\begin{equation*}
\bar{\nabla}_{H_{1}} \mathcal{E}_{0} \subseteq \mathcal{E}_{0} \tag{7.18}
\end{equation*}
$$

Consider now $V, W$ in $\mathcal{V}$ and let $X$ be in $H$. From proposition 3.2, (iii) it follows that $\left(\bar{\nabla}_{J X} \eta\right)_{V} W+J\left(\bar{\nabla}_{X} \eta\right)_{V} W=0$ hence by orthogonal projection on $H$ we obtain $\left(\tilde{\nabla}_{J X} \eta\right)_{V} W+J\left(\tilde{\nabla}_{X} \eta\right)_{V} W=0$. Because for all $k=0,1$ the distributions $\mathcal{N}_{k}$ are $\tilde{\nabla}$-parallel and satisfy $\eta_{\mathcal{N}_{k}} \mathcal{V}=F_{k}$ it follows that

$$
\tilde{\nabla}_{J X} Y+J \tilde{\nabla}_{X} Y \in F_{k}
$$

for all $X$ in $H$ and $Y$ in $F_{k}, k=0,1$. By orthogonality we also get

$$
\begin{equation*}
\tilde{\nabla}_{J X} Y-J \tilde{\nabla}_{X} Y \in F_{k} \oplus H_{1} \tag{7.19}
\end{equation*}
$$

for all $X$ in $H$ and $Y$ in $F_{k}, k=0$, 1. In particular $\tilde{\nabla}_{F_{0}} F_{0} \subseteq F_{0} \oplus H_{1}$ hence (7.18) together with the structure of $\eta$ and $\tilde{\eta}$ in lemmas 7.4 and 7.5 yield that $\mathcal{E}_{0}$ is totally geodesic, by also using that $\mathcal{N}_{0}$ is parallel w.r.t. $\tilde{\nabla}$.
(ii) we have

$$
\left(L_{X} J\right) U=-2 T_{X}(J U)-\left(\tilde{\eta}_{J U} X-J \tilde{\eta}_{U} X\right)-\left(\tilde{\nabla}_{J U} X-J \tilde{\nabla}_{U} X\right)
$$

for all $X, U$ in $T M$. The claim follows from (7.19) and by using again lemmas 7.4 and 7.5.

The foliation induced by $\mathcal{E}_{0}$ will be referred to as the canonical foliation of the normal $\mathcal{A K}_{3}$-structure $\left(M^{2 m}, g, J\right)$. Note that $\mathcal{E}_{0}$ is also totally geodesic for the canonical Hermitian connection, by lemma 7.4.

## 8. Classification Results

We begin by the following caracterisation of the examples in section 3.2. We will show that when a normal $\mathcal{A K}_{3}$-structure $(M, g, J)$ has the property that $(g, \tilde{J})$ is Kähler, or equivalently the Kähler nullity is parallel w.r.t canonical connection, if and only if it belongs to a particular class of Kähler metrics with torus symmetry; using the description of such metrics, see $[26,32]$ leads to

Proposition 8.1. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A} \mathcal{K}_{3}$ manifold. If the reversing almost complex structure $\tilde{J}$ is Kähler then $(M, g, J)$ is given locally by the construction in section 3.2 where $w: Z \rightarrow S^{2,-}\left(\mathbb{R}^{2 p}\right)$ is non-degenerate and $2 p=\operatorname{dim}_{\mathbb{R}} \mathcal{V}$.

Proof. First we show that around each point in $M$ there is an open neighborhood over which the distribution $\mathcal{V}$ is spanned by linearly independent, mutually commuting Killing vector fields. Define the linear connection $D$ in the bundle $\mathcal{V}$ by

$$
D_{V} W=\bar{\nabla}_{V} W \text { and } D_{X} V=\bar{\nabla}_{X} V+\eta_{V} X
$$

for all $V, W$ in $\mathcal{V}$ and $X$ in $H$. Note that $D$ is neither metric nor Hermitian and let us denote by $R^{D}$ its curvature tensor. In what follows we denote by $V_{1}, V_{2}, V$ respectively $X, Y$ generic vector fields in $\mathcal{V}$ and $H$ respectively.

