Maths 260 Lecture 27

- **Topic for today:** Sketching phase portraits for nonlinear systems
- **Reading for this lecture:** BDH Section 5.2
- **Suggested exercises:** BDH Section 5.2; 3, 13, 17, 19
- **Reading for next lecture:** None
- **Today's handouts:** Tutorial 10

Sketching phase portraits for nonlinear systems

We have learnt two methods for obtaining information about solutions to nonlinear systems:

- Linearisation can give information about the behaviour of solutions near an equilibrium solution.
- The method of nullclines gives information about where in the phase plane solution curves are horizontal and where they are vertical. From this we can deduce where in the phase plane solutions move up, down, left or right.

We use both of these methods to sketch phase portraits for nonlinear systems.

Outline of method

To sketch a phase portrait, it can be helpful to follow some or all of the following steps.

1. Find all equilibria. Where possible, use linearisation to determine their types (e.g., saddle, spiral source).
2. Draw the nullclines. Determine the direction of solutions in the regions between nullclines. Determine the direction of solutions on the nullclines.
3. Sketch some representative solution curves. Make sure the solution curves you sketch go in the directions determined by the nullclines and behave like the appropriate linearised system near any equilibrium.

Note: nullclines are usually **not** solution curves, and are not always straight lines.

Example 1

Sketch the phase portrait for the system

$$\frac{dx}{dt} = 2 - x - y,$$
$$\frac{dy}{dt} = x^2 - y.$$

Determine the long term behaviour of the solutions through $(x, y) = (1, 2)$ and $(-3, 4)$.

Step 1: Find equilibria and determine types

Example 1 continued

Jacobian:

Behaviour of solutions near $(1, 1)$:

$$J(1,1) = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$$

Example 1 continued

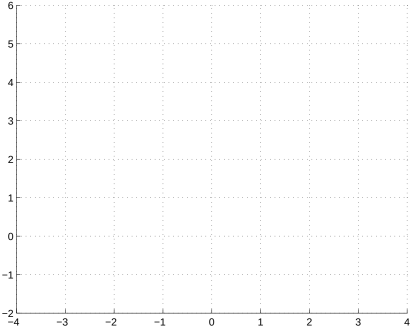
Behaviour of solutions near $(-2, 4)$:

$$J(-2,4) = \begin{pmatrix} -1 & -1 \\ -4 & -1 \end{pmatrix}$$

Example 1 continued

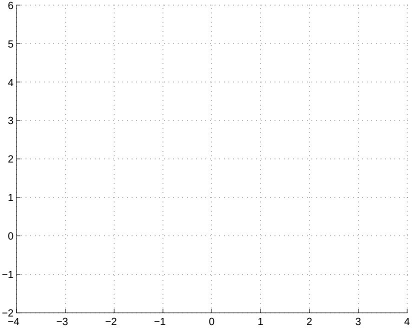
Step 2: Nullclines:

$$\dot{x} = 0 \Rightarrow y = 2 - x$$
$$\dot{y} = 0 \Rightarrow y = x^2$$



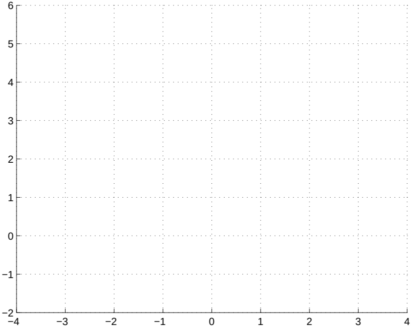
Example 1 continued

Direction of solutions on and between nullclines



Example 1 continued

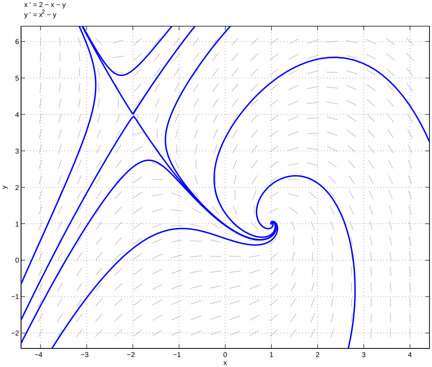
Step 3: Sketch phase portrait:



- Behaviour of solution through $(1, 2)$:
- Behaviour of solution through $(-3, 4)$:

Example 1 continued

Phase portrait from *pplane*:



Example 2:

Sketch the phase portrait for the system

$$\frac{dx}{dt} = x(x-1)$$
$$\frac{dy}{dt} = x^2 - y$$

Determine the long term behaviour of the solutions through $(x, y) = (-1, 0)$, $(0.8, 0)$ and $(1, 3)$.

