

Maths 190 Lecture 22

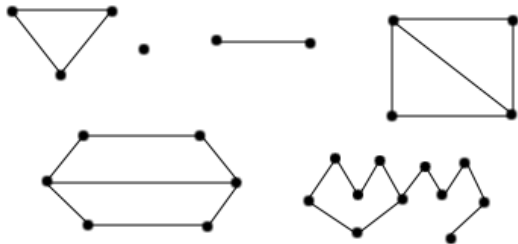
- ▶ **Topic for today:** Euler characteristic
- ▶ **Question of the day:** Why are there exactly 5 platonic solids?
- ▶ **Big idea:** The power of abstraction (again)

Graphs

Remember from the last lecture that a **graph** is a picture made from dots (called vertices) and lines between dots (called edges).

A graph is **connected** if there is a path between any two vertices.

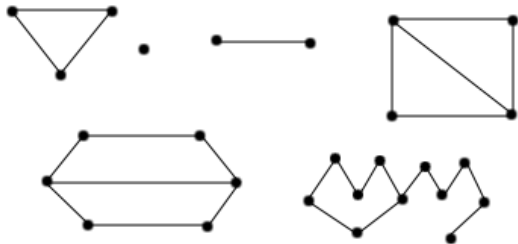
We will also restrict to graphs which can be **embedded on a sphere** without any edges crossing.



Graphs

Let E be the number of edges in a connected graph, and V the number of vertices.

We imagine the graph is drawn on a sphere (or balloon) with no edges crossing. Let F be the number of **faces** of the graph. A **face** is a region which is “enclosed” by edges. We include the “outside” face.



Euler characteristic

Determine V , E and F for these connected graphs. Put the results in a table.

V	E	F	$V - E + F$
3	3	2	2

Euler characteristic

The **Euler characteristic** is the formula $V - E + F = 2$.

(Technically, it is the value 2 of the formula.)

We will show that this formula holds for any connected graph.

I remember this using the mnemonic VErYFun.

Proof of the Euler characteristic

- ▶ Every connected graph can be built by starting with the graph with a single vertex and no edges and either adding an edge between two existing vertices or adding a new vertex and an edge between it and a previously existing vertex.
This is not surprising: It is exactly how one draws the picture.
- ▶ Graphs with a single vertex and no edges have $V = 1, E = 0, F = 1$ and so $V - E + F = 2$. Hence, the Euler characteristic holds for them.
- ▶ If we have a graph for which the Euler characteristic holds then consider what happens when we add a new edge in either of the two ways mentioned above.

Proof of the Euler characteristic (continued)

- ▶ Let E , V and F be the number of vertices, edges and faces of the graph.

By assumption, $V - E + F = 2$.

- ▶ Adding a new edge between two existing vertices leads to a new graph with $V' = V$, $E' = E + 1$ and $F' = F + 1$.

Hence

$$V' - E' + F' = V - (E + 1) + (F + 1) = V - E + F = 2.$$

- ▶ Adding a new vertex and a new edge leads to a new graph with $V' = V + 1$, $E' = E + 1$ and $F' = F$.

Hence

$$V' - E' + F' = (V + 1) - (E + 1) + F = V - E + F = 2.$$

- ▶ Hence (by the principle of mathematical induction, to be precise) the Euler characteristic holds for every graph.

Platonic solids and graphs on spheres

- ▶ Consider a platonic solid made of rubber like a balloon.
- ▶ Draw lines along the edges.
- ▶ Blow it up so it is a sphere.
- ▶ We have is a graph on a sphere.
- ▶ Hence, the Euler characteristic does hold for platonic solids.

Why only 5 platonic solids?

Solid	Vertices	Edges	Faces	Faces per Vertex	Sides per Face
Tetrahedron	4	6	4	3	3
Cube	8	12	6	3	4
Octahedron	6	12	8	4	3
Dodecahedron	20	30	12	3	5
Icosahedron	12	30	20	5	3

Why only 5 platonic solids?

- ▶ Suppose there is a hypothetical platonic solid called MYSTERAHEDRON.

Recall, a platonic solid has all faces the same, and the same number of faces meet at each vertex.

- ▶ The graph on the sphere corresponding to MYSTERAHEDRON satisfies $V - E + F = 2$.
- ▶ Let p be the number of edges of each face and q the number of faces which meet at each vertex.

(In our new language, q is the **degree** of each vertex.)

Recall that $pF = 2E$ and $qV = 2E$.

- ▶ Hence $V = 2E/q$ and $F = 2E/p$ and so

$$2 = V - E + F = E \left(\frac{2}{q} - 1 + \frac{2}{p} \right).$$

- ▶ Since $2 > 0$ and $E > 0$ it follows that

$$\frac{2}{q} + \frac{2}{p} > 1.$$

Why only 5 platonic solids?

- ▶ Now, $p \geq 3$ and $q \geq 3$.
- ▶ To have $\frac{2}{3} + \frac{2}{p} > 1$ it is necessary that $p \leq 5$.
- ▶ Hence, the MYSTERAHEDRON must have $3 \leq p, q \leq 5$.

Why only 5 platonic solids?

p	q	$2/p + 2/q$	Shape
3	3	$4/3$	Tetrahedron
3	4	$7/6$	Cube
3	5	$16/15$	Dodecahedron
4	3	$7/6$	Octahedron
4	4	1	Impossible
4	5	$9/10$	Impossible
5	3	$16/15$	Icosahedron
5	4	$9/10$	Impossible
5	5	$4/5$	Impossible

Graphs on other surfaces

The Euler characteristic does not necessarily hold for graphs which are drawn on another surface than a sphere.

See your assignment question 5.

Important ideas from today

Abstraction is a powerful tool for solving problems.

By studying a simple algebraic formula coming from the Euler characteristic we were able to prove that there are exactly 5 platonic solids.

A similar idea is used in Question 4 of the assignment.

Read Sections 5.4 of the book.