Using the definition of $D$ and the $\bar{\nabla}$-parallelism of $T M=\mathcal{V} \oplus H$ one arrives at

$$
\begin{aligned}
& R^{D}\left(V_{1}, V_{2}\right) V=\bar{R}\left(V_{1}, V_{2}\right) V \\
& R^{D}\left(V_{1}, X\right) V=\bar{R}\left(V_{1}, X\right) V-\left(\bar{\nabla}_{V_{1}} \eta\right)(V, X)
\end{aligned}
$$

Since the tensor $\tilde{\eta}$ vanishes identically lemma 7.2, (i) as well as the partial parallelism of $\eta$ in (4.1) ensure the vanishing of $R^{D}$ on $\mathcal{V} \times T M$. Again from the definition of $D$ we get

$$
R^{D}(X, Y) V=\bar{R}(X, Y) V-\left(\left(\bar{\nabla}_{X} \eta\right)(V, Y)-\left(\bar{\nabla}_{Y} \eta\right)(V, X)\right)-\left(\eta_{\eta_{V} Y} X-\eta_{\eta_{V} X} Y\right)
$$

Since $d_{\bar{\nabla}} \eta(X, V)(Y, W)=d_{\bar{\nabla}} \eta(Y, W)(X, V)$ for all $W$ in $\mathcal{V}$, by using (4.1) it follows that the middle term in the expression above vanishes. But again the vanishing of $\tilde{\eta}$ gives by means of lemma 7.2 , (iv) that $\bar{R}(V, W, X, Y)=0$ for all $W$ in $\mathcal{V}$. Therefore the symmetry property in corollary 3.1 , (i) implies that $\bar{R}(X, Y, V, W)=$ $-\left\langle\left[\eta_{V}, \eta_{W}\right] X, Y\right\rangle$ and we conclude that $R^{D}$ vanishes on $H \times H$, by also using the symmetry of $\eta$ on $\mathcal{V} \times \mathcal{V}$. We have showed that the connection $D$ is flat, hence around each point $x$ in $M$ there is an open neighborhood $U_{x}$ over which the distribution $\mathcal{V}$ is spanned by a family $\left\{V_{i}, 1 \leq i \leq 2 p\right\}$ of vector fields such that

$$
\left[V_{i}, V_{j}\right]=0, D V_{k}=0
$$

for all $1 \leq i, j, k \leq 2 p$. It is now easily checked that $D$-parallel vector fields in $\mathcal{V}$ must be Killing fields w.r.t. $g$, holomorphic w.r.t. to both $J$ and $\tilde{J}$, that is $L_{V_{i}} J=L_{V_{i}} \tilde{J}=0,1 \leq i \leq 2 p$. It follows that the matrix $\left.g\left(J V_{i}, V_{j}\right)\right)_{1 \leq i, j \leq 2 p}$ has
constant entries hence we may assume it equals $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$. The claim follows now from the well known Kähler reduction procedure, see [26, 32].
8.1. Integrability. In this section we obtain classification results for strictly normal $\mathcal{A K}_{3}$-manifolds. This will be done by showing that it is always possible to restrict, in dimension at least six, the study of strictly normal $\mathcal{A K}_{3}$-structures to the case when $\operatorname{Tr}\left(r_{1}\right)$ is constant along $\mathcal{V}$, a situation when the integrability criterion in proposition 7.5 appplies. In the case of dimension four the same conclusion can be obtained by using the work in [8].

Let therefore $\left(M^{2 m}, g, J\right)$ be a strictly normal $\mathcal{A} \mathcal{K}_{3}$-manifold, with Kähler nullity $H$ and let $\mathcal{V}$ be the distribution orthogonal to $H$. By proposition 7.3, (i) the tensor $\tilde{\eta}$ is symmetric on $H$, that is the distribution $H$ is integrable. Our analysis is build around the properties of the partial Ricci-type tensor $r_{2}$ introduced in (7.5), as follows.

