Maths 190 Assignment 2 Solutions

September 25, 2009 Due:

1. (5 marks)
   - Use proof by contradiction - start by assuming $(\sqrt{2}+\sqrt{3}) = a$ is rational.
   - Squaring both sides give $5 + 2\sqrt{6} = a^2$
   - Or $\sqrt{6} = (a^2 - 5)/2$, which is another rational number.
   - But $\sqrt{6}$ is irrational - proof as per $\sqrt{2}$ (subtract only one mark if they don't prove $\sqrt{6}$ is irrational.)
   - Contradiction: hence $\sqrt{2}+\sqrt{3}$ must be irrational.
2. (5 marks)

   (a) We want to compute $3 + 6 \times 4 \mod 9$. We get

   $$\begin{aligned} 3 + 6 \times 4 \mod 9 &= 3 + 3 \times 8 \mod 9 \\ &= 3 + 3 \times (-1) \mod 9 \\ &= 0 \mod 9 \\ &= 9 \mod 9 \end{aligned}$$

   so he finishes at 9 o'clock

   (b) Let Monday= 1, Tuesday= 2, etc. We want to compute $2 + 353 \mod 7$.

   $$\begin{aligned} 2 + 353 \mod 7 &= 2 + (7 \times 50 + 3) \mod 7 \\ &= 2 + 3 \mod 7 \\ &= 5 \mod 7 \end{aligned}$$

   Zaphod's birthday will be on a Friday next year.

   (c) Each time Ford cuts up a piece of paper, the number of pieces of paper increases by 3. So the total number of pieces of paper will be equal to $1 \mod 3$. But $3330 = 0 \mod 3$. Hence there must be a mistake either in the cutting or the counting.
3. (3 marks)

   To show that the two sets have the same cardinality I must find a one-to-one correspondence between elements in the two sets. I do this by using a shuffle, i.e., I pair the element zero in E with the element 1 in N, then pair the element 2 in E with the element 2 in N, then pair element -2 in E with element 3 in N, then 4 in E with 4 in N, -4 in E with 5 in N, and so forth.

   Alternatively, I can rewrite the set E with a new ordering: $E = \{0, 2, -2, 4, -4, 6, -6, \dots\}$ This new list contains all the elements of E precisely once, has a definite starting point and extends infinitely in one direction (unlike the original form for E, which was a list but infinite in two directions). Being able to list E in this form guarantees that E has the same cardinality as N - just pair the first element in the new form of E with the first element of N, pair the second element of E with the second element of N, and so on for ever.

4. (5 marks)

Vivien has the same number of fish and frogs in her office. I can show this by showing the set of individual frogs has the cardinality of the natural numbers and the set of individual fish also has the cardinality of the natural numbers.

To show that the set of fish has the cardinality of the natural numbers, name the packages of fish $P1, P2, P3, P4$ and so forth. Then name the individual fish within a package a, b, c, d, e, f. Then, for instance, the 5th fish in the 12th package will be named $P12e$. I can write a list that contains all the names of the fish, with the list having a definite beginning and extending infinitely in one direction:

Fish $= \{P1a, P1b, P1c, P1d, P1e, P1f, P2a, P2b, P2c, P2d, P2e, P2f, P3a, P3b, \ldots\}$.

As argued in question 3, the existence of such a list ensures that the cardinality of the set of individual fish is the same as the cardinality of the natural numbers.

In a similar way, I can write a list of all the frogs. I denote packages of frogs $F1, F2, F3$, etc and denote the frogs in a package by the letters $a, b, \ldots n$. Then I list the frogs:

Frogs $= \{F1a, F1b, \ldots F1n, F2a, P2b, \ldots F2n, F3a, F3b, \ldots\}$.

As before, the existence of this list ensures that the cardinality of the set of frogs is the same as the cardinality of the natural numbers.

Since both sets, Fish and Frogs, have the cardinality of the natural numbers, they must have the same cardinality as each other. Thus Vivien has as many fish as frogs.

5. (7 marks)

I prove this result by contradiction. First I assume that the cardinality of set D is the same as the cardinality of the natural numbers. If this is the case, then I can write a list that contains all the elements of D. Let L be such a list. Now I can write down an element of D (which I will call M) which is not anywhere in list L, as follows.

