Elliptic PDE in Conformal Geometry

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Conformal Geometry

- Given (M^n, g), g Riemannian metric

Consider $\hat{g} = \rho g$, $\rho > 0$ a positive function on M

$\hat{g}$, "angle" preserving

Denote $\rho = e^{2w}$ $\hat{g} = g_w = e^{2w} g$.

- Study "conformal invariants"
  - global invariant : e.g. $\int_{M^2} K_g \, dv_g$
  - pointwise invariant

    e.g. Weyl curvature.

    $W_{g_w} = e^{+2w} W_g$

- Using methods in P.D.E.

$$\begin{cases} \text{2nd order elliptic P.D.E.} \rightarrow \text{curvatures} \\ \text{Higher order elliptic P.D.E} \rightarrow \text{curvature polynomials.} \\ \text{Fully non-linear elliptic P.D.E} \nearrow \\ \text{Elliptic Systems} \rightarrow \end{cases}$$

Outline

§1. (M^2, g) Gaussian curvature

§2. (M^n, g) scalar curvature

§3. (M^n, g) Q-curvature

n even in particular $n=4$

§4. (M^n, g) σ_k - of Schouten tensor

§ 1. On (M^2, g) Compact surface

Gauss - Bonnet Formula:

$$2\pi\, \chi(M) = \int_{M^2} K_g \, dV_g$$

K_g = Gaussian curvature of (M^2, g)

$\chi(M)$ = Euler Characteristic

Uniformization Thm. Classify (orientable) (M^2, g)

- When $\int_M K_g \, dV_g > 0$, we can find $w \in C^\infty(M)$ so that $K_{g_w} \equiv 1$; thus (M^2, g) is diffeomorphic to (S^2, g_c)
- When $\int_M K_g \, dV_g = 0$, Solve $K_{g_w} \equiv 0$; thus (M^2, g) diffeomorphic to $(\mathbb{R}^2/\Gamma, |dx|^2)$
- When $\int_M K_g \, dV_g < 0$, Solve $K_{g_w} \equiv -1$; thus (M^2, g) diffeomorphic to $(H^2/\Gamma, h_c)$

On (M^2, g) denote $g_w = e^{2w} g$

then K_g and K_{g_w} are related by

$$(*) \qquad -\Delta_g w + K_g = K_{g_w} e^{2w}$$

P.D.E. Problem

Given the sign of $\int K_g \, dV_g$, solve

$(*)$ with $K_{g_w} \equiv$ constant c, sign c given.

Analytic Difficulties:

•). $(*)$ is an non-linear PDE in w

• An attempt to get "A priori estimate of w":

e.g. $\max w$ or $\min w$ in terms of K_g, K_{g_w}

Look at $(\mathbb{R}^2, |dx|^2)$, $K_g \equiv 0$

$$(*)' : \qquad -\Delta w = c\, e^{2w}$$

$c \equiv 1$, Sequence of solutions: $\lambda \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^2$

$$w_\lambda(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}$$

$$w_\lambda(x_0) \to -\infty \quad \text{as} \quad \lambda \to \infty$$

Actually $\int_{\mathbb{R}^2} e^{2w_\lambda(x)}\,dx = 4\pi \quad \forall \lambda > 0$

Thus no apriori estimate for solutions of (*).

<u>Remark</u> when $c \le 0$, $w \le 0$

<u>Basic Analytic tool:</u> (via variational approach)

<u>Moser – Trudinger inequality</u>:

Recall Sobolev-Imbedding: $D \subset \mathbb{R}^n$

$$W_0^{1,q}(D) \hookrightarrow L^p(D) \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$$

$q = 2$, $p = \frac{2n}{n-2}$ when $n \geq 3$

$q = 2$ $\quad p \neq \infty$ $\quad n = 2$

e.g. D = unit ball in $\mathbb{R}^2$

$$w(x) = \log \left| \log\left(e - 1 + \frac{1}{|x|}\right) \right|$$

Thm D smooth domain in $\mathbb{R}^2$, then $\exists\, c_1 > 0$ so that $\forall\, w \in W_0^{1,2}(D)$, $\int_D |\nabla w|^2 dx \le 1$, we have.

$$\int_D e^{\alpha |w|^2(x)}\,dx \le c_1 |D|$$

for any $\alpha \le 4\pi$, $\alpha = 4\pi$ being the best constant

Proof of Thm

$\boxed{\text{Step 1}}$

Prop $\int_D |Dw|^2 \le 1$, w of compact support in D

then $\exists\, \alpha > 0, C_1 > 0$ $\int_D e^{\alpha w^2(x)}\,dx \le C_1 |D|$

$\Leftrightarrow$ $\exists\, C_2 \ni \|w\|_p \le C_2 \sqrt{p}\, |D|^{1/p}$ $\forall p \ge 2$