Step 1: Find equilibria:

Example 2

Jacobian:

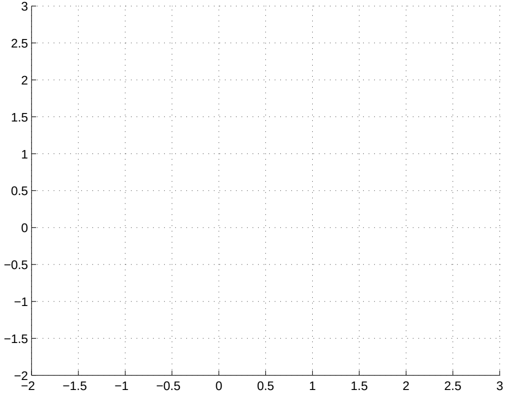
Behaviour of solutions near $(0, 0)$:

Example 2 continued

Behaviour of solutions near $(1, 1)$:

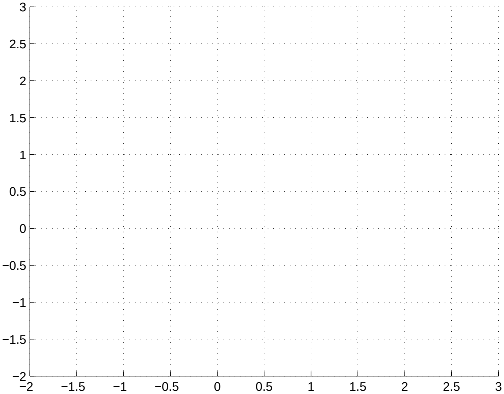
Example 2 continued

Step 2: Nullclines:



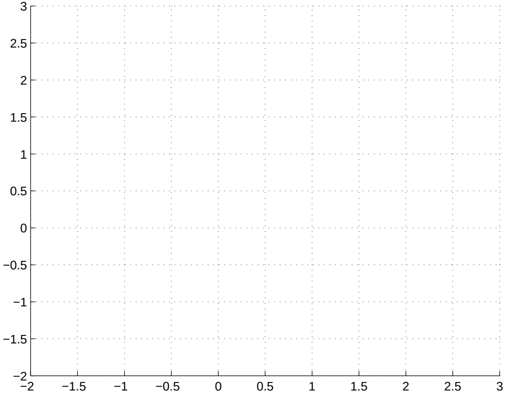
Example 2 continued

Directions of solutions on and between nullclines



Example 2 continued

Sketch of phase portrait



Example 2 continued

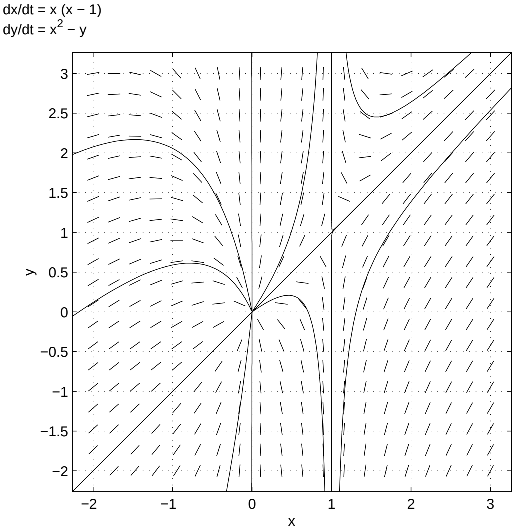
- Behaviour of solution through $(-1, 0)$:

- Behaviour of solution through $(0.8, 0)$:

- Behaviour of solution through $(1, 3)$:

Example 2 continued

Phase portrait from pplane



Important ideas from today:

To sketch a phase portrait for a nonlinear system:

1. Find all equilibria. Where possible, use linearisation to determine their types (e.g., saddle, spiral source).
2. Draw the nullclines. Determine the direction of solutions in the regions between nullclines. Determine the direction of solutions on the nullclines.
3. Sketch some representative solution curves. Make sure the solution curves you sketch go in the directions determined by the nullclines and behave like the appropriate linearised system near any equilibrium.