Differential Geometry

On the uniqueness of almost-Kähler structures ☆

Sur l'unicité de structures presque Kähleriennes

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A R T I C L E   I N F O

A B S T R A C T

We show uniqueness up to sign of positive, orthogonal almost-Kähler structures on any non-scalar flat Kähler–Einstein surface.

R É S U M É

On montre l'unicité au signe près de structures presque kähleriennes orthogonales positives sur toute surface de Kähler–Einstein de courbure scalaire non-nulle.

1. Introduction

Given a Riemannian four manifold (M, g) an orthogonal almost-Kähler structure is an orthogonal almost complex structure J such that $g(J \cdot, \cdot)$ is a closed two form. It seems to be quite difficult to determine, in general, the obstructions on the metric g to the existence of such a structure. While integrability results are available for Einstein, anti-self-dual metrics [2], familiar examples such as the Kodaira–Thurston manifold support (see e.g. [1]) a full circle of almost-Kähler structures on the same orientation.

In this Note we explore the uniqueness question of almost-Kähler structures orthogonal with respect to a Kähler metric and compatible with the positive orientation.

Theorem 1.1. *Let (M, g, J) be a connected Kähler–Einstein surface, positively oriented by J. Then any positive, orthogonal almost-Kähler structure J' is integrable; in particular $J' = \pm J$ when g has non-vanishing scalar curvature.*

The proof of Theorem 1.1 is build around the observation that the angle function between the structures J and J' is isoparametric in the sense of Cartan. The conclusion follows by investigation of the eigenvalue structure of the second fundamental form of its level sets.

In particular we obtain the following, that answers a question posed in [1]:

Corollary 1.1. *Any orthogonal almost-Kähler structure on (some open piece of) $\mathbb{CH}^1 \times \mathbb{CH}^1$ or $\mathbb{CH}^2$ equipped with their canonical metrics is integrable.*

Proof. Both spaces are equipped with Kähler–Einstein metrics of negative scalar curvature. In the case of $\mathbb{CH}^2$, an orthogonal almost-Kähler structure is either compatible with the negative orientation, where integrability follows from [2] since the metric is self-dual, or with the positive orientation, where the same conclusion is obtained by Theorem 1.1. For $\mathbb{CH}^1 \times \mathbb{CH}^1$ which admits a Kähler structure on either orientation it suffices to apply Theorem 1.1 twice. $\quad\square$

2. Almost-Kähler structures on Kähler surfaces

Let (M, g, J) be a Kähler surface, positively oriented by J and admitting a positive orthogonal almost-Kähler structure J'. Recall that the bundle $\Lambda^+ M$ of self-dual forms on M splits as

$$\Lambda^+ M = \langle \omega_J \rangle \oplus \lambda_J^2 M \tag{1}$$

where $\lambda_J^2 M = \{\alpha \in \Lambda^2 M \colon \alpha(J\cdot, J\cdot) = -\alpha\}$. The Kähler form $\omega_{J'} = g(J'\cdot, \cdot)$ of J' is self-dual hence

$$\omega_{J'} = x\omega_J + \Psi$$

along (1), where $\omega_J = g(J\cdot, \cdot)$ is the Kähler form of J and Ψ belongs to $\lambda_J^2 M$. The angle map $x \colon M \to \mathbb{R}$ is explicitly given by $x = -\frac{1}{4}\langle J', J \rangle$.

Since $\omega_{J'}$ is a closed form in $\Lambda^+ M$ it is in particular harmonic, thus $\Delta x = 0$ because the splitting (1) is parallel w.r.t. the Levi-Civita connection ∇ of g and hence preserved by the Laplace operator. At this stage it is straightforward to treat the compact case.

Proposition 2.1. *If M is compact or x is constant then either $J' = \pm J$ or the scalar curvature s of the metric g vanishes and J' is integrable.*

Proof. When x is constant the form Ψ either vanishes or it has constant, non-zero norm. As it is well known, the latter case forces $s = 0$ and $\nabla \Psi = 0$. When M is compact x must be constant since it is harmonic. $\quad\square$

Let now D be the open set where Ψ is non-zero, to be assumed not empty in what follows. After re-normalisation of Ψ we can write $J' = xJ + yI$ where I is a local gauge for J and $x^2 + y^2 = 1$. That is I is an orthogonal almost complex structure such that $IJ + JI = 0$. We obtain a local gauge I' for J' given by

$$I' = -yJ + xI, \qquad K' = I'J' = K, \tag{2}$$

where $K = IJ$.

Because J is Kähler we have $\nabla I = b \otimes K$ for some local 1-form b on M such that $db = \rho^J$, where $\rho^J = g(Ric \circ J\cdot, \cdot)$ is the Ricci form of (g, J). Here the Riemann curvature tensor is defined by $R(X, Y) = -\nabla^2_{X,Y} + \nabla^2_{Y,X}$ for all X, Y in TM and Ric is the Ricci contraction.

The local connection 1-forms a' and b' of the gauge I' are determined from the almost-Kähler condition on (g, J'), that is $d\omega_{J'} = 0$, by

$$\nabla J' = a' \otimes I' - J'a' \otimes K', \qquad \nabla I' = -a' \otimes J' + b' \otimes K'. \tag{3}$$

The action of J' on 1-forms on M is defined by $J'\alpha = \alpha(J'\cdot)$. Without loss of generality we write $x = \cos\theta$, $y = \sin\theta$ for some local function θ on M.

