A Conformally Invariant Third Order Neumann-Type Operator for Hypersurfaces

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Abstract

On $(M,[g])$, a conformal n-manifold with Σ, an embedded hypersurface (i.e. embedded submanifold of codimension 1) in M, we will construct a conformal Neumann-type operator which is third order in directions transverse to Σ. This operator acts on the same domain space, along Σ, as the conformal fourth order Paneitz operator, in all dimensions ≥ 4.

In order to generate this operator, which we will denote by P_3, we will use techniques involving the conformally invariant tractor bundle and its associated connection.

We will conclude by producing a 3rd order analogue of Branson's Q-curvature in dimension 4 and show that P_3 governs its conformal transformation.

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My parents have always encouraged my study, but over the last year they have reminded me that some things are more important.

The support of my wonderful wife Jane has enabled me to fulfill an as-yet unrealised dream; and though at times it was hard, we have both come through smiling.

Contents

Chapter 1

Introduction and Preliminaries

Conformally invariant differential operators have been important in physics since the turn of the last century. In particular, a special role has been played by those operators which have a leading term that is the power of the Laplacian. The conformally invariant wave operator was the first of these operators to be generated. Among other things, the Dirac wave equation for the neutrino depends only on conformal structure [10]. Interest in the conformal powers of the Laplacian heightened in the early eighties and a conformally invariant fourth order operator on functions was independently constructed by both Paneitz [19] and Riegert (restricted to Riemannian 4-space)[20]. This operator had leading term Δ^2. Not long after this, additional fourth order and sixth order conformal operators on both functions and forms were produced by Branson [4]. The work of Graham, Jenne, Mason & Sparling (GJMS) in the early nineties proved the existence of conformally invariant operators with a power of the Laplacian as leading term [17]. Using the ambient metric construction of Fefferman and Graham, they systematically constructed operators P_{2k}; now known as the GJMS operators. The simplest of these operators, the conformal Laplacian mentioned above, is produced when $k = 1$. Likewise, the Paneitz operator is recovered when $k = 2$.

The conformal Laplacian differs from the Laplacian by the addition of an $(n-2)$ multiple of the scalar curvature, on n-dimensional manifolds. In dimension 2, we see that this scalar curvature term drops away and hence the conformal Laplacian annihilates constant functions. Additionally, the conformal transformation of the scalar curvature is controlled by the Laplacian, when $n = 2$ [23]. Branson extended this curvature concept to the Paneitz operator, and beyond, to the other GJMS operators P_{2k}. He observed that the zero order part, which he called the Q-curvature, is composed solely of terms of curvature tensors and their derivatives. He also showed that we are able to express P_{2k} as $\mathsf{P}^1_{2k} + \frac{(n-2k)}{2}\mathsf{Q}_{2k}$, where P^1_{2k} annihilates constant functions. Hence, the Q-curvature term vanishes in dimension $2k$. Branson noted that its behaviour under conformal rescaling is governed by P_{2k} in the critical dimension $2k$ [8].

One of the drawbacks of trying to construct, and then prove the invariance of, the P_{2k} operators is that the conformal transformations required are complicated. The degree of complication is heavily dependent on the number of terms of the operator, which in turn is dependent upon the order of the operator. The conflagration of terms which are generated as a result of these conformal transformations suggest the need for a simplified algebraic structure. This is provided by a calculus now termed tractor calculus. This uses a conformally invariant connection and other basic operators on an induced invariant bundle, to produce operators which must be invariant by their very construction. There are many advantages to using tractors over more conventional methods. On the one hand the tractor algebra can easily be programmed into a computer to produce symbolic results, and on the other, tractors can be worked with directly. They do not need to use representation theory; despite their inherent link with this branch of mathematics. This link has led to the generalisation of many GJMS operators [16]; and their respective Q-curvatures.

Until recently, one relatively overlooked area of conformal geometry has been the development and classification of conformal boundary value problems and the higher order analogues of the Dirichlet-to-Neumann operator. Inroads have been made recently by Branson & Gover [5]. Their tractor construction produces families of conformally invariant boundary operators which are, in an appropriate sense, compatible with the GJMS operators. However, their construction fails to generate some boundary operators which, it is conjectured, do exist. These are the Neumann-type operators of order $2k-1$ on an embedded codimension 1 submanifold, in dimension $2k$. One example of these; the central focus of this thesis; is the P_3, the third order boundary operator which acts on the same domain as the Paneitz GJMS operator in dimension 4.

In their paper on Zeta Functional Determinants [9], Chang and Qing showed the existence of a P_3 operator and a "T curvature" term which has a conformal transformation similar to the Q curvature. Their construction involved the careful addition of local conformal invariants. However, their construction only gives P_3 for boundaries of 4-manifolds.

In this thesis, the setting for all calculations will be on smooth n-manifolds, M, for $n \geq 4$. The submanifolds we consider will be of co-dimension 1, and we will term these hypersurfaces. For conformal n-manifolds with boundary, we will assume that the conformal structure smoothly extends beyond the boundary of the manifold to some small collar. Hence when we discuss boundary differential operators, we view the boundary as a hypersurface embedded in the extended ambient manifold.

After developing the necessary tools, we will construct a conformally invariant third order Neumann-type operator which acts along the boundaries of conformal manifolds. This operator, which we denote P_3, is, in a suitable sense, compatible with the Paneitz operator and exists in all dimensions ≥ 4.

Finally we see that an application involving conformally invariant boundary problems is in the area of Electrical Impedance Tomography (EIT). One form of EIT involves the attachment of electrodes to a person's body which measure and map the electrical potential and the current on the skin as a response to a small current. In this arena, the impedance (or conversely, the conductivity) of the body is a conformal factor, which is to be reconstructed using methods in inverse problems; and the physical responses to the electrical stimulus are the Dirichlet and Neumann operators respectively [1]. Presently this only deals with the Dirichlet-to-Neumann problem and its inverse, however it is possible that higher order problems could have applications also.

1.1 Conventions

For the body of this paper the Einstein tensor convention will apply. This is described in the following way. If an identical index appears in both an upper position and a lower position in the same product then the tensor will be summed across the index noted, and in this way summation signs will be avoided.

Normally the Einstein convention is only used if a specific frame has been chosen, however this will be extended to include the indices where only the bundle is known and an abstract index will take its place, according to the Penrose abstract index notation [14]. For example, for a tangent vector field, v^a, and a field of 1-forms, u_b, the contraction of these is a scalar function represented by $v^a u_a$.

For the purposes of this paper there will be no notational distinction made between a bundle and the space of all its sections. This difference will be implied from the context. All bundles are based on smooth manifolds and will have the prefix, $\mathcal{E}$: The tangent bundle, in keeping with the index notation for tangent vectors, will adopt the symbol $\mathcal{E}^a$. Likewise, $\mathcal{E}_a$ will denote the cotangent bundle of 1-forms. The tensor product of two bundles will be represented by concatenation of their indices: e.g. $\mathcal{E}_{ab} = \mathcal{E}_a \otimes \mathcal{E}_b$.

The indices on the individual sections will only be used to denote which bundle the section originates from, and not a specific frame. (Note that letters from the start of the alphabet will be used for bundle indices, and letters from the middle will represent frame indices, when introduced).

Round brackets (..) enclosing tensor indices will be used to denote the symmetric part of the tensor or bundle section over the enclosed indices. Similarly, we will use square brackets [..] around indices to denote the skew component of the tensor, over these indices. For the purposes of this paper the scalar field for tensors will always be $\mathbb{R}$.

I have used the computer program Mathematica to carry out many of my calculations, in particular via the *Ricci* package developed by J. Lee. Any Mathematica output in this paper will be in courier font for easy recognition. The Ricci package has a concise notation with regard to indices. Namely, that derivative indices all follow the semi-colon on the tensor.

Chapter 2

Elements of Riemannian Geometry

Let (M, g) be a Riemannian n-manifold, where $n \geq 3$. (Appendix A.0.2)

Definition 2.0.1 (Connection). *[18] A connection on sections of a tensor bundle is a mapping* $\nabla_a : \mathcal{E}^* \longrightarrow \mathcal{E}_a \otimes \mathcal{E}^*$

where $\mathcal{E}^*$ represents the space of sections of an arbitrary tensor bundle such that:

$\forall\, f_1, f_2 \in \mathcal{E}^*, \sigma \in \mathcal{E}$ *and* $\xi^a \in \mathcal{E}^a$

(i)$\nabla_\xi(f_1 + f_2) = \nabla_\xi f_1 + \nabla_\xi f_2$

(ii)$\nabla_\xi(\sigma f_1) = \xi(\sigma) f_1 + \sigma \nabla_\xi f_1$

where $\xi(\sigma) = d\sigma(\xi)$

(iii) $\nabla_{\sigma\xi} f_1 = \sigma \nabla_\xi f_1$

$\nabla_\xi f_1$ *is called the Covariant Derivative of the section f_1 along ξ*

2.1 Levi-Civita Connection

Definition 2.1.1 (Christoffel symbol). *Given a connection on the tangent bundle we denote this by Γ^k_{ij}. It may be derived from the partial derivatives of the Riemannian metric, to form the necessary coordinate functions in the formula for the default connection. Let $\{x_i\}$ be a coordinate frame in a neighbourhood of a point x on (M, g) then the components Γ^k_{ij} are defined by:* $\Gamma^k_{ij}\partial_k = \nabla_i \partial_j$

Definition 2.1.2 (Torsion). *Let $u^a, v^a \in \mathcal{E}^a$ then the torsion is defined by:*
$T(u,v) = \nabla_u v - \nabla_v u - [u,v]$

We see that the Lie bracket, $[.,.]$, appears in the above definition. The definition of this appears in Appendix B.0.1.

Proposition 2.1.3 (Levi-Civita Connection). *Consider (M,g), a Riemannian manifold, and g the metric on the manifold. The Levi-Civita Connection, ∇, is the unique metric compatible, torsion free connection.*
i.e. $ug(v,w) = g(\nabla_u v, w) + g(v, \nabla_u w) \qquad \forall\, u^a, v^a, w^a \in \mathcal{E}^a$
and $T(u,v) = \nabla_u v - \nabla_v u - [u,v] = 0 \qquad \forall\, u^a, v^a \in \mathcal{E}^a$

Proof. Choose a frame $\{x_i\}$ in a neighbourhood of a point $x \in M$
Without loss of generality, let $u = \partial_i, v = \partial_j$

1: Torsion Free

$$\begin{aligned} \nabla_i \partial_j - \nabla_j \partial_i - [\partial_i, \partial_j] &= 0 \\ \Rightarrow \Gamma^k_{ij}\partial_k - \Gamma^k_{ji}\partial_k &= 0 \\ \Gamma^k_{ij} &= \Gamma^k_{ji} \end{aligned}$$

2: Metric Compatibility

Note that in the following part of the proof, the notation, $g_{ij,k}$ will be used to represent $\partial_k g_{ij}$.

Let $u = \partial_k, v = \partial_i, w = \partial_j$

$$\begin{aligned} ug(v,w) &= g(\nabla_u v, w) + g(v, \nabla_u w) \\ \Rightarrow g_{ij,k} &= g_{jl}\Gamma^l_{ki} + g_{il}\Gamma^l_{kj} \end{aligned} \tag{2.1}$$

Let us define $\Gamma_{jki} := g_{jl}\Gamma^l_{ki}$
Then by permutation of the indices:

$$\begin{aligned}
\Rightarrow g_{ij,k} &= \Gamma_{jki} + \Gamma_{ikj} \\
\Rightarrow g_{jk,i} &= \Gamma_{kij} + \Gamma_{jik} \\
\Rightarrow g_{ki,j} &= \Gamma_{ijk} + \Gamma_{kji} \\
\Gamma_{ijk} &= 1/2(g_{ij,k} + g_{ki,j} - g_{jk,i}) \qquad (2.2)
\end{aligned}$$

Thus the Christoffel function is uniquely determined for all metrics, g. Hence this connection exists for all Riemannian manifolds, and it is unique. □

Unless specifically stated otherwise (with the use of brackets and dots), the connection operator, ∇, will act on everything to its right.

2.2 Connection on a Submanifold

Take a submanifold (Appendix A.0.3) (Σ, g^{Σ}) of manifold (M, g). The metric which is intrinsic to Σ is determined by the ambient metric along Σ, acting on vector elements from the tangent bundle of Σ which is viewed as a subbundle of the restriction of the tangent bundle to M. The submanifold has co-dimension 1 and so the restriction of the ambient tangent bundle to Σ may be decomposed into [22]:

$$\mathcal{E}^a|_{\Sigma} = \mathcal{E}^a_{\Sigma} \oplus \mathcal{N}^a$$

Definition 2.2.1 (The Projection Operator). *Let $N^a \in \mathcal{N}^a$ be a unit vector field which is normal to Σ along Σ, then we define:* $\Pi^b_a = \delta^b_a - N_a N^b$

We see that for v^a, a tangent vector in the tangent bundle restricted to Σ, the decomposition follows:

$$v^a \mapsto (v^a - N^a N_b v^b) + N^a N_b v^b = \Pi^a_b v^b + N^a N_b v^b$$

The intrinsic metric g^{Σ} is $g|_{\Sigma}$ restricted to $T\Sigma$, where we view $T\Sigma$ as a subbundle of TM. Thus $g_{ab} - N_a N_b$ extends g^{Σ} to a tensor which acts on ambient tangent vectors so we identify these and write $g^{\Sigma}_{ab} = g_{ab} - N_a N_b$.

Proposition 2.2.2. *Recall that we write $\mathcal{E}^a$ for the tangent bundle of our n-dimensional manifold M and $\mathcal{E}^a_\Sigma$ for the tangent bundle of our $(n-1)$-dimensional submanifold Σ with unit normal, N^a.*
Then for $v^a \in \mathcal{E}^a_\Sigma$, $\nabla^\Sigma_a v$, defined by,

$$\nabla^\Sigma_a v^b = \Pi^b_d \, \Pi^c_a \, \nabla_c v^d$$

is the intrinsic Levi-Civita connection on Σ

Proof. Without loss of generality we will take a frame, $\{x_i\}$, in a neighbourhood of a point, x, on our submanifold Σ.
Take ξ from the tangent space of the submanifold at that point.
We will begin by showing that the formula $\Pi^i_j \, \nabla_\xi v^j$ describes a connection.
Let u^i, v^i be tangent vectors of the manifold, restricted to points on Σ, and σ and ρ be scalar functions.
Using the defn (2.0.1).

$$\begin{aligned} \Pi^j_i \, \nabla_{\rho\xi}(\sigma u^i + v^i) &= \Pi^j_i \rho \xi^k \partial^\Sigma_k(\sigma) u^i + \Pi^j_i(\sigma\rho\xi^k \nabla_k u^i) + \Pi^j_i \rho \xi^k \nabla_k v^i \\ &= \rho\big(\xi^k \partial^\Sigma_k(\sigma) u^j_\Sigma + \sigma \Pi^j_i \nabla_\xi u^i\big) + \rho \Pi^j_i \nabla_\xi v^i \end{aligned}$$

From this we can see that this does behave as we would expect a connection to. Additionally it acts only on intrinsic elements to produce an intrinsic derivative.

We wish to check that the connection is the Levi-Civita connection for g^Σ.

Step 1: Torsion Free

In our frame $\{x_i\}$, let coordinate $x_1 = 0$. Then the connection on vectors from the tangent space of Σ at the point x is given by: $\Pi^{i'}_i \Pi^{j'}_j \nabla_{j'} \partial_{i'} = \Pi^{i'}_i \Pi^{j'}_j \Gamma_{j'}{}^k{}_{i'} \partial_k$. In this instance the k index is an intrinsic index. i.e. ∂_k is a vector which is in the tangent space of the submanifold.

Hence it can be said that: $\Gamma^\Sigma_{ij}{}^k = \Pi^{j'}_j \Pi^{i'}_i \Gamma_{i'}{}^k{}_{j'}$. The commutativity of the projection tensors implies that the intrinsic Christoffel symbols are symmetric. i.e. $\Gamma^\Sigma_{ij}{}^k = \Gamma^\Sigma_{ji}{}^k$. This verifies that the connection is torsion free.