Lemma 8.1. Let $\left(M^{2 m}, g, J\right)$ be a strictly normal $\mathcal{A K}_{3}$ manifold. Then
(i) for any $X$ in $H$ the component of $\tilde{\nabla}_{X} \bar{R}$ on $\Lambda^{2} \mathcal{V} \otimes \Lambda^{2} H$ vanishes;
(ii) we have $\left(\tilde{\nabla}_{X} r_{2}\right) Y=0$ for all $X, Y$ in $H$.

Proof. (i) We recall that $\left(\bar{\nabla}_{X} \bar{R}\right)\left(V_{1}, V_{2}, V_{3}, V_{4}\right)=0$ for all $X$ in $H$ and $V_{i}$ in $\mathcal{V}, 1 \leq$ $i \leq 4$, by (ii) in proposition 5.1. Now $\bar{R}$ vanishes on $\Lambda^{2} \mathcal{V} \otimes \mathcal{V} \otimes H$ by (i) in lemma 5.1 and also on $\mathcal{V} \otimes H \otimes \Lambda^{2} \mathcal{V}$ after updating (ii) in the same lemma to the normal case. Since $\tilde{\nabla}$ preserves $\mathcal{V}$ and $H$ by construction it follows that $\tilde{\nabla}_{X} \bar{R}$ vanishes on $\mathcal{V}$. Therefore, differentiation w.r.t. $\tilde{\nabla}$ in lemma 7.3 leads to

$$
2 \bar{R}\left(V_{3}, V_{4}, V_{2},\left(\tilde{\nabla}_{X} \eta\right)_{V_{1}} Y\right)=\tilde{\nabla}_{X} \bar{R}\left(V_{3}, V_{4}, Y, \eta_{V_{1}} V_{2}\right)+\bar{R}\left(V_{3}, V_{4}, Y,\left(\tilde{\nabla}_{X} \eta\right)_{V_{1}} V_{2}\right)
$$

whenever $X, Y$ belong to $H$ and $V_{i}$ are in $\mathcal{V}, 1 \leq i \leq 4$. But

$$
\left(\tilde{\nabla}_{J X} \eta\right)_{V_{1}} J Y=-\left(\tilde{\nabla}_{X} \eta\right)_{V_{1}} Y,\left(\tilde{\nabla}_{J X} \eta\right)_{V_{1}} V_{2}=-J\left(\tilde{\nabla}_{X} \eta\right)_{V_{1}} V_{2}
$$

by taking the appropriate projections of (iii) in proposition 3.2. Thus

$$
\tilde{\nabla}_{J X} \bar{R}\left(V_{3}, V_{4}, J Y, Z\right)+\tilde{\nabla}_{X} \bar{R}\left(V_{3}, V_{4}, Y, Z\right)=0
$$

for all $Z$ in $H=\eta_{\mathcal{V}} \mathcal{V}$. Since $\bar{R}$ is $\lambda^{1,1} M$-valued we observe that the first summand above is symmetric in $(Y, Z)$ whilst the second is skew -symmetric in the same variables. We conclude that $\tilde{\nabla}_{X} \bar{R}$ vanishes on $\Lambda^{2} \mathcal{V} \otimes \Lambda^{2} H$ and the claim is proved. (ii) follows from (i) by taking the complex trace over $\mathcal{V}$.

Let $D$ be the open dense subset of $M$ such that on each connected component of $D$ the tensor $r_{2}$ diagonalises with eigenbundles of constant rank. Over such a component, say $C$, we have an orthogonal splitting:

$$
H=H_{1} \oplus \ldots \oplus H_{r}
$$

where $H_{i}, 1 \leq i \leq r$ are the eigenbundles of $r_{2}$ with corresponding pairwise distinct eigenfunctions $\lambda_{i}, 1 \leq i \leq r$.