Lemma 2.1. *The following hold on D:*

(i) $a' = d\theta$ *and* $J'a' = -\sin\theta\, b$;
(ii) $b' = \cos\theta\, b$;
(iii) $|grad\,\theta|^2 = -s\,\frac{\sin^2\theta}{4}$.

Proof. Let us record first the following inversion formulae

$$J = xJ' - yI', \qquad I = yJ' + xI'. \tag{4}$$

(i) We compute

$$\nabla J' = dx \otimes J + yb \otimes K + dy \otimes I = (x\,dy - y\,dx) \otimes I' + (yb) \otimes K'$$

after using (4). Therefore $a' = x\,dy - y\,dx = d\theta$ and $yb = -J'a'$ from the almost-Kähler condition.

(ii) As in (i), we differentiate in (2) to find

$$\nabla I' = -dy \otimes J + xb \otimes K + dx \otimes I = (y\,dx - x\,dy) \otimes J' + xb \otimes K'$$

after using (4) and $x^2 + y^2 = 1$. The claim is proved by comparison with (3).

(iii) Because (g, J) is Kähler the positive Weyl tensor is given by

$$W^+ = \begin{pmatrix} \frac{s}{6} & 0 \\ 0 & -\frac{s}{12} \end{pmatrix}$$

with respect to the decomposition (1). In particular, $W^+ \omega_{J'} = \frac{s}{6}(x\omega_J - \frac{y}{2}\omega_I)$, making that the conformal scalar curvature $\kappa = 3\langle W^+ \omega_{J'}, \omega_{J'} \rangle$ of (g, J') is given by $\kappa = (x^2 - \frac{y^2}{2})s$. The well known (see [1]) relation $\frac{\kappa - s}{3} = \frac{1}{4}|\nabla J'|^2$ combined with (3) and (i) yields the claim. $\square$

In particular, Theorem 1.1 follows immediately when $s > 0$ (see also [1]). Therefore, we assume from now on that the metric g is Einstein, that is $\rho^J = \frac{s}{4}\omega_J$ and re-normalise the scalar curvature to $s = -4$. We may assume, w.l.o.g. that $\sin\theta > 0$ so that $|grad\,\theta| = \sin\theta$ by Lemma 2.1(iii). The unit vector field $\xi = \frac{grad\,\theta}{|grad\,\theta|}$ is then totally geodesic henceforth normal to the co-dimension one Riemannian foliation induced by $\mathcal{V} = \ker d\theta$.

Let S in $S^2\mathcal{V}$ be given by $\langle SV, W \rangle = (\nabla_V \xi)W$ for all V, W in $\mathcal{V}$. It describes the second fundamental form of the distribution $\mathcal{V}$. The unit vector field $\xi_1 = J'\xi$ in $\mathcal{V}$ has dual one form ξ^1 subject to

$$d\xi^1 = -\omega_J, \tag{5}$$

since $\xi^1 = b$ by (i) in Lemma 2.1.

Lemma 2.2. *The following hold*:

(i) $tr\,S^2 = 2\cos^2\theta - 1$;
(ii) $S(\xi_1) = -(\cos\theta)\xi_1$.

Proof. (i) Because x is harmonic, we have $\Delta(dx) = 0$ and hence $(\nabla^\star\nabla)\,dx = dx$ by the Bochner formula. Taking the scalar product with dx we get further

$$\frac{1}{2}\Delta|dx|^2 = \langle \nabla^\star\nabla(dx), dx \rangle - |\nabla\,dx|^2 = |dx|^2 - \left|\nabla(dx)\right|^2.$$

Since

$$dx = -(\sin^2\theta)\xi \tag{6}$$

we have $|dx| = \sin^2\theta = 1 - x^2$. Using again that x is harmonic it follows that $\frac{1}{2}\Delta|dx|^2 = 2|dx|^2 - 6x^2|dx|^2$ hence $|\nabla(dx)|^2 = (6x^2 - 1)|dx|^2 = \sin^4\theta|\nabla\xi|^2 + 4\sin^4\theta\cos^2\theta$ by (6) and the claim follows.

(ii) By (5) we have $d\xi^1 = -\omega_J$ hence $\langle \nabla_X J'\xi, Y \rangle - \langle \nabla_Y(J'\xi), X \rangle = -\langle JX, Y \rangle$ for all X, Y in TM. Taking $Y = J'\xi$ we get $\nabla_{J'\xi}(J'\xi) = -(JJ')\xi$. Therefore

$$J'\nabla_{\xi_1}\xi = -(\nabla_{\xi_1} J')\xi - (JJ')\xi.$$

However, by means of (2) and the definition of J' we get $(\nabla_{\xi_1} J')\xi = (\sin\theta)K\xi$ while (4) yields $(JJ')\xi = -(\cos\theta)\xi - (\sin\theta)K\xi$ and the claim follows. $\square$

Proof of Theorem 1.1. On the orthogonal complement $\mathcal{V}_0$ of ξ_1 in $\mathcal{V}$ we must have $tr_{\mathcal{V}_0} S^2 = \cos^2\theta - 1 \leqslant 0$ by Lemma 2.2. Since S is symmetric and preserves $\mathcal{V}_0$ it follows that $\cos^2\theta = 0$ which is contradictory to having D non-empty. Therefore the angle function x is constant and the claim follows from Proposition 2.1. $\square$

References

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