Let L_1 be the first number on list L, let L_2 be the second number on list L, and so on.

Look at L_1. If the digit in the first decimal place of L_1 is 1, then the digit in the first decimal place of M is 2. If the digit in the first decimal place of L_1 is instead 2, then the digit in the first decimal place of M is 1.

Then look at $L2$. If the digit in the second decimal place of L_2 is 1, then the digit in the second decimal place of M is 2. If the digit in the second decimal place of L_2 is instead 2, then the digit in the second decimal place of M is 1.

And so on. If the digit in the nth decimal place of L_n is 1, then the digit in the nth decimal place of M is 2. If the digit in the nth decimal place of L_n is instead 2, then the digit in the nth decimal place of M is 1.

The number M constructed in this way is definitely not on the list L (because it differs in the first decimal place from L_1 and in the second decimal place from $L2$, etc.) This is a contradiction though, because L was assumed to be a list of all elements of D. Furthermore, since L was an arbitrary list, ANY potential list of elements of D must fail to include at least one element of D. Thus there can be no list of D and hence D is unlistable, i.e., the cardinality of D is greater than the cardinality of the natural numbers.

Tutorial questions:

Tutorial 3: (5 marks)

Consider two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c, d are integers. Then the product is

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

which is also rational since ac and bd are integers.

The sum is

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

which is also rational since $ad + bc$ and bd are integers.

The product of two irrational numbers can be either rational or irrational. For example $\sqrt{2} \times \sqrt{2} = 2$ which is rational, but $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ which is irrational. (Give 2 marks here for an example of each, no proof needed).

Tutorial 4: (5 marks)

(a) The manager could ask every guest in the hotel to move to the room next door, so that the guest originally in room 1 moves to room 2, the guest originally in room 2 moves to room 3 and so forth, with the guest in room N moving to room N+1. Then the weary traveler can be put in room 1 and everyone has a place to sleep. NOTE: the manager cannot put the weary traveler in the "last" room, because there is no "last room" in an infinite list of rooms.

(b) The manager can do something similar again. This time he moves the guests by two rooms, so that the guest in room 1 moves to room 3, the guest in room 2 moves to room 4, the guest in room N moves to room N+2 and so on. This process frees up two rooms (rooms 1 and 2) and the two new arrivals can have these rooms.

(c) The manager has to work harder this time but it can still be done, by shuffling the guests originally in the hotel with the guests from the Infinite Life Insurance Company. So, for instance, the guest originally in room 1 could move to room 2, the guest originally in room 2 could move to room 4, the guest originally in room 3 could move to room 6, the guest originally in room N could move to room 2N, and so forth. Then rooms 1, 3, 5, 7, ... will be free. The number of free rooms has the cardinality of the natural numbers and so the guests from the Infinite Life Insurance Company can be put one into each free room without ever running out of rooms.

Tutorial 5: (5 marks)

I can construct a one-to-one correspondence between points on the circle and points on the (boundary of the) square in one of two ways. First, I could put the circle inside the square. The circle will touch the square at four points. Then any line segment drawn from the point at the centre of the circle out to a point on the square will intersect the circle at precisely one point. This defines a pairing between a point on the circle and a point on the square. The pairing is clearly one-to-one, as each point on the circle corresponds to one and only one point on the square. Thus there is a one-to-one correspondence between points on the circle and points on the square and so the set of points on the circle has the same cardinality as the set of points on the square.

Alternatively, I could straighten out the circle to make a line segment of length 2π cm and straighten out the square to make a line segment of length 8 cm. As shown in class, I can always construct a one-to-one correspondence between points on two line segments of different lengths. First draw a line $S1$ joining the left ends of both line segments, then draw a second line $S2$ joining the right ends of both line segments. The lines $S1$ and $S2$ will meet at precisely one point P. Then define a one-to-one correspondence between points on the two line segments by drawing lines through P and the line segments. Each line that crosses P and one line segment will also intersect the other line segment; there will be exactly one intersection of the line with each line segment. This defines a one-to-one correspondence between points on the two line segments.