Step 2: Metric Compatibility

Metric compatibility is defined by:

$ug^\Sigma(v,w) = g^\Sigma(\nabla^\Sigma_u v, w) + g^\Sigma(v, \nabla^\Sigma_u w)$ where $u, v, w \in T\Sigma$ and g^Σ is the intrinsic metric. Let $u = \partial_k, v = \partial_i, w = \partial_j$ then

$$\begin{aligned} \partial_k g^\Sigma_{ij} &= g^\Sigma_{jl}\Gamma^{\Sigma\ l}_{i\ k} + g^\Sigma_{il}\Gamma^{\Sigma\ l}_{j\ k} \\ &= g^\Sigma_{jl}\Pi^{i'}_i\Pi^{k'}_k\Gamma^l_{i'\ k'} + g^\Sigma_{il}\Pi^{j'}_j\Pi^{k'}_k\Gamma^l_{j'\ k'} \end{aligned} \tag{2.3}$$

Now by permutation of the indices of equation (2.3) and application of the symmetry of the Christoffel symbols.

$$\partial_i g^\Sigma_{jk} = g^\Sigma_{jl}\Pi^{k'}_k\Pi^{i'}_i\Gamma^l_{k'\ i'} + g^\Sigma_{kl}\Pi^{j'}_j\Pi^{i'}_i\Gamma^l_{j'\ i'} \tag{2.4}$$

$$\partial_j g^\Sigma_{ki} = g^\Sigma_{kl}\Pi^{j'}_j\Pi^{i'}_i\Gamma^l_{j'\ i'} + g^\Sigma_{il}\Pi^{k'}_k\Pi^{j'}_j\Gamma^l_{j'\ k'} \tag{2.5}$$

We see here that adding equations (2.4) and (2.5), and subtracting (2.3) results in:

$$g^\Sigma_{kl}\Pi^{i'}_i\Pi^{j'}_j\Gamma^l_{i'\ j'} = \frac{1}{2}(\partial_i g^\Sigma_{jk} + \partial_j g^\Sigma_{ki} - \partial_k g^\Sigma_{ij})$$

Hence the intrinsic Christoffel symbol $\Gamma^{\Sigma\ l}_{ij}$, defined by $\Pi^{i'}_i\Pi^{j'}_j\Gamma^l_{i'\ j'}$, is uniquely determined by the intrinsic metric. □

2.3 Curvature on Riemannian Manifolds

Definition 2.3.1 (Riemannian Curvature Tensor). *[11] Let $v^a \in \mathcal{E}^a$ and ∇ be the Levi-Civita connection on (M, g) then:*

$$R_{ab\ d}^{\ \ c} v^d = (\nabla_a\nabla_b - \nabla_b\nabla_a)v^c$$

The Riemannian Curvature tensor, $R_{ab\ d}^{\ \ c}$, may be expanded in the following manner:

$$R_{abcd} = C_{abcd} + P_{ac}g_{bd} - P_{bc}g_{ad} + P_{bd}g_{ac} - P_{ad}g_{bc} \tag{2.6}$$

Where

C_{abcd} is the completely trace free Weyl tensor and P_{ab} is the Weyl-Schouten tensor, a symmetric 2-tensor described below.

The trace of the Riemannian Curvature Tensor, formed by contracting over the first and third indices, is the *Ricci Tensor* which is a symmetric 2-tensor denoted R_{ab}.
Let $v^a \in \mathcal{E}^a$ then $R_{ab}v^a = 2\nabla_{[a}\nabla_{b]}\ v^a$

The *Scalar Curvature* is defined by $R = R^a_a$

Proposition 2.3.2. $R_{ab} = g_{ab}J + (n-2)P_{ab}$

where $J = P_a^a$

This is easily obtained by the contraction of indices a and c of eqn (2.6).

Corollary 2.3.3. $R = 2(n-1)J$ *where* $n > 1$

One consequence of the Jacobi identity, in Appendix B.0.1, is the Bianchi identity which specifically applies to the Riemannian curvature tensor. By applying the connection to (2.6) it is not difficult to show that:

$$\nabla_{[e}R_{ab]}{}^c{}_d = 0$$

From a contraction of this we obtain:

$$\nabla_c C_{ab}{}^c{}_d = 2(n-3)\nabla_{[a}P_{b]d}$$

The tensor on the right hand side is an $(n-3)$ multiple of the *Cotton-York tensor*. When $n \leq 3$ the Weyl curvature necessarily vanishes due to symmetry conditions.
Further contraction of the a and the d index when $n > 3$ yields:

$$\nabla^b P_{ab} = \nabla_a J \tag{2.7}$$

This result will be frequently used throughout this paper.

2.4 Curvature on Submanifolds

We will need to relate the curvature of the ambient manifold to that of the embedded submanifold. Remember that our submanifold has dimension $(n-1)$.

Definition 2.4.1 (Second Fundamental Form). *We will henceforth refer to this tensor as SFF, and denote it by* L_{ab}.
Let $N^a \in \mathcal{E}^a$ *be a smooth unit vector field which is normal when restricted to* Σ, *then*

$$L_{ab} = \Pi_a^c \nabla_c N_b$$

and the mean curvature, H, *will be defined by:*

$$H = (n-1)^{-1}\nabla^a N_a$$

The SFF can be decomposed into a sum of its trace free part and the trace only part as follows:

$$L_{ab} = L_{(ab)\circ} + g^{\Sigma}_{ab}H$$

Note that the small circle following the indices represents that the tensor is trace free.

The Codazzi equation (Appendix C.0.1) provides us with a relationship between the Riemannian curvature of Σ in M and the SFF. As pointed out in the appendix, Spivak uses a different convention for his second fundamental form and hence our Codazzi equation will be expressed with the opposite sign.

$$\Pi_a^{a'}\Pi_b^{b'}\Pi_d^{d'}N_c R_{a'b'}{}^{c}{}_{d'} \quad = \quad \nabla^{\Sigma}_b L_{ad} - \nabla^{\Sigma}_a L_{bd} \tag{2.8}$$

Using this we will determine the relationship between the SFF and the Rho curvature tensor.

Proposition 2.4.2. $(n-2)\Pi_{\Sigma}N^b P_{ab} = \nabla^b_{\Sigma}L_{(ab)\circ} - (n-2)\nabla^{\Sigma}_a H$

Proof. Using the Codazzi-Mainardi equation we know that:

$$\Pi_a^{a'}\Pi_b^{b'}\Pi_d^{d'}N_c R_{a'b'}{}^{c}{}_{d'} \quad = \quad \nabla^{\Sigma}_b L_{ad} - \nabla^{\Sigma}_a L_{bd}$$

Contraction over the b and d indices results in:

$$g^{bd}_{\Sigma}\Pi_a^{a'}N_c R_{a'b}{}^{c}{}_{d} \quad = \quad \nabla^b_{\Sigma}L_{ab} - (n-1)\nabla^{\Sigma}_a H$$

The indices b and d are now intrinsic indices to Σ, hence:

$$\Rightarrow \Pi_a^{a'}N_c R_{a'}^{c} \quad = \quad \nabla^b_{\Sigma}L_{ab} - (n-1)\nabla^{\Sigma}_a H$$

Expanding the SFF into its trace and trace free components gives us: $L_{ab} = L_{(ab)\circ} + g^{\Sigma}_{ab}H$ Converting the Ricci tensor into P_{ab} results with $\Pi_a^{a'}N^c R_{a'c} = (n-2)\Pi_a^{a'}N^c P_{a'c} + \Pi_a^{a'}N_{a'}J$ so we obtain our result because $\Pi_a^{a'}N_{a'} = 0$. □

Chapter 3

Conformal Geometry

3.1 Conformal Manifold

Definition 3.1.1 (Conformal Manifold [14]). *A conformal n-Manifold, $(M, [g])$ is a smooth manifold M and an equivalence class of metrics $[g]$.*
For each pair $g, \widehat{g} \in [g]$, there exists some $\Omega \in \mathcal{E}^+$ (Ω is an $\mathbb{R}^+$-valued function)

$$\text{such that } g \sim \widehat{g} = \Omega^2 g$$

Let $(M, [g])$ be a specific conformal manifold. The *bundle of metrics*, $\mathcal{G}$, is a principal bundle with fibre $\mathbb{R}^+$. This induces line bundles of the form:

$$\mathcal{E}[w] = \mathcal{G} \times_{-w/2} \mathbb{R} = \mathcal{G} \times \mathbb{R}/\sim$$

$$\text{where } \sim \text{ is defined by } (\Omega g_x, y) \sim (g_x, (\Omega^{-1})^{-w/2} y) \qquad (g_x \in \mathcal{G}, \Omega \in \mathbb{R}^+, y \in \mathbb{R})$$

The sections (called *densities*), $\bar{\sigma}$, of $\mathcal{E}[w]$ are equivalent to $\mathbb{R}$-valued homogeneous functions, σ, acting on the bundle of metrics, where:

$$\sigma(\Omega^2 g_x, x) = \Omega^w \sigma(g_x, x)$$

The bundle of metrics, $\mathcal{G}$, is isomorphic to $\mathcal{E}^+[-2]$, where $\mathcal{E}^+[-2]$ is the ray subbundle of $\mathcal{E}[-2]$ with fibres $\mathbb{R}^+ \subset \mathbb{R}$ [16]. By choosing a metric g from $[g]$ we define a section of $\mathcal{E}^+[-2]$, $\phi : \mathcal{G} \longrightarrow \mathbb{R}$, by $\phi(\widehat{g}_x, x) = \Omega^{-2}$. Given such a section ϕ, we see that $\phi(g_x, x)g = \phi(\widehat{g}_x, x)\widehat{g}$, so it defines a metric in the conformal class. The *conformal metric* is the section, $\mathbf{g}_{ab} \in \mathcal{E}_{ab}[2]$, which represents the map, $\mathbf{g}_{ab} : \mathcal{E}^+[-2] \longrightarrow \mathcal{E}_{(ab)}$
If we choose ξ^g, a section of $\mathcal{E}[1]$, such that $g = (\xi^g)^{-1}\mathbf{g}$, then we see that the section determines a metric on M. ξ^g is called a choice of conformal scale.

The conformal metric $\mathbf{g}$ gives us a canonical isomorphism from $\mathcal{E}^a[w]$ to $\mathcal{E}_a[w+2]$, by $v^a \mapsto \mathbf{g}_{ab}v^b = v_a$. Similarly its inverse raises tensor indices.

For a particular metric, g, from the conformal class, a trivialised density, σ, is a function on M. Rescale of the metric $g \mapsto \widehat{g} = \Omega^2 g$ will lead to a conformal rescale of $\sigma \mapsto \widehat{\sigma} = \Omega^w \sigma$

The dependence of the Levi-Civita connection on the derivatives of the Riemannian metric implies that a change in metric will result in a change in the connection. After rescaling the metric, g_{ab}, to $\widehat{g}_{ab}$ we are able to see the effects on the connection by way of the following proposition.

3.2 Conformal Transformations of the Connection

Proposition 3.2.1. *Consider $(M, [g])$, a conformal manifold with Levi-Civita connection, ∇_a, for a specific metric g*
Then $\forall\, u_a \in \mathcal{E}_a$, $v^b \in \mathcal{E}^b$

$$(i)\widehat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + g_{ab}\Upsilon_d u^d$$
$$(ii)\widehat{\nabla}_a v^c = \nabla_a v^c + \Upsilon_a v^c - \Upsilon^c v_a + \delta_a^c \Upsilon^d v_d$$

Where
$$\Upsilon_a = \Omega^{-1}\nabla_a \Omega$$

Proof. Choose a coordinate frame $\{x_i\}$ in a neighbourhood of a point, x, of M.
(i)

$$\begin{aligned}
\widehat{\nabla}_i u_j &= \partial_i u_j - u_k \widehat{\Gamma_{ij}^k} \\
\text{But } \widehat{g}_{lk}\widehat{\Gamma}_{ij}^l &= 1/2(\widehat{g}_{jk,i} + \widehat{g}_{ki,j} - \widehat{g}_{ij,k}) \\
&= 1/2(2\Omega\partial_i\Omega g_{jk} + \Omega^2 g_{jk,i} + 2\Omega\partial_j\Omega g_{ki} + \Omega^2 g_{ki,j} - 2\Omega\partial_k\Omega g_{ij} - \Omega^2 g_{ij,k}) \\
\text{Let } \Upsilon_i &= \Omega^{-1}\partial_i\Omega \\
\Rightarrow \widehat{\Gamma}_{ij}^l &= \Gamma_{ij}^l + g^{lk}(\Upsilon_i g_{jk} + \Upsilon_j g_{ki} - \Upsilon_k g_{ij}) \\
\widehat{\nabla}_i u_j &= \partial_i u_j - u_k\left(\Gamma_{ij}^k + g^{lk}(\Upsilon_i g_{jl} + \Upsilon_j g_{li} - \Upsilon_l g_{ij})\right) \\
\widehat{\nabla}_i u_j &= \partial_i u_j + u_k\Gamma_{ij}^k - \delta_j^k\Upsilon_i u_k - \delta_i^k\Upsilon_j u_k + g_{ij}\Upsilon_k u^k \\
\widehat{\nabla}_i u_j &= \nabla_i u_j - \Upsilon_i u_j - \Upsilon_j u_i + g_{ij}\Upsilon_k u^k
\end{aligned} \tag{3.1}$$

(ii)

$$\begin{aligned}\widehat{\nabla}_i v^k &= \partial_i v^k + v^j \widehat{\Gamma}^k_{ij} \\ &= \partial_i v^k + v^j \big(\Gamma^k_{ij} + g^{kl}(\Upsilon_i g_{lj} + \Upsilon_j g_{li} - \Upsilon_l g_{ij})\big) \\ &= \nabla_i v^k + \Upsilon_i v^k - \Upsilon^k v_i + \delta^k_i \Upsilon_j v^j \end{aligned} \tag{3.2}$$

□

Definition 3.2.2 (Connection on Densities). *Let $g \in [g]$ be a particular metric from the conformal class, $x \in M$ and ξ be in the tangent space of M at the point x, then the connection on densities is equivalent to the homogeneous function, $\nabla^g_\xi \sigma$, on $\mathcal{G}$ such that:*

$$(\nabla^g_\xi \sigma)(g_x, x) := g^*(\nabla^g_\xi \sigma)(x) = \xi \cdot (g^*\sigma)(x)$$

Proposition 3.2.3. *Let $\bar{\sigma} \in \mathcal{E}[w]$ and ∇_a be the Levi-Civita connection for a particular metric, g, then the conformal transformation of the connection is given by:*

$$\widehat{\nabla}_a \bar{\sigma} = \nabla_a \bar{\sigma} + w \Upsilon_a \bar{\sigma}$$

Proof. Pick a particular Riemannian metric, g, and a rescaled metric $\widehat{g}$

We know $\bar{\sigma}$ is equivalent to a homogeneous function $(g^*\sigma)(x) = \sigma(g_x, x)$ such that under conformal rescaling: $\sigma(\Omega^2 g_x, x) = \Omega^w \sigma(g_x, x)$

In the following part, the g's have been included on the ∇'s to make it explicit which metric the connection is dependent on.

For $\bar{\sigma} \in \mathcal{E}[w]$, a density, and $\xi^a \in \mathcal{E}^a$, $\nabla^{\widehat{g}}_\xi \bar{\sigma}$ is equivalent to:

$$\begin{aligned}\nabla^{\widehat{g}}_\xi \sigma(\widehat{g}_x, x) &= \xi \cdot (\widehat{g}^*\sigma)(x) \\ &= \xi \cdot (\Omega^w g^*\sigma)(x) \\ &= \Omega^w(\xi \cdot (g^*\sigma))(x) + (\xi\Omega^w)(g^*\sigma)(x) \quad \text{(By the Liebniz rule)} \\ &= \Omega^w(\nabla^g_\xi \sigma)(g_x, x) + w\Omega^w \Upsilon(\xi)\sigma(g_x, x)\end{aligned}$$

However both σ and $\nabla^g_\xi \sigma$ are homogeneous functions described earlier.

$$\begin{aligned}\nabla^{\widehat{g}}_\xi \sigma(\Omega^2 g_x, x) &= (\nabla^g_\xi \sigma)(\Omega^2 g_x, x) + w\Upsilon(\xi)\sigma(\Omega^2 g_x, x) \\ &= \nabla^g_\xi \sigma(\widehat{g}_x, x) + w\Upsilon(\xi)\sigma(\widehat{g}_x, x)\end{aligned}$$

Because $\nabla^{\widehat{g}}_\xi \sigma$ is a homogeneous function on $\mathcal{G}$ this leads us to the conclusion that a conformal density transforms according to the proposition. □

We are able to use the transformations of the connection in conjunction with the Leibniz rule to generate the transformations of higher order tensors. e.g.

For $\omega_{ab} \in \mathcal{E}_{ab}[w]$

$$\widehat{\nabla_c}\omega_{ab} = \nabla_c\omega_{ab} + (w-2)\Upsilon_c\omega_{ab} - \Upsilon_b\omega_{ac} - \Upsilon_a\omega_{cb} + \Upsilon^d\omega_{ad}\mathbf{g}_{bc} + \Upsilon^d\omega_{db}\mathbf{g}_{ac}$$

3.3 Conformal Transformation of Curvature Tensors

The conformal transformation of the Levi-Civita connection induces the following transformation of the Riemannian curvature tensor [11]:

$$\begin{aligned}\widehat{R}_{abcd} &= R_{abcd} - 2g_{c\,[\,a}\nabla_{b]}\Upsilon_d + 2g_{c\,[\,a}\Upsilon_{b]}\Upsilon_d - 2g_{d\,[\,b}\nabla_{a]}\Upsilon_c \\ &\quad + 2g_{d\,[\,b}\Upsilon_{a]}\Upsilon_c - g_{c\,[\,a}g_{b]\,d}\Upsilon_e\Upsilon^e - g_{c\,[\,a}g_{b]\,d}\Upsilon_e\Upsilon^e\end{aligned}$$

We are able to see that the variation is entirely by traces, and therefore the Weyl curvature is conformally invariant because it is totally trace free. We can now easily derive the following equations for P_{ab}, and its trace J, from the above.

$$\begin{aligned}\widehat{P}_{ab} &= P_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\mathbf{g}_{ab}\Upsilon_c\Upsilon^c \\ \widehat{J} &= J - \nabla^c\Upsilon_c + \frac{2-n}{2}\Upsilon^c\Upsilon_c\end{aligned} \tag{3.3}$$

Our discussion on curvatures cannot be complete without looking at the transformation of the second fundamental form of Σ.