Proposition 8.2. Let $\left(M^{2 m}, g, J\right)$ be a strictly normal $\mathcal{A}_{3}$-manifold. The following hold over $C$ :
(i) we have an orthogonal and J-invariant decomposition $\mathcal{V}=\bigoplus_{i=1}^{r} \mathcal{V}_{i}$ where $\mathcal{V}_{i}=$ $\eta_{\nu} H_{i}$ for $1 \leq i \leq p ;$
(ii) $\eta_{\mathcal{V}_{i}} \mathcal{V}_{j}=0$ and $\eta_{\mathcal{V}_{i}} \mathcal{V}_{i}=H_{i}$ whenever $1 \leq i \neq j \leq r$;
(iii) we have $\tilde{\eta}_{H_{i}} H_{j}=0$ and $\tilde{\eta}_{H_{i}} \mathcal{V}_{j}=0$ for all $1 \leq i \neq j \leq r$;
(iv) the decomposition $T M=\bigoplus_{i=1}^{r}\left(\mathcal{V}_{i} \oplus H_{i}\right)$ defines a local splitting of $(M, g, J)$ into a Riemannian product of strictly normal $\mathcal{A K}_{3}$-manifolds, with corresponding Kähler nullities $H_{i}, 1 \leq i \leq r$.
Proof. (i) by taking traces we get from lemma 7.3 and proposition 7.4

$$
\begin{equation*}
r_{1}\left(\eta_{V} X\right)=\eta_{V}\left(r_{2} X\right) \tag{8.1}
\end{equation*}
$$

for all $V$ in $\mathcal{V}$ and $X$ in $H$. In particular $\mathcal{V}_{i}=\eta_{\mathcal{V}} H_{i} \subseteq \operatorname{Ker}\left(r_{1}-\lambda_{i}\right)$ for all $1 \leq i \leq r$. It follows that $\mathcal{V}_{i} \perp \mathcal{V}_{j}$ for all $1 \leq i \neq j \leq r$ and we conclude by using that $\eta_{\mathcal{V}} H=\mathcal{V}$.
(ii) for any $1 \leq i \neq j \leq r$ we have that $\mathcal{V}_{i}=\eta_{\mathcal{V}} H_{i}$ is orthogonal to $\mathcal{V}_{j}$ hence $\eta_{\mathcal{V}} \mathcal{V}_{j} \perp H_{i}$. Therefore $\eta_{\mathcal{V}} \mathcal{V}_{j} \subseteq H_{j}$, in particular the symmetry of $\eta$ over $\mathcal{V}$ yields $\eta_{\mathcal{V}_{i}} \mathcal{V}_{j} \subseteq H_{i} \cap H_{j}=0$. The second part of the claim follows from $\eta_{\mathcal{V}} \mathcal{V}=H$.
(iii) let $1 \leq i \neq j \leq r$. By (ii) in lemma 7.1 we get by taking into account that $\eta_{\nu_{i}} H_{j}=0$ as in (ii) above that $\eta_{\mathcal{V}_{i}}\left(\tilde{\eta}_{H} H_{j}\right)=0$. Then $\tilde{\eta}_{H} H_{j}$ is orthogonal to $\eta_{\mathcal{V}_{i}} H=\mathcal{V}_{i}$ hence it must be contained in $H_{j}$. Because $\tilde{\eta}$ is symmetric on $H$ we have in particular that $\tilde{\eta}_{H_{i}} H_{j} \subseteq \mathcal{V}_{i} \cap \mathcal{V}_{j}=0$. The second part of the claim follows similarly from $\mathcal{V}_{i}=\eta_{\mathcal{V}} H_{i}$ and again (ii) in lemma 7.1.
(iv) By lemma 8.1, (ii) the distributions $H_{i}$ are $\tilde{\nabla}$ parallel inside $H$, that is $\tilde{\nabla}_{X} X_{i}$ belongs to $H_{i}$ for all $\left(X, X_{i}\right)$ in $H \times H_{i}, 1 \leq i \leq r$. Pick now $Y$ in $H_{i}$. Then $\nabla_{X} Y=$ $\tilde{\nabla}_{X} Y+\tilde{\eta}_{X} Y$ belongs to $H_{i} \oplus \mathcal{V}_{i}$ for all $X$ in $H$ by (iii). Since $\eta$ is parallel w.r.t. $\bar{\nabla}$ along $\mathcal{V}$ we have that $\bar{\nabla}_{V}\left(\eta_{V_{i}} X_{i}\right)=\eta_{\bar{\nabla}_{V} V_{i}} X_{i}+\eta_{V_{i}} \bar{\nabla}_{V} X_{i}$ belongs to $\eta_{\mathcal{V}} H_{i}+\eta_{\mathcal{V}_{i}} H=\mathcal{V}_{i}$ for all $V$ in $\mathcal{V}$ and $V_{i}, X_{i}$ in $\mathcal{V}_{i}$ and $H_{i}, 1 \leq i \leq r$ respectively. Because $\eta_{\mathcal{V}_{i}} H_{i}=\mathcal{V}_{i}$ it follows that $\nabla_{V} V_{i}=\bar{\nabla}_{V} V_{i}-\eta_{V} V_{i}$ is in $\mathcal{V}_{i}+\eta_{\mathcal{V}} \mathcal{V}_{i}=\mathcal{V}_{i} \oplus H_{i}$ for all $V$ in $\mathcal{V}$ and whenever $V_{i}$ is in $\mathcal{V}_{i}, 1 \leq i \leq r$. A similar argument shows that $\nabla_{V} X_{i}$ belongs to $H_{i}$ for all $\left(V, X_{i}\right)$ in $\mathcal{V} \times H_{i}, 1 \leq i \leq r$.

