Maths 190 Lecture 9

Topic for today:

Sizes of Infinity

Question of the day:

Is there an infinite set bigger than the set $\{1, 2, 3, \ldots\}$ of natural numbers?

Recall from last lecture

The set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the same size as the set $\{1, 2, 3, \ldots\}$. We say the sets have the same cardinality.

Also, the set of all rational numbers (fractions) is the same size as the set $\{1, 2, 3, \ldots\}$.

This means we can write an ordered list of each set. For example:

$$\{0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3, \ldots\}$$

$$\{0, 1, -1, 1/2, 2, -1/2, -2, 1/3, 3, -1/3, -3, 2/3, 3/2, -2/3, -3/2, \ldots\}$$

To find a set bigger than the natural numbers, we need a set that cannot be listed in any way at all.

Simple Lotto

Pick four numbers from the list 1,2,3, . . . , 20. Each choice of four numbers gives you one ticket in the draw and costs $10.

If the numbers on your ticket match my four numbers, you win $50,000.

What ticket buying strategy ensures that you win?

Dodgeball

Player 1:

1						
2						
3						
4						
5						
6						

Player 2:

1	2	3	4	5	6

Dodgeball

Player 1:

1						
2						
3						
4						
5						
6						

Player 2:

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Dodgeball

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Playing Dodgeball: rule change

Player 1 can pick the number of rows in the table at the start of game. There are still only six columns.

Is there now a strategy for Player 1 to always win?

You can change the rule about the order of turns if you want.

Playing Infinite Dodgeball

Each row can have an infinite number of Os and Xs. There can be infinitely many rows. Players 1 and 2 alternate turns as in the original 6×6 version of the game.

Who wins?

Given any list of infinite sequences of Os and Xs, it is always possible to write down a sequence that is not on the list.

We conclude that the set of infinite sequences of Os and Xs cannot be listed.

Thus, the set of infinite sequences of Os and Xs is a bigger set than the set $\{1, 2, 3, \ldots\}$.

How many real numbers are there? Is the set of real numbers bigger than the set $\{1, 2, 3, \dots\}$?

Important ideas from today:

- There are sets bigger than the set $\{1, 2, 3, \dots\}$.
- We found two such sets:
  1. the set of infinite sequences of O's and X's
  2. the set of numbers in the interval [0,1]
- These ideas about different sizes of infinity are challenging. Many mathematicians did not believe these ideas for many years after Cantor described them, and Cantor was ridiculed and belittled for his ideas.

For next time

- Read §3.3 in the textbook.
- Try some Mindscapes at the end of §3.3 of the textbook.
- Try to explain why mathematicians believe there are different sizes of infinity to a friend who is not in Maths 190.