Proposition 3.3.1. *The trace free component of the second fundamental form (SFF), is invariant*

Proof. Recall that the SFF of a hypersurface is a projection of the Levi-Civita connection acting on a unit cotangent vector, $N_a \in \mathcal{E}_a[1]$, which is normal to the hypersurface. Equation (3.1) shows us how this derivative transforms. Here we are looking at the restriction of the connection along Σ.

$$L_{ab} \in \mathcal{E}^{\Sigma}_{ab}[1]$$

$$\begin{aligned}\widehat{L}_{ab} &= \Pi^c_a\widehat{\nabla}_c N_b = \Pi^c_a(\nabla_c N_b - \Upsilon_b N_c + g_{cb}\Upsilon_d N^d) \\ &= L_{ab} + g^{\Sigma}_{ab}\Upsilon_c N^c \\ \widehat{L}_{(ab)\circ} &= L_{(ab)\circ}\end{aligned}$$

□

From this we can see that the transformation of the mean curvature is $\widehat{H} = H + \Upsilon_a N^a$

3.4 Conformal Operators

The conformal operators which we are particularly interested in are those which are called GJMS operators [17]. These are essentially the conformally invariant powers of the Laplacian. The first known of these was the conformal wave equation, which we will construct below.

Definition 3.4.1 (□ Operator). *$\Box := \Delta + wJ$ where w is the weight of the domain space of densities it acts on.*

Proposition 3.4.2. *Let $\bar{\sigma} \in \mathcal{E}[w]$ then*

$\Box\bar{\sigma} = (\Delta + wJ)\bar{\sigma}$ is conformally invariant

if and only if $w = \frac{2-n}{2}$.

This is the conformal Laplacian.

Proof. By repeated application of equations (3.1),(3.2),(3.3) & Prop. (3.2.3) we are able to demonstrate this invariance.

Let $\bar{\sigma} \in \mathcal{E}[w]$

$$\begin{aligned}
\Box\bar{\sigma} &= (\Delta + wJ)\bar{\sigma} \\
&= \nabla^a \nabla_a \bar{\sigma} + wJ\bar{\sigma} \\
\widehat{\Box\bar{\sigma}} &= \widehat{\nabla}^a \widehat{\nabla}_a \bar{\sigma} + w\widehat{J}\bar{\sigma} \\
&= \mathbf{g}^{ab}(\widehat{\nabla}_b \widehat{\nabla}_a \bar{\sigma} + w\widehat{P}_{ab}\bar{\sigma}) \\
&= \mathbf{g}^{ab}\big(\nabla_b \widehat{\nabla}_a \bar{\sigma} - \Upsilon_b \widehat{\nabla}_a \bar{\sigma} - \Upsilon_a \widehat{\nabla}_b \bar{\sigma} + \mathbf{g}_{ab}\Upsilon^c \widehat{\nabla}_c \bar{\sigma} + w\Upsilon_b \widehat{\nabla}_a \bar{\sigma} + \\
&\quad + w\bar{\sigma}(P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\mathbf{g}_{ab}\Upsilon^c \Upsilon_c)\big) \\
&= \mathbf{g}^{ab}\big(\nabla_b(\nabla_a \bar{\sigma} + w\Upsilon_a \bar{\sigma}) - \Upsilon_b(\nabla_a \bar{\sigma} + w\Upsilon_a \bar{\sigma}) - \Upsilon_a(\nabla_b \bar{\sigma} + w\Upsilon_b \bar{\sigma}) + \\
&\quad + \mathbf{g}_{ab}\Upsilon^c(\nabla_c \bar{\sigma} + w\Upsilon_c \bar{\sigma}) + w\Upsilon_b(\nabla_a \bar{\sigma} + w\Upsilon_a \bar{\sigma}) + \\
&\quad + w\bar{\sigma}(P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\mathbf{g}_{ab}\Upsilon^c \Upsilon_c)\big)
\end{aligned}$$

Use of the Leibniz rule and careful expansion will generate:

$$\begin{aligned} &= \mathbf{g}^{ab}\big(\nabla_b\nabla_a\bar{\sigma} + wP_{ab}\bar{\sigma} + \mathbf{g}_{ab}\Upsilon^c\nabla_c\bar{\sigma} + w/2\mathbf{g}_{ab}\Upsilon^c\Upsilon_c\bar{\sigma} + \\ &\quad (2w-2)\Upsilon_b\nabla_a\bar{\sigma} + (w^2-w)\Upsilon_b\Upsilon_a\bar{\sigma}\big) \\ &= \Delta\bar{\sigma} + wJ\bar{\sigma} + (n+2w-2)(\Upsilon\cdot\nabla)\bar{\sigma} + \frac{w}{2}(n+2w-2)(\Upsilon\cdot\Upsilon)\bar{\sigma} \end{aligned} \tag{3.4}$$

In order for the $\Box$ operator to be invariant, w, the weight of the density must be equal to $\frac{2-n}{2}$. $\Box$

We can see that the number of terms would increase exponentially with the order of the operator and this makes for very complicated calculations. This, of itself, is a strong motivation towards developing a calculus which works within the conformal structure of the manifold. The following chapter will look into this.

As an aside, we will look at two features of the conformal Laplacian in dimension 2. The first of these is that in this critical dimension we see that the Laplacian, Δ is conformally invariant. The scalar curvature term drops away and the operator annihilates constant functions. Secondly, as briefly stated in the introduction, the transformation of the curvature term, J, can be obtained from the Yamabe operator itself [23]. By rewriting Υ_a as $\nabla_a\Upsilon$, where $\Upsilon = \log_e(\Omega)$, we observe that the trace only version of equation (3.3) is expressed as $\widehat{J} = J + \nabla^a\nabla_a\Upsilon$. This may be identified with $\widehat{J} = J + \Box\Upsilon$ in this dimension. This is the characteristic of Branson's Q-curvatures (of which the scalar curvature is the simplest) which will be explained in greater depth in later chapters.

Generalising back to arbitrary dimension ≥ 4, the Yamabe operator applied to densities of weight 1 is not invariant. This can be seen by substituting $w = 1$ into the last proposition. Note that despite this non-invariance, the following proposition reveals that the *trace free* Yamabe operator is invariant on weight-1 densities. This prepares us for the conformal calculus.

Proposition 3.4.3. $\nabla_{(a}\nabla_{b)\circ} + P_{(ab)\circ}$ *is conformally invariant when acting on densities from* $\mathcal{E}[1]$.

Proof.

$$
\begin{aligned}
&\text{Let } \bar{\sigma} \in \mathcal{E}[1] \\
\widehat{\nabla}_a \widehat{\nabla}_b \bar{\sigma} + \widehat{P_{ab}}\bar{\sigma} &= \widehat{\nabla}_a(\nabla_b \bar{\sigma} + \Upsilon_b \bar{\sigma}) + (P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{\mathbf{g}_{ab}}{2}\Upsilon^c \Upsilon_c)\bar{\sigma} \\
&= \nabla_a \nabla_b \bar{\sigma} - \Upsilon_a \nabla_b \bar{\sigma} + \mathbf{g}_{ab} \Upsilon^c \nabla_c \bar{\sigma} \\
&\quad + \nabla_a \Upsilon_b \bar{\sigma} - \Upsilon_b \Upsilon_a \bar{\sigma} + \mathbf{g}_{ab} \Upsilon^c \Upsilon_c \bar{\sigma} \\
&\quad + P_{ab}\bar{\sigma} - \bar{\sigma} \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b \bar{\sigma} - \frac{\mathbf{g}_{ab}}{2} \Upsilon^c \Upsilon_c \bar{\sigma} \\
&= \nabla_a \nabla_b \bar{\sigma} + P_{ab}\bar{\sigma} + \mathbf{g}_{ab}(\Upsilon^c \nabla_c \bar{\sigma} + \frac{1}{2}\Upsilon^c \Upsilon_c \bar{\sigma})
\end{aligned}
$$

$\nabla_a \nabla_b \bar{\sigma} + P_{ab}\bar{\sigma}$ conformally transforms by trace only and hence the trace free operator $\nabla_{(a}\nabla_{b)_\circ} + P_{(ab)_\circ}$ is invariant.

□

Chapter 4

Tractor Calculus

At the end of the last chapter we saw that when carrying out the transformations on a differential operator of order 2 the equations can become complicated. In order to avoid these extensive calculations for higher order differential operators, the concept of the *Tractor Calculus* and its associated bundle will be introduced. Note that from this point onwards, the bars will be omitted from the sections of weighted bundles. This has been done because we will have no further need of identifying the conformal factors with $\mathbb{R}$-valued homogeneous functions. We will work with them exclusively in the conformal setting.

4.1 Construction of the Tractor Bundle

We know that the trace free part of $(\nabla_a \nabla_b + P_{ab})\sigma$ is conformally invariant for $\sigma \in \mathcal{E}[1]$.

If σ is a solution of the equation, $\nabla_{(a}\nabla_{b)_\circ}\sigma + P_{(ab)_\circ}\sigma = 0$
then this is equivalent to the following pair of equations (this is explained in greater detail on pg 6 of [2]):

$$\begin{aligned} \nabla_b \sigma - \mu_b &= 0 \\ \nabla_b \mu^a + \delta^a_b \rho + P^a_b \sigma &= 0 \end{aligned} \tag{4.1}$$

where $\mu^a \in \mathcal{E}^a[-1]$ and $\rho \in \mathcal{E}[-1]$

It is not a difficult exercise to determine the transformation formulae for μ and ρ, in order that this system is conformally invariant:

$$\begin{aligned} \hat{\mu}^a &= \mu^a + \Upsilon_a \sigma \\ \hat{\rho} &= \rho - \Upsilon_b \mu^b - \frac{1}{2}\Upsilon^b \Upsilon_b \sigma \end{aligned} \tag{4.2}$$

The three functions σ, μ^a, ρ, and the system of differential equations (4.1) are sufficient to generate the conformally invariant tractor bundle. We will adopt the definition in Prop 2.2 of [2].

Definition 4.1.1 (Standard Tractor Bundle). *For a particular choice of conformal scale, g,* $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$

A particular section, $[V^A]_g \in [\mathcal{E}^A]_g$, is defined by:

$$[V^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}$$

Where σ, μ^a, ρ satisfy the condition of conformal invariance of the set of differential equations (4.1). In a new scale the trivialised tractor transforms to:

$$[V^A]_{\hat{g}} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a \sigma \\ \rho - \Upsilon_b \mu^b - \frac{1}{2}\Upsilon^b \Upsilon_b \sigma \end{pmatrix} \tag{4.3}$$

The tractor $V^A \in \mathcal{E}^A$ is conformally invariant, so $[V^A]_g$ is identified with $[V^A]_{\hat{g}}$ in the tractor section.

Definition 4.1.2 (Coupled Levi-Civita Tractor Connection). *[2]*
If we take a tractor $[V^A]_g \in [\mathcal{E}^A]_g$ then the connection operates in the following way:

$$[\nabla_b V^A]_g = \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu^a + \delta^a_b \rho + P^a_b \sigma \\ \nabla_b \rho - P_{cb} \mu^c \end{pmatrix} \tag{4.4}$$

The term in the bottom slot is generated as a consequence of differentiating the second equation.

The connection acting on a tractor is defined by the system of differential equations described on page 21. This is invariant because we know that the terms of the terms of the tractor transform according to (4.3).

4.2 The Splitting Operators

The power of the tractor bundle lies in the fact that the structure does not require a chosen metric from the conformal class. However, this can be difficult to explicitly work with, so using the splitting for a particular metric is a clear way of seeing how the components of the tractors transform.

Definition 4.2.1 (Splitting Operators). *[16] For a particular choice of conformal scale we define the following set of three operators X^A, Y^A & Z_a^A below.*
We know that for a chosen $g \in [g]$ the tractor splits by:

$$[V^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}$$

where σ, μ^a, ρ are defined by Defn. (4.1.1).
We define $[X^A]_g$ to be the invariant injection:

$$\rho \mapsto \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}$$

and $[Y^A]_g$ and $[Z^A_a]_g$ are the non-invariant mappings:

$$\sigma \mapsto \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} \quad \textit{and} \quad \mu^a \mapsto \begin{pmatrix} 0 \\ \mu^a \\ 0 \end{pmatrix}$$

Using the conformal metric $\mathbf{g}_{ab}$ the splitting operators may be described by:

$$X^A \in \mathcal{E}^A \otimes (\mathcal{E}[-1])^* \cong \mathcal{E}^A[1]$$
$$X^A : \mathcal{E}[-1] \longrightarrow \mathcal{E}^A$$
$$\rho \longmapsto X^A \rho$$

$$Y^A \in \mathcal{E}^A[-1]$$
$$Y^A : \mathcal{E}[1] \longrightarrow \mathcal{E}^A$$
$$\sigma \longmapsto Y^A \sigma$$

$$Z^A_a \in \mathcal{E}^A_a[1]$$
$$Z^A_a : \mathcal{E}^a[-1] \longrightarrow \mathcal{E}^A$$
$$\mu^a \longmapsto Z^A_a \mu^a$$

Let $V^A \in \mathcal{E}^A$, be an invariant tractor section, $V^A = Y^A\sigma + Z^A_a \mu^a + X^A \rho$, where σ, μ^a, ρ are in the sense of Defn (4.1.1), then we can deduce the transformations of the splitting operators.

X^A is invariant by definition and Y^A and Z^A_a transform in the following manner:

$$\hat{Y^A} = Y^A - \Upsilon^a Z^A_a - \frac{1}{2}\Upsilon_b \Upsilon^b X^A \tag{4.5}$$
$$\hat{Z^A_a} = Z^A_a + \Upsilon_a X^A \tag{4.6}$$

This is not difficult to prove using the conformal transformations described by equations (4.3).

Definition 4.2.2 (Tractor Metric). *The tractor bundle has an invariant metric composed of a sum of the splitting operators,* $h_{AB} = X_A Y_B + Z^a_A Z_{Ba} + Y_A X_B$.

Note that the Z tractors are contracted over the tensor index a via the conformal metric, $\mathbf{g}_{ab}$.

The tractor metric and its inverse respectively lower and raise tractor indices just as the Riemannian metric does.

e.g. $$h_{AB}U^B V^A = U_A V^A$$

The following table is derived by taking the inner product of the metric with each of the splitting operators in turn.

	Y^A	Z_i^A	X^A
Y_A	0	0	1
Z_{Aj}	0	g_{ij}	0
X_A	1	0	0

Proposition 4.2.3. *[16] Let ∇_a be the coupled Levi-Civita tractor connection then*

$$\begin{aligned} \nabla_b X^A &= Z_b^A \\ \nabla_b Y^A &= P_b^c Z_c^A \\ \nabla_b Z_a^A &= -P_{ab} X^A - \mathbf{g}_{ab} Y^A \end{aligned} \tag{4.7}$$

Proof. Let $V^A = Y^A\sigma + Z_b^A\mu^b + X^A\rho$ be an invariant tractor section in $\mathcal{E}^A$. Using the Liebniz rule to expand the connection acting on V^A we obtain:

$$\nabla_a V^A = Y^A\nabla_a\sigma + \sigma\nabla_a Y^A + Z_b^A\nabla_a\mu^b + \mu^b\nabla_a Z_b^A + X^A\nabla_a\rho + \rho\nabla_a X^A$$

But the definition of the connection acting on V^A gives us:

$$\nabla_a V^A = Y^A(\nabla_a\sigma - \mu_a) + Z_b^A(\nabla_a\mu^b + \delta_a^b\rho + P_a^b\sigma) + X^A(\nabla_a\rho - P_{ac}\mu^c)$$

Hence by expanding and equating the two equations, we are able to determine the derivative of each of the respective splitting operators. □

Corollary 4.2.4. *The tractor metric, h_{AB}, is preserved by the connection and therefore commutes with it.*

i.e.

$$(i)\nabla_c h_{AB} = 0$$
$$(ii)\nabla_c V_A = h_{AB}\nabla_c V^B$$

Proof. For (i), all terms in the expansion of the metric cancel by way of proposition (4.2.3).
(ii) is a direct consequence of (i). □

The use of X, Y, Z and their associated contraction and differentiation rules now allow us to write computer code which will symbolically solve tractor problems. This leads to significant time saving in the calculation of conformal operators. Appendix D contains all of the macros which I have used or written to verify my hand calculations in the remainder of this paper.

4.3 Tractor Operators

From this point, a tractor, or tractor section, refers to a tractor from a bundle with any number of tractor indices and with any weight. We will use the notation $\mathcal{E}^*[w]$ to represent any tractor bundle of weight w. All tractors are conformally invariant however the Levi-Civita - tractor connection is only conformally invariant when acting on tractors of weight zero. This leaves an opportunity to try and develop differential operators which are invariant on arbitrary tractor bundles. One way of producing these invariant weighted operators is to use the tractor-D operator, which will be derived shortly. To begin with, we will work with an intermediary operator, $\tilde{D}^A$ [14].

Definition 4.3.1. *Let* $\psi \in \mathcal{E}^*[w]$ *then*

$$\tilde{D}^A : \mathcal{E}^*[w] \longrightarrow \mathcal{E}^A[-1] \otimes \mathcal{E}^*[w]$$

is defined by $\tilde{D}^A\psi = Y^A w\psi + Z^{Aa}\nabla_a\psi$

It is an interesting aside to note that the contracted action of $\tilde{D}$ on itself produces the □ operator defined on page 17. We will quickly carry out a calculation to verify this.