Let now $X$ belong to $H$ and $\left(V_{i}, X_{i}\right)$ be in $\mathcal{V}_{i} \times H_{i}, 1 \leq i \leq r$. By proposition 3.2, (iii) we have $\left(\bar{\nabla}_{J X} \eta\right)_{J V_{i}} X_{i}+\left(\bar{\nabla}_{X} \eta\right)_{V_{i}} X_{i}=0$. Because the connection $\tilde{\nabla}_{\tilde{\nabla}}$ preserves the splitting $T M=\mathcal{V} \oplus H$ after projection on $\mathcal{V}$ we get $\left(\tilde{\nabla}_{J X} \eta\right)_{J V_{i}} X_{i}+\left(\tilde{\nabla}_{X} \eta\right)_{V_{i}} X_{i}=0$. But

$$
\left(\tilde{\nabla}_{X} \eta\right)_{V_{i}} X_{i}=\tilde{\nabla}_{X}\left(\eta_{V_{i}} X\right)-\eta_{\tilde{\nabla}_{X} V_{i}} X_{i}-\eta_{V_{i}} \tilde{\nabla}_{X} X_{i}
$$

where the last two terms belong to $\eta_{\mathcal{V}} H_{i}=\mathcal{V}_{i}$ and $\eta_{\nu_{i}} H_{i}=H_{i}$ respectively. Therefore we end up with

$$
\tilde{\nabla}_{J X} V_{i}+J \tilde{\nabla}_{X} V_{i} \text { in } \mathcal{V}_{i}
$$

Because the connection $\tilde{\nabla}$ preserves $\mathcal{V}$ by definition, the orthogonal counterpart of the relation above is $\tilde{\nabla}_{J X} V_{i}-J \tilde{\nabla}_{X} V_{i}$ in $\mathcal{V}_{i}$. It follows that $\tilde{\nabla}_{X} V_{i}$ belongs to $\mathcal{V}_{i}$ and using again (iii) we get that in fact $\nabla_{X} V_{i}=\tilde{\nabla}_{X} V_{i}-\tilde{\eta}_{X} V_{i}$ is in $\mathcal{V}_{i} \oplus H_{i}$. By collecting the facts above we see that each of the distributions $\mathcal{V}_{i} \oplus H_{i}, 1 \leq i \leq r$ is parallel w.r.t the Levi-Civita connection of $g$ and the claim follows from the deRham splitting theorem.

