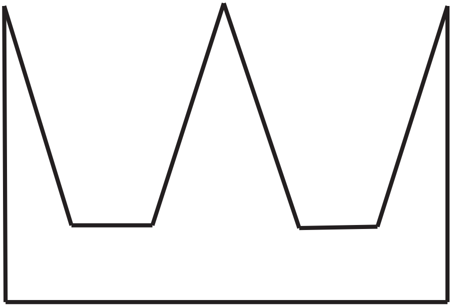
Maths 190 Lecture 13

REMINDER: Test Thursday April 29 at 6:15. The room is to be announced.

Question of the day: Here is a floor plan for an art gallery. How many guards would you need, each standing at a corner, so that each wall could be seen by at least one of the guards? The guards can turn their heads but they cannot leave their corners.



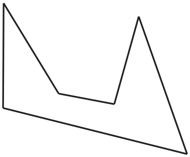
A **polygonal closed curve** is any figure made up of straight line pieces that are connected end-to-end to form a loop.

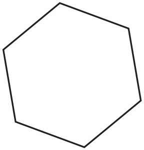
The corners where the walls meet are called **vertices**.

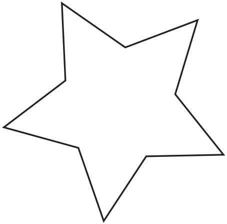
We are interested in art galleries whose floor plans are polygonal closed curves, with no interior walls or partitions.

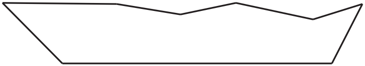
Try this in pairs:

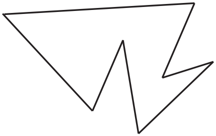
For the following art galleries, put guards at the vertices so that every interior point is seen by at least one guard. Use as few guards as possible and count the number of guards needed.











Now try this in pairs:

Can you draw a gallery with 7 vertices that needs only 1 guard? 2 guards? 3 guards?

Can you draw a gallery with 9 vertices that needs only 1 guard? 2 guards? 3 guards?

Can you draw a gallery with 11 vertices that needs only 1 guard? 2 guards? 3 guards? 4 guards?

Data from example art galleries

Number of vertices (v)	Smallest number of guards needed	

The Art Gallery Theorem

Suppose we have a polygonal closed curve in the plane with v vertices. Then the number of guards needed is at most $v/3$.

In other words, there are $v/3$ vertices from which it is possible to view every point on the interior of the curve. If $v/3$ is not an integer, then the number of vertices we need is the biggest integer less than $v/3$.

Why is the theorem true?:

We use a very powerful idea called **divide and conquer**.

Why is the theorem true?:

- The simplest gallery is a triangle. The theorem is clearly true for any triangular gallery.

- We can divide any other polygonal gallery into a collection of triangles.
  Note: Each vertex of the original gallery must be a vertex of at least one triangle and each vertex of a triangle must be a vertex of the original gallery.

Why is the theorem true? - continued:

- Now colour each vertex of the gallery so that each triangle has three colours (say, red, yellow and blue).
- If R is the number of red vertices, B the number of blue vertices and Y the number of yellow vertices then

$$R + B + Y = v.$$

  Hence, at least one of these numbers is less than $v/3$.

- Choose one of the 3 colours and place guards at each vertex which has been given that colour.
  Since each point in the gallery is contained in at least one triangle and since each triangle has a vertex of each colour, it follows that all points in the gallery can be seen.

What the theorem tells us:

The Art Gallery theorem tells us that a gallery with n vertices needs **AT MOST** $v/3$ guards.

The theorem **DOES NOT** tell us the minimum number of guards needed - some galleries with v vertices will need fewer than $v/3$ guards. For example, one guard will be able to see all walls of a regular octagonal gallery (where $v = 8$).

To determine the minimum number of guards needed for a gallery is non-trivial. There are complicated algorithms to do this, but for small examples one can work it out by trial and error.

Related questions:

- How many guards are needed if we are interested only in galleries where the walls meet at right angles?

- If all the internal walls of a polygonal gallery are mirrored, is it possible to find a point in the gallery from which all of the gallery is visible?

Important ideas from today:

- If we have a polygonal closed curve in the plane with v vertices, then it is possible to find $v/3$ vertices from which every point inside the curve is visible. If $v/3$ is not an integer then the number of vertices we need is the largest integer less than $v/3$.
- This means that **at most** $v/3$ guards are needed to guard a gallery with v vertices. In fact, fewer guards might be needed for a particular gallery.
- Our discussion of this example illustrates a useful process in mathematics.
  - First we looked at lots of examples and looked for patterns.
  - Then we tackled the complex cases (polygonal curves with many vertices) by dividing them into many simpler pieces (triangles).

  This process is useful in many areas of mathematics and elsewhere.

For next time

- Read §4.2 in the textbook.
- Try some Mindscapes at the end of §4.2 of textbook.
- Draw a polygonal art gallery floor plan for a friend and ask them how many guards should be placed at corners in order to keep an eye on all parts of the gallery. Tell them about the Art Gallery theorem and try to explain why it is true.