Proof

"$\Rightarrow$" $\frac{1}{k!}\int_D (\alpha w^2)^k(x)\,dx \le C_1 |D| \quad \forall k \in \mathbb{Z}$

$\Downarrow$

$$\left(\int w^{2k}(x)\,dx\right)^{\frac{1}{2k}} \le \left(\frac{k!}{\alpha^k} C_1 |D|\right)^{\frac{1}{2k}}$$

$(k!)^{\frac{1}{k}} \le k$ $\quad \le \sqrt{2k}\, C_3 |D|^{\frac{1}{2k}}$

$p = 2k$

Thus Prop. holds $\forall k$, hence for all p.

$\Leftarrow$ $\int e^{\alpha w^2} \le \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}\left(C_2 \sqrt{2k}\, |\Omega|^{\frac{1}{2k}}\right)^{2k}$

Choose $2\alpha C_2^2 < \frac{1}{e}$ Apply Stirling's formula

$\le C_1 |D|$.

$\boxed{\text{Step 2}}$ To prove that

$$\|w\|_p \le C\sqrt{p}\, |D|^{1/p} \qquad w|_{\partial D} = 0$$

Apply Green's representating formula for $D = B_1(0)$

$$w(x) = -\frac{1}{2\pi}\int_{B_1(0)} \Delta w(y) \log|x-y|\, dy$$

$$\Rightarrow |w(x)| \le C \int_{B_1} |\nabla w(y)|\, |x-y|^{-1}\, dy$$

$$\le C\left(\int_{B_1} |\nabla w(y)|^2 |x-y|^{-a} dy\right)^{1/p}$$

$$\cdot\left(\int_{B_1} |x-y|^{-a}\, dy\right)^{1/2}$$

$$\cdot\left(\int_{B_1} |\nabla w(y)|^2\, dy\right)^{\frac{1}{2}-\frac{1}{p}}$$

$\frac{a}{p} + \frac{a}{2} = 1$

$$\Rightarrow \int_{B_1(0)} |x-y|^{-a} dy \le \int_{B_2(x)} |x-y|^{-a} dy$$

$$= C \left.\frac{r^{2-a}}{2-a}\right|_{r=0}^{2} \le C\,p$$

$$\Rightarrow \|w\|_p \le C\sqrt{p}\,.$$

Above proof does not give "sharp" p

$\boxed{\text{Step 3}}$ To see $\alpha \le 4\pi$

Use symmetrization technique reduce to O.D.E. of

$$\int_0^\infty \dot{w}^2(t)\,dt = 1$$

$$\Downarrow \qquad\qquad C_0 > 1+e$$

$$I[w] = \int_0^\infty e^{w^2(t)-t}\,dt \le C_0 < \infty$$

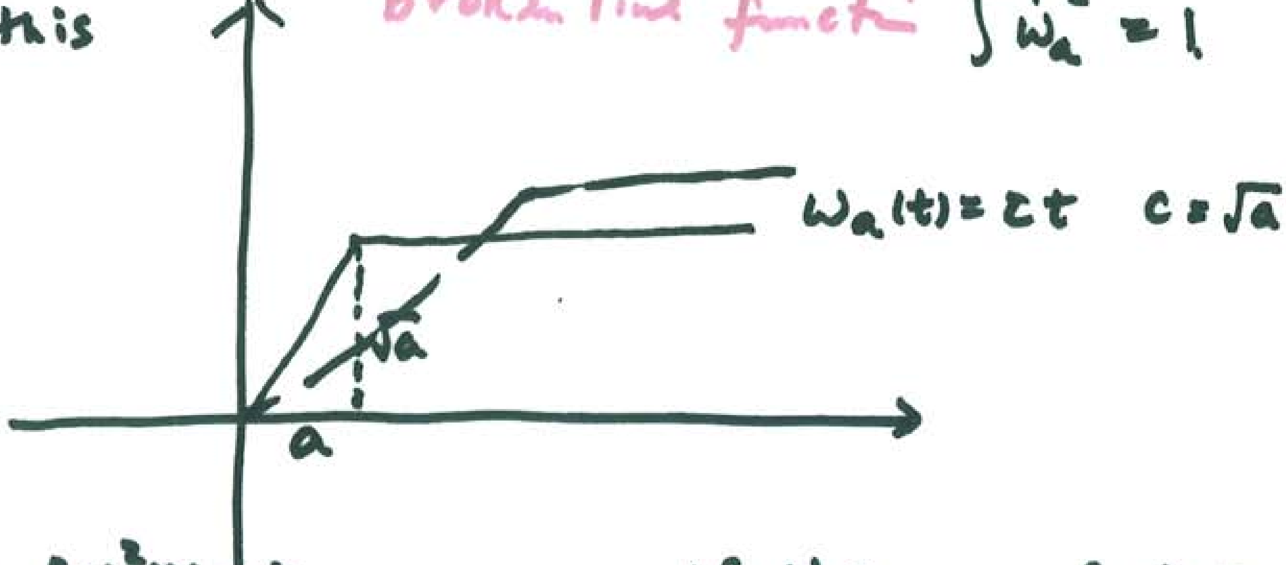
Denote

$$I_\beta(w) = \int_0^\infty e^{\beta \dot{w}^2(t)-t}\,dt$$

$$\sup_{\int_0^\infty \dot{w}^2=1} I_\beta(w) = \infty \qquad \forall\, \beta > 1$$

To see this

broken line function $\int \dot{w}_a^2 = 1$



$$\int_0^\infty e^{\beta w_a^2(t)-t}\,dt \ge e^{(\beta-1)a} \xrightarrow{a\to\infty} \infty$$

Cor linearized form of Moser's inequality

$$\log \frac{1}{|D|}\int_D e^{2w}\,dx \le \frac{1}{4\pi}\int_D |\nabla w|^2\,dx + C$$