Theorem 8.1. Let $\left(M^{2 m}, g, J\right)$ be a strictly normal $\mathcal{A K}_{3}$-manifold. Then $(g, \tilde{J})$ is a Kähler structure or equivalently $\mathcal{V}$ is parallel w.r.t $\bar{\nabla}$. Moreover $(g, J)$ is locally obtained by the construction in section 3.2 where $w: Z \rightarrow S^{2,-}\left(\mathbb{R}^{2 p}\right)$ is non-degenerate and immersive.

Proof. By proposition 8.2 we can assume that over each connected component of some dense open set in $M$ we have $r_{2}=\lambda 1_{H}$ for some function $\lambda$ on $M$. Then (8.1) and the fact that $\eta_{\mathcal{V}} H=\mathcal{V}$ yield $r_{1}=\lambda 1_{\mathcal{V}}$ whence

$$
\sum_{v_{k} \in \mathcal{V}} \bar{R}\left(v_{k}, J v_{k}, V, W\right)=\lambda\langle J V, W\rangle
$$

for all $V, W$ in $\mathcal{V}$. If $\operatorname{dim}_{\mathbb{R}} \mathcal{V}=2$ having $(g, J)$ strictly normal implies that $\operatorname{dim}_{\mathbb{R}} H=$ 2 and we know by the work in [8] that $(g, \tilde{J})$ must be Kähler. If the rank of $\mathcal{V}$ is at least 4 , using the differential Bianchi identity for $\bar{\nabla}$ over $\mathcal{V}$, which does not involve torsion terms since $T(\mathcal{V}, \mathcal{V})=0$, one obtains that $L_{V} \lambda=0$ for all $V$ in $\mathcal{V}$. Proposition 7.5, (ii) implies then the vanishing of $\tilde{\eta}$, hence $(g, \tilde{J})$ is Kähler. By continuity this extends to $M$ and the claim follows from proposition 8.1.

In the more rigid case of $\mathcal{A} \mathcal{K}_{2}$-structures we prove the following.
Theorem 8.2. Let $\left(M^{2 m}, g, J\right)$ be a normal $\mathcal{A} \mathcal{K}_{2}$-manifold. Then:
(i) $(g, J)$ has parallel intrinsic torsion tensor with respect to the first Hermitian connection;
(ii) if $(g, J)$ is strictly normal then $(M, g)$ is a locally 3-symmetric space.

Proof. (i) by proposition 7.2 , (i) it is enough to prove the claim when $(g, J)$ is strictly normal; in this case the previous result says that $\tilde{\eta}$ vanishes hence $\tilde{\nabla}=\bar{\nabla}$ and then $\bar{\nabla} \eta=0$ by proposition 7.2 , (i).
(ii) we compute fully the curvature tensor of the canonical connection, taking into account that $T M=\mathcal{V} \oplus H$ is a $\bar{\nabla}$-parallel decomposition. The vanishing of $\tilde{\eta}$ implies by lemma 7.2 , (i) and (iv) and lemma 7.3 that curvature terms of the form

$$
\bar{R}\left(V_{1}, X, V_{2}, Y\right), \bar{R}\left(V_{1}, V_{2}, X, Y\right) \text { and } \bar{R}\left(V_{1}, V_{2}, V_{3}, V_{4}\right)
$$

where $V_{i}, 1 \leq i \leq 4$ are in $\mathcal{V}$ and $X, Y$ in $H$ equally vanish. Let us compute the restriction of $\bar{R}$ to $H$. Because $\bar{R}$ vanishes on $\Lambda^{2} \mathcal{V} \otimes \Lambda^{2} H$ the symmetry property in (i) of corollary 3.1 yields