Let $\psi \in \mathcal{E}^*[w]$

$$\begin{aligned}
\tilde{D}^A \tilde{D}_A \psi &= Y^A (w-1) \tilde{D}_A \psi + Z^{Aa} \nabla_a \tilde{D}_A \psi \\
&= Y^A (w-1)(Y_A w \psi + Z_{Ab} \nabla^b \psi) + Z^{Aa} \nabla_a (Y_A w \psi + Z_{Ab} \nabla^b \psi) \\
&= Y^A Y_A w(w-1)\psi + Y^A Z_{Ab}(w-1)\nabla^b \psi + Z^{Aa}\big(Y_A w \nabla_a \psi + Z_A^b w P_{ab} \psi \\
&\quad + Z_{Ab} \nabla_a \nabla^b \psi + (-P_{ab} X_A - \mathbf{g}_{ab} Y_A) \nabla^b \psi\big) \\
&= wJ\psi + \Delta\psi \\
&= \Box \psi
\end{aligned}$$

Using $\tilde{D}^2$ provides us with a clue on how to produce the higher order conformal operators. The conformal transformation of $\tilde{D}$ is very important for the construction of the tractor-D operator so we will look at this below.

Proposition 4.3.2. $\widehat{\tilde{D}_A \psi} = \tilde{D}_A \psi + X_A (\Upsilon^a \nabla_a \psi + \frac{w}{2} \Upsilon^a \Upsilon_a \psi)$

Proof.

$$\begin{aligned}
\widehat{\tilde{D}_A \psi} &= \hat{Y}_A w \psi + \hat{Z}_A^a \hat{\nabla}_a \psi \\
&= (Y_A - Z_A^b \Upsilon_b - \tfrac{1}{2} X_A \Upsilon^b \Upsilon_b) w \psi + (Z_A^a + X_A \Upsilon^a)(\nabla_a \psi + w \Upsilon_a \psi)
\end{aligned}$$

Simplifying gives us the required result. □

Using $\tilde{D}^A$ we are able to generate a conformally invariant "Derivative" tractor-D operator which acts on tractors of any weight. Some of the properties of the tractor-D operator will be explored here, but none of the history of its derivation. See [14] for a rigorous derivation of the tractor-D operator.

Definition 4.3.3 (Tractor D Operator). *Let* $D^A : \mathcal{E}^*[w] \longrightarrow \mathcal{E}^{A\,*}[w-1]$ *be defined by*

$$D^A \psi := (n + 2w - 2) \tilde{D}^A \psi - X^A \Box \psi$$

Corollary 4.3.4. *The tractor-D is conformally invariant on weighted tractor sections.*

Proof. Recall that the transformation of the Yamabe operator, equation (3.4), is:

$$\widehat{\Box \psi} = \Box \psi + (n + 2w - 2)(\Upsilon \cdot \nabla)\psi + \tfrac{w}{2}(n + 2w - 2)(\Upsilon \cdot \Upsilon)\psi$$

Let $\psi \in \mathcal{E}^*[w]$ then:

$$\begin{aligned}
\widehat{D_A\psi} &= (n+2w-2)\widehat{\tilde{D}_A\psi} - X_A\widehat{\Box\psi} \\
&= (n+2w-2)\big(\tilde{D}_A\psi + X_A(\Upsilon^a\nabla_a\psi + \tfrac{w}{2}\Upsilon^a\Upsilon_a\psi)\big) \\
&\quad -X_A\big(\Box\psi + (n+2w-2)(\Upsilon^a\nabla_a)\psi + \tfrac{w}{2}(n+2w-2)(\Upsilon^a\Upsilon_a)\psi\big) \\
&= (n+2w-2)\tilde{D}^A\psi - X^A\Box\psi
\end{aligned}$$

□

Using the definition of the tractor-D operator, we observe that the contracted action of the operator on each of the splitting operators is given in the following way.

If $\psi \in \mathcal{E}^*[w]$ then

$$D^AY_A\psi = (n+w-2)J\psi - \Delta\psi \tag{4.8}$$

$$D^AZ_A^a\psi = (n+2w-2)\nabla^a\psi \tag{4.9}$$

$$D^AX_A\psi = (n+2w+2)(n+w)\psi \tag{4.10}$$

This result allows the tractor-D to act on the terms independently and produces predictable outcomes. It provides some simple rules to work with and prevents the need in the future to completely expand the formula involving the tractor-D.

Here we will digress briefly in order to look at another invariant operator. This operator bears a resemblance to the tractor-D, however it acts on 1-forms instead of tractor densities.

Proposition 4.3.5. *[6] Let $\mu_a \in \mathcal{E}_a[w]$ then the operator, $A_A{}^a$, defined by the mapping*

$$\mu_a \longrightarrow \begin{pmatrix} 0 \\ (n+w-2)\mu_a \\ -\nabla^b\mu_b \end{pmatrix} \tag{4.11}$$

is conformally invariant.

Proof. Written in terms of the tractor splitting operators the A^a_A operator looks like:

$$(n+w-2)Z^a_A\mu_a - X_A\nabla^b\mu_b$$

The transformation under conformal rescale can be carried out by applying equations (3.1) and (4.6) to the above equation.

$$\begin{aligned}(n+w-2)\widehat{Z^a_A\mu_a} - \widehat{X_A\nabla^b}\mu_b &= (n+w-2)(Z^a_A + \Upsilon^a X_A)\mu_a \\ &\quad -X_A(\nabla^b\mu_b + (n+w-2)\Upsilon^b\mu_b) \\ &= (n+w-2)Z^a_A\mu_a - X_A\nabla^b\mu_b\end{aligned}$$

□

The tractor $A_A{}^a\mu_a \in \mathcal{E}_A[w-1]$, so when $w = \frac{4-n}{2}$ we know that the conformal Laplacian acting on this tractor is invariant, by Prop. 3.4.2. On 4-manifolds this generates the following tractor, described on page 240 of [6]:

$$\begin{pmatrix} 0 \\ 2\nabla^b\nabla_{[b}\mu_{a]} \\ \frac{-1}{2}\nabla_b(\nabla^b\nabla^c + 4P^{bc} - 2Jg^{bc})\mu_c \end{pmatrix}$$

The bottom slot contains an extension to the Maxwell operator (in the secondary slot). In other dimensions, we know that there are curvature terms associated with the operator in the secondary slot. We will begin by expanding the equation:

$$\Box A_A{}^a\mu_a = \frac{n}{2}\Delta Z^a_A\mu_a - \Delta X_A\nabla^b\mu_b + \frac{n(2-n)}{4}JZ^a_A\mu_a - \frac{2-n}{2}JX_A\nabla^b\mu_b$$

This has been calculated using Ricci to produce the tractor:

$$\begin{gathered}Z^a_A(\tfrac{n}{2}\Delta\mu_a - 2\nabla^b\nabla_a\mu_b - \tfrac{(n-4)(n+2)}{4}J\mu_a + (n-4)P_{ab}\mu^b) \\ -X_A\big(\nabla_c(\nabla^c\nabla^b - \tfrac{n}{2}g^{bc}J + nP^{bc})\mu_c\big)\end{gathered}$$

When $n = 4$, the curvature terms in the Z^a_A coefficient fall away, and (up to scale) there is agreement with the tractor in the middle of the page.

4.4 Tractor Curvature

As we saw earlier the Riemannian curvature tensor measures the deviation of a particular Riemannian structure from the flat case. Similarly, the variance of conformal manifolds from conformally flat models can be measured by the *tractor curvature tensor* [2].

Definition 4.4.1 (Tractor Curvature). *Let $V^A \in \mathcal{E}^A$ and ∇ be the coupled Levi-Civita tractor connection then the tractor curvature, ${\Omega_{ab}}^C{}_D$ is defined by:*

$$ {\Omega_{ab}}^C{}_D V^D = (\nabla_a \nabla_b - \nabla_b \nabla_a) V^C $$

Proposition 4.4.2. *Let $V^C = Y^C \sigma + Z^C_k \mu^k + X^C \rho \quad \in \mathcal{E}^C$ then:*

$$ [{\Omega_{ij}}^C{}_D V^D]_g \quad = \begin{pmatrix} 0 & 0 & 0 \\ 2\nabla_{[i} {P_{j]}}^k & {C_{ij}}^k{}_l & 0 \\ 0 & -2\nabla_{[i} P_{j]l} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^l \\ \rho \end{pmatrix} $$

Proof. Let $V^C = Y^C \sigma + Z^C_k \mu^k + X^C \rho \quad \in \mathcal{E}^C$
then by definition:

$$ {\Omega_{ji}}^C{}_D V^D = (\nabla_j \nabla_i - \nabla_i \nabla_j) V^C $$

The expansion of the right hand side of the above equation was carried out using Ricci. The code is in Appendix D and the output is Fig. 4.1

Out[3]=
$$ -mu_k \, {P_i}^k{}_{;j} \, X^{AA} + mu_k \, {P_j}^k{}_{;i} \, X^{AA} + rho_{;i\,j} \, X^{AA} - rho_{;j\,i} \, X^{AA} + sigma_{;i\,j} \, Y^{AA} - sigma_{;j\,i} \, Y^{AA} + mu_k \, {P_j}^k \, {Z^{AA}}_i - mu_k \, {P_i}^k \, {Z^{AA}}_j + mu_{k\,;i\,j} \, Z^{AA\,k} - mu_{k\,;j\,i} \, Z^{AA\,k} + mu_j \, P_{i\,k} \, Z^{AA\,k} - mu_i \, P_{j\,k} \, Z^{AA\,k} + P_{i\,k\,;j} \, sigma \, Z^{AA\,k} - P_{j\,k\,;i} \, sigma \, Z^{AA\,k} $$

Figure 4.1: Ω - the tractor curvature

The output requires some simplification, but we know that $\nabla_a \nabla_b \sigma = \nabla_b \nabla_a \sigma$ when σ is a scalar function and by (2.6) we can see that:

$$C_{ji}{}^{k}{}_{l}\, \mu^l = 2\nabla_{[j}\nabla_{i]}\, \mu^k - \mu^l g_{il} P_j^k + \mu^l g_{jl} P_i^k - \mu^l \delta_j^k P_{il} + \mu^l \delta_i^k P_{jl}$$

Therefore:

$$\Omega_{ji}{}^{C}{}_{D} V^D \quad = \quad -2\nabla_{[j} P_{i]\,l}\, \mu^l X^C + (C_{ji}{}^{k}{}_{l} \mu^l + 2\nabla_{[j} P_{i]}{}^{k} \sigma) Z_k^C$$

This is equivalent to the proposition, using the X, Y, Z notation. □

When $n \geq 4$ we know that the following is true [2]:
The coupled Levi-Civita tractor connection is flat if the Weyl curvature is zero. (Recall that $2(n-3)\nabla_{[a} P_{b]d} = \nabla_c C_{ab}{}^{c}{}_{d}$).

Chapter 5

Tractor Operators on a Submanifold

5.1 Construction of an Intrinsic Tractor Bundle

Let $(M, [g])$ be a conformal n-manifold, and Σ be an embedded hypersurface of M of co-dimension 1. The pullback, e^*, of the embedding operator, $e : \Sigma \hookrightarrow M$, gives Σ a conformal structure.

In a Riemannian setting, $g^\Sigma = e^* g$, where g is a metric on M and g^Σ is a metric on Σ, so for each pair $g, \hat{g} \in [g]$, where $\hat{g} = \Omega^2 g$, we have $\widehat{g^\Sigma} = e^* \hat{g} = e^*(\Omega^2 g) = (e^*\Omega)^2 g^\Sigma$

This now describes the conformal $(n-1)$-submanifold, $(\Sigma, [g_\Sigma])$. Along Σ, Branson & Gover [5] show that we are able to decompose our tractor bundle $\mathcal{E}_A|_\Sigma$ into a "tangential" component and a "normal" component. The bundle restricted to the hypersurface Σ, is $\mathcal{E}_A|_\Sigma = \mathcal{E}_A^\Sigma \oplus \mathcal{N}_A$.

This will be illustrated in the next couple of pages.

With regards to the notation, in most cases the restriction to Σ will be implied by the context rather than explicitly written.

Definition 5.1.1 (The Normal Tractor). *Let $N_a \in \mathcal{E}_a[1]$, be a unit vector field along Σ, which is normal to Σ, and $H \in \mathcal{E}[-1]$ be the mean curvature*
Then:

$N_A = Z_A^a N_a - X_A H \in \mathcal{N}_A$ *or*

$$[N_A]_g = \begin{pmatrix} 0 \\ N_a \\ -H \end{pmatrix} \tag{5.1}$$

Proposition 5.1.2. *The normal tractor is conformally invariant and has unit length.*

Proof. We will work with the splitting operator notation to keep the proof compact.

$$\begin{aligned} \widehat{N}_A &= \widehat{Z}_A^a \widehat{N}_a - \widehat{X}_A \widehat{H} \\ &= (Z_A^a + X_A \Upsilon^a) N_a - X_A(H + \Upsilon^b N_b) \\ &= N_A \\ N_A N^A &= (Z_A^a N_a - X_A H)(Z_b^A N^b - X^A H) \\ &= \delta_b^a N_a N^b = 1 \end{aligned}$$

□

Let us identify the intrinsic tractor bundle to Σ with that tractor subbundle which is orthogonal to N^A. The normal tractor gives us the ability to project tractors into the intrinsic tractor bundle using the projection defined by:

$\Pi_\Sigma = (\delta_A^B - N_A N^B)$

We see that for V^A, a tractor in the ambient tractor bundle restricted to Σ, the decomposition follows:

$$V^A \mapsto (V^A - N^A N_B V^B) + N^A N_B V^B = \Pi_\Sigma V^A + N^A N_B V^B$$

When acting on weighted tractors, the projection sends the ambient bundle restricted to Σ to the bundle intrinsic to the submanifold, $\Pi_\Sigma(\mathcal{E}_A) = \mathcal{E}_A^\Sigma$.

Definition 5.1.3 (Intrinsic Metric on Σ). $h_{AB}^\Sigma = h_{AB} - N_A N_B$

It is interesting to note that along Σ the splitting tractor, X_A, agrees with its projection into the intrinsic tractor bundle of the hypersurface. N_A is orthogonal to $\mathcal{E}^{\Sigma}_A$, so contraction of N_A with X^A is zero. This clearly follows from the fact that there is no Y component in N_A. When the second fundamental form is trace free, Y^A is equivalent to Y^A_{Σ} and hence contraction with the normal tractor is also zero in this case.

5.2 The Splitting Operators of the Intrinsic Tractor Bundle

The next step will be to determine a set of tractors in the intrinsic tractor bundle which act like the ambient tractor splitting operators when differentiated and contracted. The method used when working in the tangent space is to project using the projection tensor. Hence one possibility will be to pursue an analogous approach.

Consider $(\delta^B_A - N_A N^B) Y_B$

$$\begin{aligned}(\delta^B_A - N_A N^B) Y_B (\delta^A_C - N^A N_C) Y^C &= Y_B (\delta^B_A - N_A N^B)(\delta^A_C - N^A N_C) Y^C \\ &= Y_B (\delta^B_C - N^B N_C) Y^C \\ &= (Y_C + H N_C) Y^C \\ &= -H^2\end{aligned}$$

However if $Y^{\Sigma}_A \in \mathcal{E}^{\Sigma}_A$ then the tractor identities state that $Y^{\Sigma}_A Y^A_{\Sigma} = 0$.

As stated earlier, when the mean curvature is zero our result is correct. However, we see clearly that Y^{Σ} is not simply a projection of the ambient splitting operator, Y, in all other cases.

In order to be in the set of splitting operators, Y^{Σ} must meet the identity conditions described on pg 25 when contracted [15]. In each of the following propositions it is assumed that the ambient splitting operators are acting along Σ.

Proposition 5.2.1. *The following are true:*

i) $Y^{\Sigma}_A X^A_{\Sigma} = 1$

ii) $Y^{\Sigma}_{A} Y^{A}_{\Sigma} = 0$

iii) $Y^{\Sigma}_{A} N^{A} = 0$

if and only if

$$Y^{\Sigma}_{A} = \gamma Y_A + Z^{a}_{A} N_a \beta + X_A \alpha$$
where $\gamma = 1$, $\beta = H$ *and* $\alpha = -\frac{H^2}{2}$

Proof. i) $Y^{\Sigma}_{A} X^{A}_{\Sigma} = 1$ is true for all values of α and β, but γ must be equal to 1.

ii)

$$\begin{aligned} Y^{\Sigma}_{A} Y^{A}_{\Sigma} &= 0 \\ (Y_A + \alpha X_A + \beta Z_{Aa} N^a)(Y^A + \alpha X^A + \beta Z^{A}_{b} N^b) &= 0 \\ 2\alpha + \beta^2 &= 0 \\ \Rightarrow \alpha &= -\tfrac{\beta^2}{2} \end{aligned}$$

iii) Finally, the Y^{Σ} is intrinsic to Σ and hence orthogonal to N^A

$$\begin{aligned} Y^{\Sigma}_{A} N^{A} &= 0 \\ (Y_A + \beta Z^{b}_{A} N_b - \tfrac{\beta^2}{2} X_A)(Z^{A}_{c} N^c - X^A H) &= 0 \\ -H + \beta N_b N^b &= 0 \\ \beta &= H \end{aligned}$$

□

Proposition 5.2.2. *The following are true:*

i) $Z^{\Sigma}_{A\,a} Z^{A}_{\Sigma\,b} = g^{\Sigma}_{ab}$

ii) $Z^{\Sigma a}_{A} X^{A}_{\Sigma} = 0$

iii) $Z^{\Sigma a}_{A} Y^{A}_{\Sigma} = 0$

iv) $Z^{\Sigma a}_{A} N^{A} = 0$

if

$$Z^{\Sigma a}_{A} = \Pi^{a}_{b} Z^{b}_{A}$$

Proof. i)

$$\begin{aligned} Z^{\Sigma}_{A\,a} Z^{A}_{\Sigma\,b} &= \Pi^c_a Z_{A\,c} Z^A_d \Pi^d_b \\ &= (\delta^c_a - N_a N^c) g_{cd} (\delta^d_b - N_b N^d) \\ &= g_{ab} - N_a N_b \\ &= g^{\Sigma}_{ab} \end{aligned}$$

ii) There is no Y component in Z^{Σ} so the inner product of Z^{Σ} and X^{Σ} is trivially null.
iii) Using the decomposition of Y^{Σ} described by Prop. (5.2.1).

$$\begin{aligned} Z^{\Sigma a}_{A} Y^A_{\Sigma} &= \Pi^c_a Z_{A\,c} (Y^A + Z^A_b N^b H - X^A \tfrac{H^2}{2}) \\ &= \Pi^c_a Z_{A\,c} Z^A_b N^b H \\ &= \Pi^c_a N_c H = 0 \end{aligned}$$

iv)

$$\begin{aligned} Z^{\Sigma\,a}_{A} N^A &= (\delta^a_b - N^a N_b) Z^b_A (Z^A_c N^c - X^A H) \\ &= (\delta^a_b - N^a N_b) N^b \\ &= 0 \end{aligned}$$

□

If the operators Y^{Σ}, Z^{Σ}, are to be the usual operators acting on tensor fields along Σ then they need to transform correctly under a conformal rescaling of the metric.