best constant.

Thm (Moser's) On $(S^2, d\mu)$ $d\mu$ surface measure.

$$\log \frac{1}{4\pi}\int_{S^2} e^{2w}\,dx \le \frac{1}{4\pi}\int_{S^2}|\nabla w|^2 + 2\frac{1}{4\pi}\int_{S^2} w\,d\mu + C_1$$

best

$\forall\, w \in W^{1,2}(S^2)$.

Return to the study of $\quad g_w = e^{2w}g$

(*) $$-\Delta_g w + K_g = K_{g_w} e^{2w}$$

Variational Approach Study the functional

Denote $K = K_{g_w}$; when K positive

$$J_K[w] = \int_M |\nabla w|^2\,dV_g + 2\int_M K_g\,w\,dV_g - \log\int_M K e^{2w}\,dV_g$$

$J_K[w+c] = J_K[w]$

Geometric Content of Moser's exponent:

A) In the problem of " Prescribing Gaussian curvature " on S^2, what function K is allowed to be the Gaussian curvature of a metric $g_w = e^{2w} du$ on S^2?

i.e.

$$(*)' \qquad -\Delta w + 1 = K e^{2w} \quad \text{on } S^2 \qquad \Delta = \Delta_{du}$$

When is $(*)'$ Solvable ?

• Kazdan-Warner Obstruction : $(*)' \Rightarrow$

$$\int_{S^2} \langle \nabla K, \nabla x \rangle e^{2w} = 0$$

i.e. $K = 1 + \varepsilon x_i$ not solvable.

$K(\xi) = K$

Thm (Moser) K is even, $K > 0$

then $(*)'$ solvable with w an even function.

Morse

(Index Thm for general K.)

ω is even, then $\quad (\alpha = 8\pi)$

$$\log \frac{1}{4\pi} \int_{S^2} e^{2w}\, d\mu \leq \frac{1}{8\pi} \int_{S^2} |\nabla w|^2\, d\mu + \frac{2}{4\pi} \int_{S^2} w\, d\mu + C_1$$

Hence a minimal sequence of w_ℓ of $J[w]$

with $\int K e^{2w_\ell}\, d\mu = 4\pi$ have $(K_{g_0} \equiv 1)$

then $J[w_\ell] = \frac{1}{4\pi} \int_{S^2} |\nabla w_\ell|^2 + \frac{2}{4\pi} \int_{S^2} w_\ell\, d\mu \leq C(K)$

while $K \geq \underline{K} > 0$ by ① we have,

$$C'(K) \lesssim \frac{1}{8\pi} \int_{S^2} |\nabla w_\ell|^2 + \frac{2}{4\pi} \int_{S^2} w_\ell\, d\mu$$

Hence $\left(\frac{1}{4\pi} - \frac{1}{8\pi}\right) \int |\nabla w_\ell|^2 \lesssim C(K)$

w_ℓ converges weakly (and strongly) in $W^{1,2}$

(B)

- When $\int K_g \, dV_g = 4\pi$, i.e. $M = S^2$

Thm Onofri, T. Aubin
$J[w] \geq 0$ on S^2

$J[w] = 0$ precisely for conformal factors w of the form $e^{2w} g_c = T^* g_c$, T is a Mobius transformation of the 2-sphere.