$$
\bar{R}\left(X, Y, W, \eta_{V} Z\right)=-\left\langle\left[\eta_{W}, \eta_{\eta_{V} Z}\right] X, Y\right\rangle
$$

since $H$ is the Kähler nullity of $(g, J)$. At the same time, proposition 3.6 gives

$$
\left[\bar{R}(X, Y), \eta_{V}\right]+\left[\bar{R}(Y, V), \eta_{X}\right]+\left[\bar{R}(V, X), \eta_{Y}\right]=\eta_{\bar{R}(X, Y) V+\bar{R}(Y, V) X+\bar{R}(V, X) Y}
$$

The last two terms of the first summand vanish by the definition of $H$ whilst the use of the first Bianchi identity for $\bar{\nabla}$ in corollary 3.1, (iii) leads after a short computation to $\left[\bar{R}(X, Y), \eta_{V}\right]=\eta_{\eta_{\eta_{V} Y} X-\eta_{\eta_{V} X} Y \text {. Therefore }}$

$$
\begin{aligned}
\bar{R}\left(X, Y, \eta_{V} W, Z\right) & =-\bar{R}\left(X, Y, W, \eta_{V} Z\right)+\left\langle\eta_{\eta_{\eta_{V}} X-\eta_{\eta_{V}} Y} W, Z\right\rangle \\
& =\left\langle\eta_{\eta_{\eta_{V}} X} X-\eta_{\eta_{V} X} Y, Z\right\rangle+\left\langle\left[\eta_{W}, \eta_{\eta_{V}} Z\right] X, Y\right\rangle .
\end{aligned}
$$

In a more condensed form this is re-written as

$$
\begin{equation*}
\bar{R}\left(X, Y, \eta_{V} W, Z\right)=\left\langle\left[\left[\gamma_{X}, \gamma_{Y}\right], \gamma_{Z}\right] V, W\right\rangle \tag{8.2}
\end{equation*}
$$

where for every $X$ in $H$ we have defined $\gamma_{X}: \mathcal{V} \rightarrow \mathcal{V}$ by $\gamma_{X} V=\eta_{V} X$. Note that the vanishing of the torsion on $\mathcal{V}$ implies that $\gamma_{X}$ is symmetric for all $X$ in $H$. The $\bar{\nabla}$-parallelism of $\eta$ leads now to $\overline{\nabla R}=0$, in other words $\bar{\nabla}$ is an Ambrose-Singer connection. We conclude that $\left(M^{2 m}, g\right)$ is locally 3 -symmetric by [24].

The proof of theorem 1.4 in the introduction is now completed by using theorem 1.1 and a density argument.

Remark 8.1. From the proof of theorem 8.2 we get the explicit dependence on the torsion of the curvature tensor $\bar{R}$ of a strictly normal $\mathcal{A K}_{2}$-manifold. This can be used to get algebraic caracterizations as a homogeoneous space of such a manifold. Since this is beyond the scope of this paper it will be omitted.

To finish let us prove theorem 1.6.
Theorem 8.3. (i) Let $\left(M^{2 m}, g\right)$ be a manifold of constant sectional curvature. If there exists an almost complex structure $J$ such that $(g, J)$ is almost Kähler, then $g$ is a flat metric.
(ii) Let $\left(M^{2 m}, g, J\right)$ be a Kähler manifold of constant negative holomorphic sectional curvature. If I is an almost complex structure such that $(g, I)$ is almost Kähler and $[I, J]=0$, then $(g, I)$ must be Kähler.
Proof. Direct verification on the curvature tensor shows that almost Kähler structures as in (i) or (ii) satisfy the second Gray curvature condition and therefore must have parallel torsion by theorem 1.4. We conclude by recalling (cf. theorem 5.1, (i)) that any almost Kähler structure compatible with an Einstein metric and having parallel intrinsic torsion is Kähler.

Note that the result in (i) is not new, with exception of its proof. In dimension beyond 8 it was proven in [34], and in dimension 4 and 6 in [10].

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