Proposition 5.2.3. *i)* $\widehat{Y}^{\Sigma}_A = Y^{\Sigma}_A - Z^{\Sigma a}_A \Upsilon^{\Sigma}_a - \frac{1}{2} X^{\Sigma}_A \Upsilon^{\Sigma}_b \Upsilon^b_{\Sigma}$
ii) $\widehat{Z}^{\Sigma a}_A = Z^{\Sigma\,a}_A + \Upsilon^b_{\Sigma} X^{\Sigma}_A$

Proof. i)

$$\begin{aligned} \widehat{Y}^{\Sigma}_A &= \widehat{Y}_A + \widehat{Z}^b_A (H + \Upsilon^c N_c) N_b - \tfrac{1}{2}(H + \Upsilon^d N_d)^2 X_A \\ &= (Y_A - \Upsilon_a Z^a_A - \tfrac{\Upsilon^2}{2} X_A) + (Z^b_A + \Upsilon^b X_A)(H + \Upsilon^c N_c) N_b - \\ &\quad \tfrac{1}{2}(H + \Upsilon^d N_d)^2 X_A \\ &= (Y_A + Z^b_A H N_b - \tfrac{H^2}{2} X_A) - \Upsilon^b (g_{ab} - N_a N_b) Z^a_A \\ &\quad -\tfrac{1}{2} \Upsilon^b (g_{ab} - N_a N_b) \Upsilon^a X_A \end{aligned}$$

Now $g^{\Sigma}_{ab} = g_{ab} - N_a N_b$ so this implies that $\Upsilon^b(g_{ab} - N_a N_b)\Upsilon^a = \Upsilon^2_{\Sigma}$.

$$\begin{aligned}\widehat{Y}^{\Sigma}_A &= Y^{\Sigma}_A - g^{\Sigma}_{ab}\Upsilon^b Z^a_A - \tfrac{1}{2}\Upsilon^2_{\Sigma} X_A \\ &= Y^{\Sigma}_A - \Upsilon^c_{\Sigma} Z^{\Sigma}_{Ac} - \tfrac{1}{2}\Upsilon^2_{\Sigma} X^{\Sigma}_A\end{aligned}$$

ii)

$$\begin{aligned}\widehat{Z}^{\Sigma\,a}_A &= (\delta^a_b - N^a N_b)\widehat{Z}^b_A \\ &= (\delta^a_b - N^a N_b)(Z^b_A + \Upsilon^b X_A) \\ &= Z^{\Sigma\,a}_A + \Upsilon^b_{\Sigma} X^{\Sigma}_A\end{aligned}$$

□

Using the three intrinsic splitting operators we should be able to confirm the relationship between them in terms of the intrinsic tractor metric, h^{Σ}_{AB}, defined earlier.

Corollary 5.2.4. $h^{\Sigma}_{AB} = X^{\Sigma}_A Y^{\Sigma}_B + Y^{\Sigma}_A X^{\Sigma}_B + Z^{\Sigma a}_A Z^{\Sigma}_{Ba}$

Proof. From the definition of the intrinsic tractor metric:

$$\begin{aligned}h^{\Sigma}_{AB} &= h_{AB} - N_A N_B \\ &= X_A Y_B + Z^c_A Z_{Bc} + Y_A X_B - (N^b Z_{Ab} - X_A H)(N^a Z_{Ba} - X_B H) \\ &= X_A(Y_B + H N^a Z_{Ba} - \tfrac{H^2}{2} X_B) + (g^{ab} - N^a N^b) Z_{Ab} Z_{Ba} \\ &\quad + (Y_A + H N^b Z_{Ab} - \tfrac{H^2}{2} X_A) X_B \\ &= X^{\Sigma}_A Y^{\Sigma}_B + Z^{\Sigma\,c}_A Z^{\Sigma}_{Bc} + Y^{\Sigma}_A X^{\Sigma}_B\end{aligned}$$

□

5.3 Tractor Second Fundamental Form

Earlier we saw that the second fundamental form was defined by the projected connection acting on the normal vector. In light of this, it is very interesting to determine what the projected coupled Levi-Civita - Tractor connection acting on the N^B tractor looks like.

Proposition 5.3.1. *Let N^B be the normal tractor.* $N^B \in \mathcal{E}^B$

$$\Pi^c_a \nabla_c N^B = Z^{Bb} L_{(ab)\circ} - (n-2)^{-1} X^B \nabla^b_{\Sigma} L_{(ab)\circ}$$

Proof. We will begin by looking at the ambient connection on N^B. This is simply obtained by applying eqn (4.4) to the normal tractor. This is written in column vector notation as follows [2]:

$$[\nabla_a N^B]_g = \begin{pmatrix} -N_a \\ \nabla_a N^b - \delta_a^b H \\ -\nabla_a H - P_{ab} N^b \end{pmatrix}$$

Our interest lies with the connection which is orthogonal to the unit normal field along Σ. Action of the tensor projection operator on our a index will produce the following tractor:

$$\Pi_a^c \nabla_c N^B = -Y^B \Pi_a^c N_c + Z^{Bb}(\Pi_a^c \nabla_c N_b - \Pi_a^c g_{cb} H) - X^B(\Pi_a^c \nabla_a H + \Pi_a^c P_{cb} N^b)$$

which when simplified using the relationship between the second fundamental form and the mean curvature, generates:

$$\Pi_a^c \nabla_c N^B = Z^{Bb} L_{(ab)\circ} - X^B(\nabla_a^\Sigma H + \Pi_a^c P_{cb} N^b).$$

It is clear that the projection kills the primary slot.
The application of Codazzi equation, $(n-2)\Pi_\Sigma N^b P_{ab} = \nabla_\Sigma^b L_{(ab)\circ} - (n-2)\nabla_a^\Sigma H$, further simplifies this to the tractor-tensor field in the proposition. □

The tractor coefficients are tensors in the submanifold, and so the tractor itself may be identified with its Σ component in the intrinsic tractor bundle, $\mathcal{E}_\Sigma^A$. Hence we can add Σ's to the splitting operators as follows:

$$\Pi_a^c \nabla_c N^B = Z_\Sigma^{Bb} L_{(ab)\circ} - (n-2)^{-1} X_\Sigma^B \nabla_\Sigma^b L_{(ab)\circ}$$

We know that the Levi-Civita - tractor connection acts invariantly on tractors with no weight (such as the normal tractor), so by construction, the above tractor-tensor field has weight zero and is conformally invariant.

Let us define an operator [15] which we will call the Tractor Second Fundamental Form, and denote it by L_{AB}. Recall the tractor operator, A_A^a described by (4.11). If we apply the $(\Sigma, [g_\Sigma])$ version of this operator to the differentiated normal tractor above, we will obtain a conformally invariant tractor in $\mathcal{E}_{AB}^\Sigma[-1]$.

Definition 5.3.2 (Tractor Second Fundamental Form). *Let N_B be a unit normal* *tractor on* Σ. $N_B \in \mathcal{N}_B$

$$[L_{AB}]_g = \begin{pmatrix} 0 \\ (n^\Sigma - 2)\Pi_a^c \nabla_c N_B \\ -\nabla_\Sigma^b \Pi_b^d \nabla_b N_B \end{pmatrix}$$

We know that this is conformally invariant because the operator described by the mapping (4.11) is invariant on weighted forms, and it is straight forward to see that this operator has weight -1.

An interesting exercise for our purposes is to convert the column vector notation into that involving the splitting operators.

$$\begin{aligned} L_{AB} &= Z_A^{\Sigma\,a}(n^\Sigma - 2)\Pi_a^c \nabla_c (Z_B^b N_b - HX_B) - X_A^\Sigma \nabla_\Sigma^b \Pi_b^d \nabla_d (Z_B^a N_a - HX_B) \\ &= (n-3) Z_A^{\Sigma\,a}(Z_B^{\Sigma\,b} L_{ab} - X_B \Pi_a^c P_{cb} N^b - Y_B \Pi_a^b N_b - X_B \nabla_a^\Sigma H - \Pi_a^c Z_{B\,c} H) \\ &\quad - X_A^\Sigma \nabla_\Sigma^b (Z_B^{\Sigma\,a} L_{ab} - X_B \Pi_b^d P_d^a N_a - Y_B \Pi_b^a N_a - X_B \nabla_b^\Sigma H - \Pi_b^c Z_{B\,c} H) \end{aligned}$$

Applying the rules for the intrinsic splitting operators, eqns (5.2.1) & (5.2.2), helps us to determine the following:

$$\begin{aligned} L_{AB} &= (n-3) Z_A^{\Sigma\,a}(Z_B^{\Sigma\,b}(L_{(ab)\circ} + g_{ab}^\Sigma H) - X_B \Pi_a^c P_{cb} N^b - X_B \nabla_a^\Sigma H - Z_{B\,a}^\Sigma H) \\ &\quad - X_A^\Sigma Z_B^{\Sigma\,a} \nabla_\Sigma^b L_{ab} + X_A^\Sigma X_B^\Sigma P_\Sigma^{ab} L_{ab} + (n-1) X_A^\Sigma Y_B^\Sigma H + X_A^\Sigma X_B^\Sigma \nabla_\Sigma^b \Pi_b^d P_d^a N_a \\ &\quad + X_A^\Sigma Z_B^{\Sigma b} \Pi_b^d P_d^a N_a + X_A^\Sigma X_B^\Sigma \Delta^\Sigma H + X_A^\Sigma Z_B^{\Sigma\,b} \nabla_b^\Sigma H + X_A^\Sigma Z_B^{\Sigma\,b} \nabla_b^\Sigma H \\ &\quad - X_A^\Sigma X_B^\Sigma J^\Sigma H - (n-1) X_A^\Sigma Y_B^\Sigma H \end{aligned}$$

In the process of simplifying the above formula we will use the form of the Codazzi equation, $(n-2)\nabla_\Sigma^a \Pi_\Sigma N^b P_{ab} = \nabla_\Sigma^a \nabla_\Sigma^b L_{(ab)\circ} - (n-2)\Delta^\Sigma H$, as well as the equation (2.8). After substitution and simplification we end up with the tractor:

$$\begin{aligned} L_{AB} &= (n-3) Z_A^{\Sigma\,a} Z_B^{\Sigma\,b} L_{(ab)\circ} - \frac{n-3}{n-2} Z_A^{\Sigma\,a} X_B^\Sigma \nabla_\Sigma^b L_{(ab)\circ} \\ &\quad - \frac{n-3}{n-2} X_A^\Sigma Z_B^{\Sigma\,b} \nabla_\Sigma^a L_{(ab)\circ} + X_A^\Sigma X_B^\Sigma (P_{ab}^\Sigma L^{(ab)\circ} + (n-2)^{-1} \nabla_\Sigma^a \nabla_\Sigma^b L_{(ab)\circ}) \end{aligned} \quad (5.2)$$

The tractor second fundamental form is completely symmetric and trace free and could just as readily be denoted by $L_{(AB)\circ}$ creating a tractor analogue to $L_{(ab)\circ}$.

Let $V^B \in \mathcal{E}^B[w]$ then $S_A = L_{AB}V^B \in \mathcal{E}^\Sigma_A[w-1]$. This tractor, S_A, resides in the tractor bundle along Σ which is orthogonal to the normal tractor.

5.4 Tractor D Operator acting along a Hypersurface

Recall that the intrinsic-to-Σ tractor bundle may be identified with the projection by Π^Σ of $\mathcal{E}^A$ along Σ. It follows then that the operator $D^A_\Sigma \Pi^\Sigma : \mathcal{E}_A[w] \longrightarrow \mathcal{E}[w-1]$ given by $V_A \mapsto D^A_\Sigma \Pi^\Sigma V_A$ is conformally invariant.

We will verify this directly.

Part 1: Expansion of the operator

For $V_A \in \mathcal{E}_A[w]$

There exist sections

$$\begin{aligned}
&\sigma \in \mathcal{E}[w+1]\\
&\mu_a \in \mathcal{E}_a[w+1]\\
&\rho \in \mathcal{E}[w-1]\\
&\text{such that } V_A = Y_A\sigma + Z^a_A\mu_a + X_A\rho
\end{aligned}$$

$$D^A_\Sigma \Pi^\Sigma V_A = D^A_\Sigma \Pi^\Sigma (Y_A\sigma + Z^a_A\mu_a + X_A\rho)$$

In order to be able to use the intrinsic-to-Σ tractor-D operator we need to express $\Pi^\Sigma V_A$ in terms of the usual intrinsic splitting operators, X^Σ_A, Y^Σ_A and $Z^{\Sigma\, a}_A$. Props (5.2.1) and (5.2.2) show how these intrinsic operators are related to the splitting operators in the ambient space along Σ. Using this knowledge, in conjunction with the respective expansions of $\Pi^\Sigma X_A, \Pi^\Sigma Y_A$ and $\Pi^\Sigma Z^a_A$, we are able to determine the following relationships:

$$\begin{aligned}
\Pi^{\Sigma\ B}_A X_B &= X^\Sigma_A\\
\Pi^{\Sigma\ B}_A Z^a_B &= Z^{\Sigma\ a}_A + X^\Sigma_A H N^a\\
\Pi^{\Sigma\ B}_A Y_B &= Y^\Sigma_A - \tfrac{1}{2} X^\Sigma_A H^2
\end{aligned}$$

$$\Rightarrow D^A_\Sigma \Pi^\Sigma V_A = D^A_\Sigma\big(Y^\Sigma_A\sigma + Z^{\Sigma\ a}_A \mu^\Sigma_a + X^\Sigma_A(\rho + HN^b\mu_b - \tfrac{1}{2}H^2\sigma)\big)$$

Although the tractor-D is heavily dependent on the coupled tensor-tractor connection, it is not necessary to obtain an explicit expansion for this connection. We know that the tractor connection ∇^Σ is induced from the ambient connection ∇ [5], and our only requirement is that the intrinsic connection is Leibniz and satisfies the Σ equivalent of Prop. (4.2.3).

Earlier we established that the intrinsic splitting operators have the usual contraction rules, and we have implicitly defined the intrinsic coupled Levi-Civita - tractor connection to have the usual rules of differentiation; so there exists a $(\Sigma, [g_\Sigma])$ version of equations (4.8), (4.9) and (4.10). They reveal to us how the intrinsic tractor-D acts on each term individually.

$$
\begin{aligned}
D_\Sigma^A Y_A^\Sigma \sigma &= (n^\Sigma + (w+1) - 2) J^\Sigma \sigma - \Delta^\Sigma \sigma \\
D_\Sigma^A Z_A^{\Sigma\, a} \mu_a^\Sigma &= (n^\Sigma + 2(w+1) - 2) \nabla_\Sigma^a \mu_a^\Sigma \\
D_\Sigma^A X_A \tau &= (n^\Sigma + 2(w-1) + 2)(n^\Sigma + w - 1)\ \tau \\
& \quad \textit{where}\ \ \tau = \rho + H N^b \mu_b - \tfrac{1}{2} H^2 \sigma
\end{aligned}
$$

Combining these we obtain:

$$
\begin{aligned}
D_\Sigma^A \Pi^\Sigma V_A &= (n+w-2) J^\Sigma \sigma - \Delta^\Sigma \sigma + (n+2w-1) \nabla_\Sigma^a \mu_a^\Sigma \\
& \quad + (n+2w-1)(n+w-2)(\rho + H N^b \mu_b - \frac{1}{2} H^2 \sigma) \quad (5.3)
\end{aligned}
$$

Part 2: Conformal Invariance

We will apply the conformal transformation rules specified in Chapter 3 to the above operator, and ignore all Υ^2 terms.

$$
\begin{aligned}
\widehat{D_\Sigma^A \Pi^\Sigma V_A} &= (n+w-2)(J^\Sigma - \nabla_\Sigma^b \Upsilon_b^\Sigma + \tfrac{3-n}{2} \Upsilon_\Sigma^b \Upsilon_b^\Sigma)\sigma - \nabla_\Sigma^b (\nabla_b^\Sigma \sigma + (w+1) \Upsilon_b^\Sigma \sigma) \\
& \quad -(n+w-2) \Upsilon_\Sigma^b (\nabla_b^\Sigma \sigma + (w+1) \Upsilon_b^\Sigma \sigma) + (n+2w-1) \nabla_\Sigma^a (\mu_a^\Sigma + \Upsilon_a^\Sigma \sigma) \\
& \quad +(n+2w-1)(n+w-2) \Upsilon_\Sigma^a (\mu_a^\Sigma + \Upsilon_a^\Sigma \sigma) + (n+2w-1)(n+w-2) \\
& \quad \big((\rho - \Upsilon^c \mu_c - \tfrac{1}{2} \Upsilon^c \Upsilon_c \sigma) + (H + \Upsilon^c N_c) N^b (\mu_b + \Upsilon_b \sigma) - \tfrac{1}{2}(H + \Upsilon^c N_c)^2 \sigma\big)
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow \widehat{D^A_\Sigma \Pi^\Sigma V_A} &= (n+w-2)J^\Sigma\sigma - (n+w-2)(\nabla^\Sigma_b \Upsilon^b_\Sigma)\sigma - \Delta^\Sigma\sigma \\
&-(w+1)(\nabla^b_\Sigma \Upsilon^\Sigma_b)\sigma - (w+1)\Upsilon^\Sigma_b \nabla^b_\Sigma\sigma - (n+w-2)\Upsilon^b_\Sigma \nabla^\Sigma_b \sigma \\
&+(n+2w-1)\nabla^a_\Sigma \mu^\Sigma_a + (n+2w-1)(\nabla^\Sigma_b \Upsilon^b_\Sigma)\sigma + (n+2w-1)\Upsilon^\Sigma_b \nabla^b_\Sigma \sigma \\
&+(n+2w-1)(n+w-2)\Upsilon^a_\Sigma \mu^\Sigma_a + (n+2w-1)(n+w-2)\rho \\
&-(n+2w-1)(n+w-2)\Upsilon^a \mu_a + (n+2w-1)(n+w-2)HN^a\mu_a \\
&+(n+2w-1)(n+w-2)HN^a\Upsilon_a\sigma + (n+2w-1)(n+w-2)\Upsilon^b N_b N^a \mu_a \\
&-(n+2w-1)(n+w-2)\tfrac{H^2}{2}\sigma - (n+2w-1)(n+w-2)HN_b\Upsilon^b\sigma
\end{aligned}$$

Careful simplification will result in the cancellation of all terms involving Υ_b and its submanifold equivalent. This completes the explicit check of the invariance.