- Geometric Content for $J[w]$ when $Vol(g_w) = Vol(g)$

$$J[w] = 12\pi \log \frac{\det(-\Delta_g)}{\det(-\Delta_{g_w})}$$

where the determinant of the Laplacian $\det(-\Delta_g)$ is defined by Ray-Singer as

$$\log \det(-\Delta_g) = -\zeta'(0)$$

$$\left(\zeta(s) \doteq \sum_{\lambda_k \neq 0} \lambda_k^{-s} \right). \quad \text{Zeta function}$$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \cdots$$

eigenvalue of $(-\Delta_g)$

Formal differentiation

$$\zeta'(s) = \sum_{\lambda_k \neq 0} -(\log \lambda_k)\, \lambda_k^{-s}$$

hence $\zeta'(0) = -\sum_{\lambda_k \neq 0} \log \lambda_k = -\log \prod_{k=1}^{\infty} \lambda_k$

$$\text{“}=\text{”} -\log \det(-\Delta_g)$$

Apply Mellin transform, for all $x > 0$

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}\, t^{s-1}\, dt.$$

We can rewrite $\zeta(s)$ in terms of Gamma functi

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k=1}^{\infty} e^{-\lambda_k t}\, t^{s-1}\, dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty (Z(t) - 1)\, t^{s-1}\, dt$$

$Z(t)$ denotes the Heat Kernel. The existance of $\zeta'(0)$ can be justified via Weyl's Asympototic formula of the heat kernel.

Use

$$2ab \le a^2 + b^2$$

$$\Rightarrow \quad 2w \le 4\pi \frac{w^2}{\int |Dw|^2} + \frac{1}{4\pi} \int |Dw|^2$$

$$\Rightarrow \quad e^{2w} \le e^{4\pi \frac{w^2}{\int |Dw|^2}} \, e^{\frac{1}{4\pi} \int |Dw|^2}$$

$$\Rightarrow \quad \int_D e^{2w} dx \le \left(\int e^{4\pi \frac{w^2}{\int |Dw|^2}} dx \right) e^{\frac{1}{4\pi} \int |Dw|^2 dx}$$

by Thm

$$\Rightarrow \quad \log \frac{1}{|D|} \int_D e^{2w} dx \le \log c_1 + \frac{1}{4\pi} \int_D |Dw|^2 dx$$

§2. Conformal Laplacian operator
scalar curvature

On (M^2, g), $\Delta_{g_w} = e^{-2w} \Delta_g$ for $g_w = e^{2w} g$

As $\Delta_g = \frac{1}{(\det g)^{1/2}} \partial_i \left(g^{ij} (\det g)^{1/2} \partial_j \right)$

On (M^n, g), $n \geq 3$, denote

$$L_g = -\Delta_g + c_n R_g \qquad c_n = \frac{(n-2)}{4(n-1)}$$

R_g = scalar curvature.

satisfies "covariant" property:

$$L_{g_w}(\phi) = e^{-\frac{n+2}{2} w} L_g \left(e^{\frac{n-2}{2} w} \phi \right) \quad \forall \phi \in C$$

Take $\phi \equiv 1$ we get

$$c_n R_{g_w} = e^{-\frac{n+2}{2} w} L_g \left(e^{\frac{n-2}{2} w} \right) \qquad \frac{4}{n-2}$$

Denote $u = e^{\frac{n-2}{2} w}$, we get $\hat{g} = e^{2w} g = u^{\cancel{\frac{n-2}{4}}} g$

$$(**) \qquad c_n R_{\hat{g}} = u^{-\frac{n+2}{n-2}} L_g (u \phi)$$
$$= u^{-\frac{n+2}{n-2}} L_g (u)$$

Scalar curvature equation

Yamabe problem: Solve (***) with $R_{\hat{g}} \equiv$ constant.

(Yamabe, Aubin, Trudinger, Schoen 80's).

Yamabe constant

$$Y(M, [g]) = \inf_{\hat{g} \in [g]} \frac{\int R_{\hat{g}} \, dV_{\hat{g}}}{Vol(\hat{g})^{\frac{n-2}{n}}}$$

plays role like that of Moser's constant.

$\boxed{\text{Key Property:}}$ $Y(M, [g]) \leq Y(S^n, g_e)$

"=" only on (S^n, g_e).

separate (M^n, g) from (S^n, g_e)

Topological result: (Schoen-Yau)

On (M^n, g), if $R_g > 0$, and (M^n, g) is locally conformally flat (i.e. locally $g = \rho |dx|^2$) then $\tilde{M}^n$ is conformally embedded in (S^n, g_e).