As an example, if we take the ambient tractor-D acting on some density, ψ, of weight w, then we know that:

$$\begin{aligned}
D^A_\Sigma \Pi^\Sigma D_A &= (n+w-3)\Big((n+2w-2)\Box^\Sigma \\
&+(n+2w-3)\big((n+2w-2)(H\nabla_N - \tfrac{w}{2}H^2) - \Box\big)\Big)
\end{aligned}$$

is a conformally invariant operator which has weight $w-2$.

Chapter 6

The $(\mathsf{P}_4, \mathsf{P}_3)$ Problem

Definition 6.0.1 (Normal Order). *[5] Let $B : \mathcal{E}^*[w] \longrightarrow \mathcal{E}^{*'}[w']$ be a kth-order operator, and Σ be the kernel of a defining function, f. B has normal order m at $x \in \Sigma$ if there exists a section $\phi \in \mathcal{E}^*[w]$ such that $B(f^m \phi)(x) \neq 0$ however for any section $\phi' \in \mathcal{E}^*[w]$, $B(f^{m+1}\phi')(x) = 0$. If B has normal order m for all $x \in \Sigma$ then B has normal order m.*

For example, differential operators intrinsic to Σ have normal order zero.

Given the conformally invariant Laplacian, $\Box$, on a manifold M, we are interested in the first order conformally invariant operator, δ, defined along the boundary Σ of M, which has normal order 1 and is compatible with $\Box$. By compatible we mean that, along Σ, this operator acts on densities from the domain $\mathcal{E}[\frac{2-n}{2}]$; the same domain as $\Box$. $\delta = N^a \nabla_a - wH$ is the conformally invariant Neumann operator, called the Robin operator, which acts on a domain space of densities of weight $w \in \mathbb{R}$ [5]. We can see that the Robin operator has normal order of 1 on Σ. The pair $(\Box, \delta)$ can then be used to set up a conformally invariant Neumann problem.

For each GJMS operator, P_{2k}, an analogous Neumann problem is established by obtaining each of the compatible conformally invariant boundary operators of odd normal order $< 2k$. Each m-order Neumann-type operator has normal order equal to m. P_4 is one such GJMS operator which has been studied extensively that we will look at shortly. The Robin operator acts as the compatible first order boundary operator, however in order to complete the Neumann data we also require the 3rd order Neumann-type operator. In their paper on Conformal Non-Local Operators, Branson & Gover describe tractor methods for generalising the Robin operator above to higher orders, m, creating an m-th order Neumann-type operator, δ_m [5]. Unfortunately, their tractor construction for δ_3 does not have normal order 3 in dimension 4.

A slightly alternative tractor construction will be applied in order to generate P_3 and complete the pair, $(\mathsf{P}_4, \mathsf{P}_3)$ for all dimensions ≥ 4. Note that this can be extended to include ambient dimension 3, but it is not looked at here.

We will begin by explicitly calculating the P_4 operator.

6.1 Part 1: The Paneitz operator, P_4

Definition 6.1.1 (Paneitz Operator). *[19] The Paneitz Operator is a conformally invariant fourth order differential operator which acts on densities of weight, $2 - \frac{n}{2}$.*
It is described by the equation:

$$\Delta^2 + (2-n)J\Delta + 4P^{ab}\nabla_a\nabla_b + (6-n)(\nabla^a J)\nabla_a + \tfrac{n-4}{2}(\tfrac{n}{2}J^2 - \Delta J - 2P^{ab}P_{ab})$$

In [16] it is shown that the GJMS operators can be generated using the tractor calculus. I have adopted the method used in that paper to produce the P_4 below.

Proposition 6.1.2. *[16] Let $\psi \in \mathcal{E}[\frac{4-n}{2}]$ then*

$$(n-4)\mathsf{P}_4\psi \quad = \quad D^A \Box D_A \psi \tag{6.1}$$

where P_4 is the Paneitz operator.

Proof. Step 1: The invariant tractor-D is applied to ψ, where

$$D_A : \mathcal{E}[\tfrac{4-n}{2}] \longrightarrow \mathcal{E}_A[\tfrac{2-n}{2}]$$
$$D_A\psi = (4-n)Y_A\psi + 2Z_A^a\nabla_a\psi - X_A\Box\psi$$

Step 2: The tractor-coupled Yamabe operator is applied to the resultant operator. We know that this is the conformal Laplacian only when acting on densities of weight $\frac{2-n}{2}$.

$$\Box : \mathcal{E}_A[\tfrac{2-n}{2}] \longrightarrow \mathcal{E}_A[\tfrac{-2-n}{2}]$$

$$\begin{aligned}
\Box D_A\psi &= \nabla_c\nabla^c((4-n)Y_A\psi + 2Z_A^a\nabla_a\psi - X_A\Box\psi) \\
&\quad + \frac{2-n}{2}J((4-n)Y_A\psi + 2Z_A^a\nabla_a\psi - X_A\Box\psi) \\
&= -X_A\big(\Delta^2\psi + (2-n)J\Delta\psi + 4P^{ab}\nabla_a\nabla_b\psi + (6-n)(\nabla^a J)\nabla_a\psi \\
&\quad + \frac{n-4}{2}(\frac{n}{2}J^2 - \Delta J - 2P^{ab}P_{ab})\psi\big) \qquad (6.2)
\end{aligned}$$

Step 3: The final tractor-D is applied, via the mapping below, to produce the 4th order operator.

$D^A : \mathcal{E}_A[\frac{-2-n}{2}] \longrightarrow \mathcal{E}[\frac{-4-n}{2}]$

After applying (4.10) and the Bianchi identity to (6.2) we end up with

$$\begin{aligned}
D^A\Box D_A\psi &= (n-4)((\Delta^2\psi + (2-n)J\Delta\psi + 4P^{ab}\nabla_a\nabla_b\psi + (6-n)(\nabla^a J)\nabla_a\psi \\
&\quad + \frac{n-4}{2}(\frac{n}{2}J^2 - \Delta J - 2P^{ab}P_{ab})\psi)
\end{aligned}$$

$$\begin{aligned}
&(-4+n)\,Fn_{;i}{}^{i}{}_{j}{}^{j} - (-4+n)\,(-2+n)\,Fn_{;i}{}^{i}\,J + \\
&\frac{(-4+n)^2\,n}{4}\,Fn\,J^2 - (-4+n)^2\,Fn_{;i}\,J_{;}{}^{i} - \\
&\frac{(-4+n)^2}{2}\,Fn\,J_{;i}{}^{i} + 2\,(-4+n)\,Fn_{;i}\,P^{i}{}_{j;}{}^{j} + \\
&4\,(-4+n)\,Fn_{;i\,j}\,P^{i\,j} - (-4+n)^2\,Fn\,P_{i\,j}\,P^{i\,j}
\end{aligned}$$

Figure 6.1: Ricci Output of $(n-4)P_4$ operator

The tractor calculus gives a multiple of the Paneitz operator which is polynomial in $(n-4)$. It has also been generated in Ricci. The output is Fig. 6.1 (not yet simplified by the Bianchi identity), and the code can be found in Appendix Fig. D.7. □

6.2 Q_4 Curvature

A differential operator is *natural* if it can be given by a universal polynomial expression in g, its inverse g^{-1}, the Levi-Civita connection ∇, and the Riemannian curvature tensor [8]. The operator P_4 is natural by this definition, because it is polynomial in the metric

g, its inverse, the Levi-Civita connection ∇, and P_{ab} and its trace. Keeping this in mind, let us proceed to describe the Q curvature.

We begin by separating the above formula into two components. The first component $\mathsf{Q}_{4,n}$ is the part of the Paneitz operator which is scalar in dimension n on the density ψ, and the second part, denoted $\mathsf{P}^1_{4,n}$ is the part with order of at least 1 which annihilates constant functions. Hence we will describe the Paneitz operator by:

$$\mathsf{P}_{4,n} = \mathsf{P}^1_{4,n} + \tfrac{n-4}{2}\mathsf{Q}_{4,n}$$

The Q component is a zeroth order term which is composed entirely of curvature terms and their derivatives.

In his paper on Sharp Inequalities [8], T. Branson describes the conformal transformation of the Q component provided that both P and Q are natural operators. He worked in the covariant setting for a particular metric in the conformal class. This is equivalent to identifying our densities in the conformal structure to homogeneous functions in the Riemannian structure. In this setting the P_4 is not invariant but rather conformally covariant. He proceeds to show that $\mathsf{Q}_{4,n}$ transforms as follows in dimension 4. Note that we have written the formula in terms of densities, and not their trivialised functions, and we rewrite Υ_a as $\nabla_a\Upsilon$, where $\Upsilon = \log_e(\Omega)$.

$$\widehat{\mathsf{Q}}_{4,4} = \mathsf{P}^1_{4,4}\Upsilon + \mathsf{Q}_{4,4} \tag{6.3}$$

In dimension 4, $\mathsf{P}^1_{4,4}$ may be identified with $\mathsf{P}_{4,4}$ because the Q component is eliminated. Recall the transformation of the scalar curvature in dimension 2 and compare this with (6.3).

Theorem 1.1 of [8] states that the P_4 operator defined above must be the 4th order GJMS operator. Note that we drop the second subscript on the operator for general n.

The $\mathsf{Q}_{4,4}$ *Curvature* is the scalar function explicitly described by:

$$2J^2 - \Delta J - 2P^{ab}P_{ab} \tag{6.4}$$

The Paneitz operator acting on Υ (in dimension 4) is:

$$\mathsf{P}_{4,4}\Upsilon = \Delta^2\Upsilon - 2J\Delta\Upsilon + 4P_{ab}\nabla^a\nabla^b\Upsilon + 2(\nabla_b J)\nabla^b\Upsilon$$

We will conformally transform equation (6.4) in order to illustrate that its transformation is governed by equation (6.3)..

$$\begin{aligned}\widehat{\mathsf{Q}}_{4,4} &= 2\widehat{J}^2 - \widehat{\Delta J} - 2\widehat{P^{ab}}\widehat{P_{ab}} \\ &= 2(J-\Delta\Upsilon)^2 - \nabla^b\big(\nabla_b(J-\Delta\Upsilon) - 2\Upsilon_b(J-\Delta\Upsilon)\big) \\ &\quad -2(P^{ab} - \nabla^a\nabla^b\Upsilon)(P_{ab} - \nabla_a\nabla_b\Upsilon)\end{aligned}$$

$$\begin{aligned}\Rightarrow \widehat{\mathsf{Q}}_{4,4} &= \mathsf{Q}_{4,4} + \Big(-2J\Delta\Upsilon + \Delta^2\Upsilon + 4P_{ab}\nabla^a\nabla^b\Upsilon + 2(\nabla_b J)\nabla^b\Upsilon\Big) \\ &= \mathsf{Q}_{4,4} + \mathsf{P}_{4,4}\Upsilon\end{aligned}$$

This has confirmed that P_4 controls the transformation of Q_4 in dimension 4.

Another interesting point to note is that $\mathsf{P}^1_{4,4}$ is a conformal divergence of a weighted 1-form. By Stokes Theorem, this implies that $\int_M \mathsf{Q}_{4,4} d\epsilon$ is a global invariant [7].

6.3 Part 2: The P_3 Operator Acting on a Hypersurface

The P_3 operator is the third normal order boundary operator compatible with the Paneitz operator. It acts (along Σ) invariantly on the same domain space as the Paneitz operator.

We have seen that the P_4 can be derived by the tractor formula (6.1). In Theorem 5.1 of the Branson & Gover paper [5] a similar tractor formula is proposed to generate the Neumann operator analogue, δ_3. This construction fails to produce a third normal order operator when $n = 4$, though we shall see that one exists for all dimensions $n > 4$. The P_3 operator which we generate here will incorporate δ_3.

Chang & Qing have generated one form of the P_3 operator to act on the boundary of compact 4-manifolds [9], however a remark made by them indicates that they cannot tell if the P_3 exists in other dimensions. Our construction will produce an operator which is invariant in ambient dimensions $n \geq 4$. Unfortunately it is difficult to compare their operator with that produced by myself in dimension 4, because of the differences in convention and notation used.

Proposition 6.3.1. *Let $\psi \in \mathcal{E}[\frac{4-n}{2}]$, then*

$$(4-n)\mathsf{P}_3\psi = \delta_3\psi + D^A_\Sigma L_{AB} D^B\psi$$

where $\delta_3 = D^A_\Sigma \Pi_\Sigma \, \delta D_A$, is the invariant third order Robin analogue.

Proof. The proof will proceed over the following three pages. Each of the terms on the right hand side will be explicitly calculated separately.

i) $\delta_3\psi$

Step 1: Expand the operator described by:

$$\delta D_A : \mathcal{E}[\tfrac{4-n}{2}] \longrightarrow \mathcal{E}_A[\tfrac{-n}{2}]$$

This operator is conformally invariant and has normal order 3.

$$\begin{aligned}
\delta D_A\psi &= \delta\Big((4-n)Y_A\psi + 2Z_A^a\nabla_a\psi - X_A\Box\psi\Big) \\
\delta D_A\psi &= N^b\nabla_b((4-n)Y_A\psi + 2Z_A^a\nabla_a\psi - X_A\Box\psi) \\
&\quad \frac{2-n}{2}H((4-n)Y_A\psi + 2Z_A^a\nabla_a\psi - X_A\Box\psi) \\
&= (2-n)Y_A(N^b\nabla_b\psi - \frac{4-n}{2}H\psi) \\
&\quad +Z_A^a(2N^b\nabla_b\nabla_a\psi - (2-n)H\nabla_a\psi + (4-n)N^bP_{ab}\psi - N_a\Box\psi) \\
&\quad -X_A(N^b\nabla_b\Box\psi - \frac{2-n}{2}H\Box\psi + 2N^bP_b^a\nabla_a\psi) \qquad (6.5)
\end{aligned}$$

We are able to see that the X coefficient contains a term which has a normal order of 3.

Step 2: Expand the operator described by:

$$D^A_\Sigma\Pi^\Sigma : \mathcal{E}_A[\tfrac{-n}{2}] \longrightarrow \mathcal{E}[\tfrac{-2-n}{2}]$$

We will begin by obtaining the coefficients of the splitting operators of eqn (6.5), which are intrinsic along Σ. (Note that σ is the Y coefficient, μ_a is the Z coefficient, and ρ is the X coefficient).

$$\begin{aligned}
\sigma &= (2-n)(N^b\nabla_b\psi - \tfrac{4-n}{2}H\psi) && \in \mathcal{E}^\Sigma[\tfrac{2-n}{2}] \\
\mu_a^\Sigma &= \Pi_a^c\mu_c \\
&= \Pi_a^c\big(2N^b\nabla_b\nabla_c\psi - (2-n)H\nabla_c\psi + (4-n)N^bP_{cb}\psi - N_c\Box\psi\big) \\
&= 2\Pi_a^cN^b\nabla_b\nabla_c\psi - (2-n)H\nabla_a^\Sigma\psi + (4-n)\Pi_a^cN^bP_{cb}\psi && \in \mathcal{E}_a^\Sigma[\tfrac{2-n}{2}] \\
\rho &= -N^b\nabla_b\Box\psi + \tfrac{2-n}{2}H\Box\psi - 2N^bP_{ab}\nabla^a\psi && \in \mathcal{E}^\Sigma[\tfrac{-2-n}{2}]
\end{aligned}$$

The explicit equation for $D^A_\Sigma\Pi^\Sigma\delta D_A\psi$ is obtained by inserting the above three formulae into eqn (5.3).

$$\begin{aligned} D^A_\Sigma \Pi^\Sigma \delta D_A \psi &= (n-4)\Big(\frac{1}{2} N^b \nabla_b \Delta\psi + \Delta^\Sigma N^b \nabla_b \psi + \frac{n}{4} H \Delta\psi + \frac{n-2}{2} \Delta^\Sigma H \psi \\ &-\nabla^a_\Sigma H \nabla^\Sigma_a \psi - H N^a N^b \nabla_a \nabla_b \psi - \frac{3(n-2)}{4} H^2 N^b \nabla_b \psi \\ &+N^b P_{ab} \nabla^a \psi - \frac{n-4}{4} N^b \nabla_b J \psi - \frac{n-2}{2} J^\Sigma N^b \nabla_b \psi \\ &-\frac{(n-2)(n-4)}{4} J^\Sigma H \psi + \frac{n-4}{2} H N^a N^b P_{ab} \psi + \nabla^a_\Sigma \Pi^c_a N^b P_{bc} \psi \\ &-\frac{(n-2)(n-4)}{8} H^3 \psi - \frac{n(n-4)}{8} H J \psi\Big) + 2\nabla^a_\Sigma L_{(ab)\circ} \nabla^b_\Sigma \psi \end{aligned} \tag{6.6}$$

The leading term has normal order 3 and the composition of invariant operators tells us that the resultant operator is conformally invariant.

Unlike the Paneitz tractor formula, the right hand side does not drop away in the critical dimension, 4. In this dimension the remaining term, involving the derivative of the trace free part of the second fundamental form, does not have a normal order of 3. We will show that the intrinsic tractor D acting on L_{AB} (the tractor SFF) also produces a multiple of this term in dimension 4.

ii) $D^A_\Sigma L_{AB} D^B \psi$

When acting on the above density ψ, along Σ, the operator $D^A_\Sigma L_{AB} D^B$ is conformally invariant and has weight $\frac{-2-n}{2}$

Based upon the definition of L_{AB} (equation 5.2) the operator can be explicitly calculated.

$$L_{AB} D^B \psi = Z^{\Sigma\,a}_A \zeta^\Sigma_a + X^\Sigma_A \xi$$

Where, along Σ:

$$\begin{aligned} \zeta^\Sigma_a &= 2(n-3) L_{(ab)\circ} \nabla^b_\Sigma \psi + \tfrac{(n-4)(n-3)}{n-2} (\nabla^b_\Sigma L_{(ab)\circ}) \psi \\ \xi &= -(n-4) P^\Sigma_{ab} L^{(ab)\circ} \psi - \tfrac{n-4}{n-2} (\nabla^a_\Sigma \nabla^b_\Sigma L_{(ab)\circ}) \psi \\ &\quad -2\big(\tfrac{n-3}{n-2}\big) (\nabla^a_\Sigma L_{(ab)\circ}) \nabla^b_\Sigma \psi \end{aligned}$$

Given that $\psi \in \mathcal{E}[\frac{4-n}{2}]$, and both the D^B operator and L_{AB} tractor lower the weight by 1, it is seen that $\zeta^\Sigma_a \in \mathcal{E}^\Sigma_a[\frac{2-n}{2}]$ and $\xi \in \mathcal{E}[\frac{-2-n}{2}]$.

$$
\begin{aligned}
D^A_\Sigma(Z^{\Sigma\ a}_A \zeta^\Sigma_a) &= \big(n^\Sigma + 2(\tfrac{2-n}{2}) - 2\big)\nabla^a_\Sigma\, \zeta^\Sigma_a \\
D^A_\Sigma(X^\Sigma_A \xi) &= \big(n^\Sigma + 2(\tfrac{-2-n}{2}) + 2\big)(n^\Sigma + \tfrac{-2-n}{2})\xi
\end{aligned}
$$

Adding the two terms together we obtain:

$$
\begin{aligned}
D^A_\Sigma L_{AB} D^B \psi &= -\nabla^a_\Sigma \zeta^\Sigma_a - \tfrac{n-4}{2}\xi \\
&= 2(3-n)\nabla^a_\Sigma L_{(ab)\circ}\nabla^b_\Sigma \psi + (n-4)\Big(-\tfrac{1}{2}(\nabla^a_\Sigma \nabla^b_\Sigma L_{(ab)\circ})\psi + \tfrac{n-4}{2} P^\Sigma_{ab} L^{(ab)\circ}\psi\Big)
\end{aligned}
$$

In dimension 4, the remaining term is identical, up to sign, as the remainder of equation (6.6).

Our proposition states that the Paneitz compatible operator, P_3, is the combination of the $D^A_\Sigma \Pi^\Sigma \delta D_A$ operator and $D^A_\Sigma L_{AB} D^B$.

Addition of the two operators and some simplification using the Codazzi equation, creates an $(n-4)$ multiple of a P_3. One form of this P_3 operator is shown below.

$$
\begin{aligned}
&\tfrac{1}{2}N^b\nabla_b\Delta\psi + \Delta^\Sigma N^b \nabla_b\psi + \tfrac{n}{4}H\Delta\psi + \tfrac{n-4}{2}H\Delta^\Sigma\psi - 2L_{(ab)\circ}\nabla^a_\Sigma\nabla^b_\Sigma\psi \\
&-HN^aN^b\nabla_b\nabla_a\psi + N^bP_{ab}\nabla^a\psi - \tfrac{n-2}{2}J^\Sigma N^b\nabla_b\psi - \tfrac{3(n-2)}{4}H^2N^b\nabla_b\psi - \tfrac{n-4}{4}JN^b\nabla_b\psi \\
&+(n-4)(\nabla^a_\Sigma H)\nabla^\Sigma_a\psi + \tfrac{5-2n}{n-2}(\nabla^b_\Sigma L_{(ab)\circ})\nabla^a_\Sigma\psi + \tfrac{n-4}{2}\big(\Delta^\Sigma H - \tfrac{1}{n-2}\nabla^a_\Sigma\nabla^b_\Sigma L_{(ab)\circ} \\
&-\tfrac{1}{2}N^b(\nabla_b J) + HN^aN^bP_{ab} - \tfrac{n}{4}HJ - \tfrac{n-2}{2}HJ^\Sigma + P^\Sigma_{ab}L^{(ab)\circ} - \tfrac{n-2}{4}H^3\big)\psi
\end{aligned}
$$

The addition of the two operators has not affected the normal order of the resulting operator, which has a normal order of 3 regardless of dimension. □

This operator annihilates constant functions in dimension 4, in an analogous way to P_4. We also see that this P_3 operator has a universal expression which is polynomial in g, g^{-1}, N^a, ∇, P_{ab} and L_{ab}. Note that the submanifold curvature terms are related to the Riemannian curvature via Gauss' Theorem [22].

6.4 Q_3 Curvature

The part of the Paneitz operator which is zero order on ψ is the scalar, integral invariant Q_4 curvature term which drops away in dimension 4. A similar property of P_3 is that it is associated with a natural curvature operator, Q_3, which has the relationship described by Branson. That is, there exists an odd dimensional analogue of the Q-curvature.

Definition 6.4.1. *The scalar function:*

$$\Delta^\Sigma H - \tfrac{1}{2}\nabla^a_\Sigma\nabla^b_\Sigma L_{(ab)\circ} - \tfrac{1}{2}N^b\nabla_b J + HN^aN^bP_{ab} - \tfrac{1}{2}H^3 + P^{ab}_\Sigma L_{(ab)\circ} - HJ - J^\Sigma H$$

is $\mathsf{Q}_{3,4}$

In general dimension, $\mathsf{P}_3 = \mathsf{P}^1_3 + \frac{(n-4)}{2}\mathsf{Q}_3$, so by an easy adaptation of Branson's argument for the recovery of Q-curvature from GJMS used earlier, we are able to conclude that in dimension 4, $\mathsf{Q}_{3,4}$ must transform according to:

$$\widehat{\mathsf{Q}}_{3,4} = \mathsf{P}_{3,4}\Upsilon + \mathsf{Q}_{3,4}$$

Here we recall that $\Upsilon = \log_e(\Omega)$ and by this, $\Upsilon_a = \nabla_a\Upsilon$. As a check of our formulae let us verify that $\mathsf{Q}_{3,4}$ does indeed transform correctly. Under a rescale of the metric g to $\hat{g}$, $\mathsf{Q}_{3,4}$ transforms by:

$$\begin{aligned}
\widehat{\mathsf{Q}}_{3,4} &= \widehat{\Delta}^\Sigma\widehat{H} - \tfrac{1}{2}\widehat{\nabla^a_\Sigma}\widehat{\nabla^b_\Sigma}L_{(ab)\circ} - \tfrac{1}{2}N^b\widehat{\nabla_b}\widehat{J} + \widehat{H}N^aN^b\widehat{P}_{ab} - \tfrac{1}{2}\widehat{H}^3 + \widehat{P}^{ab}_\Sigma L_{(ab)\circ} - \widehat{H}\widehat{J} - \widehat{J}^\Sigma\widehat{H} \\
&= (\Delta^\Sigma H + \Delta^\Sigma N^b\nabla_b\Upsilon - H\Delta^\Sigma\Upsilon - (\nabla^b_\Sigma H)\nabla^\Sigma_b\Upsilon) \\
&\quad -(\tfrac{1}{2}\nabla^a_\Sigma\nabla^b_\Sigma L_{(ab)\circ} + L_{(ab)\circ}\nabla^a_\Sigma\nabla^b_\Sigma\Upsilon + (\nabla^b_\Sigma L_{(ab)\circ})\nabla^a_\Sigma\Upsilon) \\
&\quad -(\tfrac{1}{2}N^b\nabla_b J - \tfrac{1}{2}N^b\nabla_b\Delta\Upsilon - \tfrac{1}{2}JN^b\nabla_b\Upsilon) + (HN^aN^bP_{ab} - HN^aN^b\nabla_b\nabla_a\Upsilon \\
&\quad +N^aN^bP_{ab}N^c\nabla_c\Upsilon) - (\tfrac{1}{2}H^3 + \tfrac{3}{2}H^2N^b\nabla_b\Upsilon) + (P^{ab}_\Sigma L_{(ab)\circ} - L_{(ab)\circ}\nabla^a_\Sigma\nabla^b_\Sigma\Upsilon) \\
&\quad -(HJ + JN^b\nabla_b\Upsilon - H\Delta\Upsilon) - (J^\Sigma H + J^\Sigma N^b\nabla_b\Upsilon - H\Delta^\Sigma\Upsilon)
\end{aligned}$$

Here we note that:

$$\begin{aligned}
\mathsf{P}_{3,4}\Upsilon &= \tfrac{1}{2}N^b\nabla_b\Delta\Upsilon + \Delta^\Sigma N^b\nabla_b\Upsilon + H\Delta\Upsilon - 2L_{(ab)\circ}\nabla^a_\Sigma\nabla^b_\Sigma\Upsilon - HN^aN^b\nabla_b\nabla_a\Upsilon \\
&\quad +N^bP_{ab}\nabla^a\Upsilon - J^\Sigma N^b\nabla_b\Upsilon - \tfrac{3}{2}H^2N^b\nabla_b\Upsilon - \tfrac{3}{2}(\nabla^b_\Sigma L_{(ab)\circ})\nabla^a_\Sigma\Upsilon
\end{aligned}$$

$$\Rightarrow \widehat{\mathsf{Q}}_{3,4} = \mathsf{Q}_{3,4} + \mathsf{P}_{3,4}\Upsilon + N^aN^bP_{ab}N^c\nabla_c\Upsilon - N^bP_{ab}\nabla^a\Upsilon + \tfrac{1}{2}(\nabla^b_\Sigma L_{(ab)\circ})\nabla^a_\Sigma\Upsilon - (\nabla^b_\Sigma H)\nabla^\Sigma_b\Upsilon$$

The terms involving P_{ab} can be simplified as follows:

$$N^bP_{ab}\nabla^a\Upsilon - N^aN^bP_{ab}N^c\nabla_c\Upsilon = g^{ac}_\Sigma N^bP_{ab}\nabla_c\Upsilon = g^{ad}_\Sigma N^bP_{ab}\nabla^\Sigma_d\Upsilon$$

Now by the Codazzi equation (in dimension 4), $\Pi_a^d N^b P_{db} = \frac{1}{2}\nabla^b_\Sigma L_{(ab)\circ} - \nabla^\Sigma_a H$, we have shown that :

$$-(\Pi_a^d N^b P_{db})\nabla^a_\Sigma \Upsilon \;=\; -\tfrac{1}{2}(\nabla^b_\Sigma L_{(ab)\circ})\nabla^a_\Sigma \Upsilon + (\nabla^\Sigma_a H)\nabla^a_\Sigma \Upsilon$$

This conveniently cancels the remainder of the terms in the transformation equation above.

Hence $\widehat{\mathsf{Q}}_{3,4} = \mathsf{Q}_{3,4} + \mathsf{P}_{3,4}\Upsilon$ as claimed.

To summarise, the tractor calculus has provided us with a concise, specific formula for a third normal order boundary operator as the $n^\Sigma = 3$ specialisation of a formula that gives P_3 in all dimensions $n^\Sigma \geq 3$. Additionally, the $\mathsf{P}_{3,4}$ operator governs the transformation of an odd order Q-curvature analogue, $\mathsf{Q}_{3,4}$, in dimension 4. It is unknown whether the integral of this $\mathsf{Q}_{3,4}$ function is a globally invariant curvature term.

It is also unknown how to generate the higher order P_{2k-1} operators of $2k$-order GJMS operators in the critical dimension $2k$. One drawback to the procedure used for the construction of P_3 is that it has only dealt with this unique situation and it would be difficult to generalise the method which we have used to all P_{2k-1} operators.

The above problems could form the basis of further research. Future work could look at extending the ideas presented here to CR-manifolds and complex scalar fields. On another hand, the digression of page 29 provides us with a brief introduction to looking at tractor-tensor operators acting on 1-forms. There is still work to be done on the Q-curvature analogues of these operators, and also on the Σ versions of these operators.

Appendix A

Manifolds and Riemannian Manifolds

Definition A.0.1 (Manifold). *[18, 21] Let M be a topological space, then if each of the following hold true M is called a smooth n-dimensional manifold:*

1) $\{U_i\}$ forms a covering of M, where $U_i \subset M$

2) $\forall$ U_i and U_j, open sets $\in M$ there exist homeomorphisms:

$h_i : U_i \longrightarrow V_i \subset \mathbb{R}^n$,

$h_j : U_j \longrightarrow V_j \subset \mathbb{R}^n$,

3) Let $U_{ij} = U_i \bigcap U_j$ then h_{ij} exists where

$$h_{ij} = h_j \circ h_i^{-1} : h_i(U_{ij}) \longrightarrow h_j(U_{ij})$$

Definition A.0.2 (Riemannian Manifold). *Let M be a smooth n-dimensional manifold, and g be a smooth positive definite metric acting on M. Then (M, g) is called a Riemannian Manifold.*

Definition A.0.3 (Riemannian Submanifold). *Let (M, g) be a Riemannian manifold and $\Sigma \subset M$ with $T\Sigma$ a subbundle of $TM|_\Sigma$, then (Σ, g^Σ) is a Riemannian submanifold, where g^Σ is the restriction of g to $T\Sigma \subseteq TM|_\Sigma$.*

Appendix B

Lie Algebras

Definition B.0.1 (Lie Algebra). *[18, 12] A Lie algebra, $\mathfrak{g}$, is a real vector space with a bilinear operator,*

$$[\,.\,,.\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

such that for all A, B & $C \in \mathfrak{g}$:

$$[A, B] = -[B, A]$$

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0$$

Appendix C

Fundamental Equations of Submanifolds

Theorem C.0.1 (Codazzi-Mainardi Equation). *[22] Let U, V & $W \in \Gamma(T\Sigma)$ then the following equation is true:*

$$\perp R(U,V)W = (\nabla^{\Sigma}_U II)(V,W) - (\nabla^{\Sigma}_V II)(U,W)$$

$$\textit{Where } II(U,V) = -(\nabla_U N)\cdot V \textit{ is the second fundamental form}$$

Proof.

$$\begin{aligned}
R(U,V)W &= \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W \\
&= \nabla_U(\nabla^{\Sigma}_V W) - \nabla_U(II(V,W)N) - \nabla_V(\nabla^{\Sigma}_U W) + \nabla_V(II(U,W)N) - \\
&\quad \nabla^{\Sigma}_{[U,V]}W + II([U,V],W)N \\
&= \nabla^{\Sigma}_U\nabla^{\Sigma}_V W - II(U,\nabla^{\Sigma}_V W)N - \nabla_U(II(V,W)N) - \\
&\quad \nabla^{\Sigma}_V\nabla^{\Sigma}_U W + II(V,\nabla^{\Sigma}_U W)N + \nabla_V(II(U,W)N) - \\
&\quad \nabla^{\Sigma}_{[U,V]}W + II(\nabla^{\Sigma}_U V,W)N - II(\nabla^{\Sigma}_V U,W)N \\
&= R^{\Sigma}(U,V)W - II(U,\nabla^{\Sigma}_V W)N - \nabla_U(II(V,W)N) + II(V,\nabla^{\Sigma}_U W)N + \\
&\quad \nabla_V(II(U,W)N) + II(\nabla^{\Sigma}_U V,W)N - II(\nabla^{\Sigma}_V U,W)N
\end{aligned}$$

According to the Liebniz rule:

$$\nabla_X(II(Y,Z)) = \nabla_X(II)(Y,Z) + II(\nabla_X Y,Z) + II(Y,\nabla_X Z)$$

$$\begin{aligned}
\perp R(U,V)W &= -II(U,\nabla^{\Sigma}_{V}W) - N\nabla_{U}(II(V,W)N) + II(V,\nabla^{\Sigma}_{U}W) + \\
&\quad N\nabla_{V}(II(U,W)N) + II(\nabla^{\Sigma}_{U}V,W) - II(\nabla^{\Sigma}_{V}U,W) \\
&= (\nabla^{\Sigma}_{U}II)(V,W) - (\nabla^{\Sigma}_{V}II)(U,W)
\end{aligned}$$

□

C.0.1 Note:

Let $X^a, Y^b \in \mathcal{E}^a_{\Sigma}$ then

$$\begin{aligned}
II(X,Y) &= -(\nabla_X N)\cdot Y \\
&= -X^a Y^b \nabla_a N_b
\end{aligned}$$

By virtue of the fact that X^a is intrinsic to Σ we are confident that the connection is the projection of the ambient connection onto our hypersurface. The negative sign clearly indicates that Spivak's interpretation of the Second Fundamental Form is the opposite to that which has been adopted for this paper.

Appendix D

Ricci Code in Mathematica

All of the programming code was done in Ricci, a differential geometry package designed for use in Mathematica. To save time, I borrowed some of the initial macros from Larry Peterson. They are the following:
Generation of the tangent bundle of an n-dimensional manifold: tangent.m
Generation of the tractor bundle for the associated tangent bundle: tracbund.m
Assignment of the tractor rules: tracbase.m

All subsequent code has been written by myself:
Definition of the V tractor

```
(* V Tractor & Projections *)
(* Second Fundamental Form & Codazzi Rules *)
(* Last updated 5/6/03 *)
DefineTensor[V, 1, Bundle -> tractor1, Variance -> Con]

DefineTensor[sigma, 0]; DefineTensor[mu, 1,
  Variance -> Co]; DefineTensor[rho, 0];

DefineRule[VToBase, V[U[AA]],
  sigma*Y[U[AA]] + mu[U[i]]*Z[U[AA], L[i]] + rho*X[U[AA]]];

Projn := n - 1;
```

Figure D.1: V Tractor

```
(* Curvature.m *)
(* file last updated 4/6/03 *)

DefineTensor[P, 2, Symmetries -> Symmetric];
DefineTensor[J, 0];

DefineTensor[Weyl, 4, Symmetries -> RiemannSymmetries];

DefineRule[RmToPC, Rm[L[i], L[j], L[k], L[l]],
    Weyl[L[i], L[j], L[k], L[l]] + g[L[k], L[i]]*P[L[j], L[l]] -
      g[L[k], L[j]]*P[L[i], L[l]] + g[L[l], L[j]]*P[L[i], L[k]] -
      g[L[l], L[i]]*P[L[j], L[k]]];

DefineRule[CToRm, Weyl[L[i],L[j],L[k],L[l]],
    Rm[L[i],L[j],L[k],L[l]] - g[L[k], L[i]]*P[L[j], L[l]] +
      g[L[k], L[j]]*P[L[i], L[l]] - g[L[l], L[j]]*P[L[i], L[k]] +
      g[L[l], L[i]]*P[L[j], L[k]]];

DefineRule[PToRc, P[L[i], L[j]],
  (1/(n - 2))(Rc[L[i], L[j]] + g[L[i], L[j]]*Sc/(2 - 2n))];
DefineRule[RcToP, Rc[L[i], L[j]],
  (n - 2)P[L[i], L[j]] + J*g[L[i], L[j]]];

DefineRule[PToJ, P[L[i], U[i]],J];

DefineRule[JToP, J, P[L[i], U[i]]];

DefineRule[PDerivative, P[L[i],L[j]][U[j]],J[L[i]]];

DefineTensor[CY,3,Variance->{Co,Co,Con}];

DefineRule[CottonYork,CY[L[i],L[j],U[k]],P[L[j],U[k]][L[i]]-
P[L[i],U[k]][L[j]]];


(* TractorD Operator *)
(* Robin Operator *)
(* Updated by D Grant     4/6/03 *)

dgTractorD[operand_, weight_, index_] =
    Module[{LocOp, LocX, LocY, LocZ},
      (
        LocOp = NewDummy[operand];
          LocY = (weight(n + 2*weight - 2)*LocOp*Y[index]);
          LocZ = ((n+2*weight-2)*NewDummy[LocOp[L[i]]]*Z[index, U[i]]);
          LocX = (-(NewDummy[LocOp[L[i], U[i]]] +
                      weight*J*LocOp)*X[index]);

        NewDummy[LocY + LocZ + LocX]
        )
      ];

dgRobin[operand_,weight_] =
    Module[{LocOp,LocRobin},
    (
        LocOp = NewDummy[operand];
            LocRobin = (NewDummy[norm[U[i]]*LocOp[L[i]]] - weight*H*LocOp);
        NewDummy[LocRobin]
        )
    ];
```

Figure D.2: Assignment of the curvature rules

```
(* This package sets up the sub bundles to the tangent *)
(* and the tractor bundle.                              *)
(* It defines the intrinsic calculus on both.           *)
(*    D Grant 21/6/03 *)

DefineTensor[H, 0];
DefineTensor[norm, 1, Variance -> Con];
DefineTensor[UN, 1, Bundle -> tractor1, Variance -> Con];

(* Intrinsic Tractor bases *)
DefineTensor[SX, 1, Bundle -> tractor1];
DefineTensor[SZ, 2, Bundle -> {tractor1, tangent}];
DefineTensor[SY, 1, Bundle -> tractor1];
DefineTensor[SP, 2, Bundle->{tangent,tangent}];
DefineTensor[SJ, 0];

DefineTensor[ProjTang, 2, Bundle->{tangent,tangent},Variance -> {Co, Con}];
DefineTensor[ProjTract, 2, Bundle -> {tractor1, tractor1}, Variance -> {Co, Con}];

DefineRule[Project2,ProjTang[L[i], U[j]],
    Kronecker[L[i], U[j]] - norm[L[i]]*norm[U[j]]];
DefineRule[Project2,ProjTract[L[AA], U[BB]],
    Kronecker[L[AA], U[BB]] - UN[L[AA]]*UN[U[BB]]];

DefineRule[IProd,UN[U[AA]]*SZ[L[AA],U[i]],0];
DefineRule[IProd,UN[U[AA]]*SX[L[AA]],0];
DefineRule[IProd,UN[U[AA]]*SY[L[AA]],0];
DefineRule[NormIProd,ProjTang[L[j],U[i]]*norm[L[i]],0];
DefineRule[NormIProd,norm[U[i]]*ProjTang[L[i],U[j]],0];


DefineRule[GradXYZ, SX[L[AA]][L[i]], SZ[L[AA],L[i]]];
DefineRule[GradXYZ, SX[L[AA]][U[i]], SZ[L[AA],U[i]]];
DefineRule[GradXYZ, SZ[L[AA],L[i]][L[j]], -SP[L[i],L[j]]*SX[L[AA]]-
        ProjTang[L[i],L[j]]*SY[L[AA]]];
DefineRule[GradXYZ, SZ[L[AA],L[i]][U[j]], -SP[L[i],U[j]]*SX[L[AA]]-
        ProjTang[L[i],U[j]]*SY[L[AA]]];
DefineRule[GradXYZ, SY[L[AA]][L[i]], SP[L[i],U[j]]*SZ[L[AA],L[j]]];
DefineRule[GradXYZ, SY[L[AA]][U[i]], SP[U[i],L[j]]*SZ[L[AA],U[j]]];

DefineRule[PToJ,SP[L[i],U[i]],SJ];

(* DefineRule[Project,ProjTang[L[a],U[j]]*mu[L[j]],Smu[L[a]]]; *)
DefineRule[Project,ProjTang[L[a],U[j]]*Fn[L[j]],Fn[L[a]]];

(* The purpose of the projection tractors *)
DefineRule[Intrinsic, ProjTract[L[AA],U[BB]]*X[L[BB]],SX[L[AA]]];
DefineRule[Intrinsic,ProjTract[L[AA],U[BB]]*Z[L[BB],U[i]],ProjTang[U[i],L[j]]*
        SZ[L[AA],U[j]] + H*norm[U[i]]*SX[L[AA]]];
DefineRule[Intrinsic,ProjTract[L[AA],U[BB]]*Y[L[BB]],SY[L[AA]]-H^2/2*SX[L[AA]]];

(*The intrinsic to Sigma tractor D operator is defined - analogous to the original *)
dgSigmaTractorD[operand_, weight_, index_] =
    Module[{LocOp,k,LocX,LocY,LocZ},
      (
        LocOp = NewDummy[operand];
        k=Projn+2*weight-2;
        LocY = (weight*k*LocOp*SY[index]);
          LocZ = (k*NewDummy[LocOp[L[i]]]*SZ[index, U[i]]);
          LocX = (-(NewDummy[LocOp[L[j], U[j]]] +
                      weight*SJ*LocOp)*SX[index]);
        NewDummy[LocY + LocZ + LocX]  )
      ];

(* The second fundamental form and its trace free (TF) component are created *)
DefineTensor[Lab, 2,Bundle ->{tangent,tangent},Variance->{Co,Co},Symmetries->
        Symmetric];
DefineTensor[TFLab,2,Bundle->{tangent,tangent},Variance->{Co,Co},Symmetries->
        Symmetric];

(* The second fundamental form and its decomposition are defined *)
DefineRule[TraceFree,g[U[i],U[j]]*TFLab[L[i],L[j]],0];
DefineRule[MeanCurv,Lab[U[i],L[i]],(n-1)H];
DefineRule[MeanCurvInv, H, norm[U[i]][L[i]]/(n - 1)];
DefineRule[SecFundForm,ProjTang[L[j],U[k]]*norm[L[i]][L[k]],Lab[L[i],L[j]]];
DefineRule[ExpandSec,Lab[L[i],L[j]],TFLab[L[i],L[j]]+ProjTang[L[i],L[j]]*H];
DefineRule[NormTract, UN[U[AA]], Z[U[AA],L[i]]*norm[U[i]] - X[U[AA]]*H];
DefineRule[NormIProd,norm[L[i]]*norm[U[i]],1];
DefineRule[NormIProd, Lab[L[i], L[j]]*norm[U[i]], 0];
DefineRule[NormIProd, TFLab[L[i], L[j]]*norm[U[i]], 0];
DefineRule[NormIProd,H[L[i]]*norm[U[i]],0];

(* The projected Codazzi rule is defined *)
DefineRule[Codazzi, ProjTang[L[i],U[k]]*norm[U[j]]*P[L[k], L[j]],
    1/(n - 2)TFLab[L[i], L[j]][U[j]] - H[L[i]]];
```

Figure D.3: Assignment of the intrinsic rules

```
(* This package defines the inner products on Normal tractors *)
(* This package defines the hypersurface intrinsic inner products *)
(* Last updated 13/10/03 *)

DefineRule[IProd, UN[L[AA]]*X[U[AA]], 0];
DefineRule[IProd, UN[L[AA]]*Y[U[AA]], -H];
DefineRule[IProd, UN[L[AA]]*Z[U[AA], L[i]], norm[L[i]]];
DefineRule[IProd, UN[L[AA]]*UN[U[AA]], 1];

DefineRule[IProd,SX[L[AA]]*SY[U[AA]],1];
DefineRule[IProd,SZ[L[AA],L[i]]*SY[U[AA]],0];
DefineRule[IProd,SY[L[AA]]*SY[U[AA]],0];
DefineRule[IProd,SZ[L[AA],L[i]]*SZ[U[AA],L[j]],ProjTang[L[i],L[j]]];
DefineRule[IProd,SZ[L[AA],L[i]]*SX[U[AA]],0];
DefineRule[IProd,SX[L[AA]]*SX[U[AA]],0];
```

Figure D.4: Inner Product rules on submanifold

```
(*
<< Ricci.m; << tangent.m; << tracbund.m; << tracbase.m; << dgCurvature.m; <<
    dgVTractor.m; << dgConformal.m; << dgTractorD.m; << dgInnerProduct.m;

*)
(*
Let us generate the Tractor Curvature  $\Omega$
*)
(*
V[U[AA]][L[i]][L[j]] - V[U[AA]][L[j]][L[i]]
*)
(*
TensorSimplify[
  TensorSimplify[
      TensorSimplify[
          TensorSimplify[
              % /. VToBase]
            /. GradXYZ]
        /. GradXYZ]
    /. IProd]
*)
```

Figure D.5: Generation of the tractor curvature

```
(*
<< Ricci.m; << tangent.m; << tracbund.m; << tracbase.m; << dgCurvature.m; <<
    dgVTractor.m; << dgConformal.m; << dgTractorD.m; << dgInnerProduct.m;

*)
(*
This notebook will be used to generate the action of the Tractor D operator
on the splitting operators
*)
(*
Tractor D on $Y_A$
*)
(*
dgTractorD[Y[L[AA]]*Fn, w - 1, U[AA]]
*)
(*
TensorSimplify[
  TensorSimplify[
      TensorSimplify[
          TensorSimplify[
              % /. GradXYZ]
            /. GradXYZ]
        /. PToJ]
    /. IProd]
*)
(*
TensorSimplify[% /. PToJ]
*)
(*
Tractor D on $Z_A^a$
*)
(*
dgTractorD[Z[L[AA], U[i]]*Fn[L[i]], w - 1, U[AA]]
*)
(*
TensorSimplify[
  TensorSimplify[
      TensorSimplify[
          TensorSimplify[
              % /. GradXYZ]
            /. GradXYZ]
        /. IProd]
    /. PToJ]
*)
(*
Tractor D on $X_A$
*)
(*
dgTractorD[X[L[AA]]*Fn, w + 1, U[AA]]
*)
(*
TensorSimplify[
  TensorSimplify[
      TensorSimplify[
          TensorSimplify[
              % /. GradXYZ]
            /. GradXYZ]
        /. IProd]
    /. PToJ]
*)
```

Figure D.6: The tractor D acting on the splitting operators X, Y and Z

```
(* Generation of the $P_4$ operator, or Paneitz operator *)
(* Step 1: Take the Tractor D of the Fn *)

dgTractorD[Fn, (4 − n)/2, L[AA]]

(* Step 2: Take the Yamabe operator of the above result *)

%[L[i], U[i]] + (2 − n)/2*J*%

TensorSimplify[
  TensorSimplify[
      TensorSimplify[
          % /. GradXYZ]
        /. GradXYZ]
    /. IProd]

(* Step 3: Take the final Tractor D − This will be done using identities *)
(* Tractor D on X *)

Y[U[AA]]*%4

TensorSimplify[
  TensorSimplify[
      %]
    /. IProd]

(n + 2w + 2)(n + w)*%

(* Tractor D on Z *)

Z[U[AA], L[i]]*%4

TensorSimplify[
  TensorSimplify[
      %]
    /. IProd]

(n + 2w − 2)*%[U[j]]

(* Tractor D on Y *)

X[U[AA]]*%4

TensorSimplify[
  TensorSimplify[
      %]
    /. IProd]

TensorSimplify[% /. PToJ]

(* Assign a specific weight *)

w = −n/2

Simplify[%19]
Expand[%]
TensorSimplify[% /. PToJ]

w = (−4 − n)/2

Simplify[%16]
TensorSimplify[% /. PToJ]

(* As can be seen this is  $(n−4)P_4$ *)
```

Figure D.7: Generation of an $(n-4)$ multiple of the P_4 operator

```
(* The purpose of this notebook is to generate the \delta_3 operator while using \
the intrinsic tractor calculus.
    D Grant : 21/6/03 *)

DefineTensor[Fn, 0];

(* Step 1: Apply Tractor D *)
dgTractorD[Fn, (4 - n)/2, L[BB]]

(* Step 2: Apply Robin operator and simplify *)
dgRobin[%, (2 - n)/2]
TensorSimplify[
  TensorSimplify[
      %]
    /. GradXYZ]

(* Step 3: Project onto the submanifold *)
TensorSimplify[ProjTract[L[AA], U[BB]] %]
TensorSimplify[% /. Intrinsic]
TensorSimplify[% /. NormIProd]

(* Isolate the coefficients of the above tractor *)
(* Coefficients of X *)
TensorSimplify[
  TensorSimplify
        [SY[U[AA]]*%8
      ] /. IProd
  ]

(* Coefficients of Y *)
TensorSimplify[
    TensorSimplify[
        SX[U[AA]]*%8
      ] /. IProd
  ]

(* Coefficients of Z *)
TensorSimplify[
  TensorSimplify[
      SZ[U[AA], U[l]]*%8
      ] /. IProd
  ]
DefineRule[Substitute, rho, %9]
DefineRule[Substitute, mu[U[l]], %11]
DefineRule[Substitute, sigma, %10]

(* Step 4: Apply intrinsic Tractor D - the rules are established in dgTractorD *)
(* Intrinsic tractor D acting on Y *)
(Projn + (2 - n)/2 - 2)*SJ*sigma -
  ProjTang[L[k], U[l]]*ProjTang[L[r], U[s]]*sigma[L[l], L[s]]
TensorSimplify[
  % /. Substitute]

(* Intrinsic D acting on Z *)
(Projn + (2 - n) - 2)ProjTang[L[l], U[p]]*mu[U[l]][L[p]]
TensorSimplify[% /. Substitute]
TensorSimplify[% /. SecFundForm]

(* Intrinsic D acting on X *)
(Projn + 2(-2 - n)/2 + 2)(Projn + (-2 - n)/2)*rho
TensorSimplify[% /. Substitute]

(* Sum the three operators *)
%16 + %19 + %21
```

Figure D.8: Generation of the δ_3 operator

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