Chapter 10

Connections with totally skew-symmetric torsion and nearly-Kähler geometry

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Contents

1 Introduction

For a given Riemannian manifold (M^n, g), the holonomy group $\mathrm{Hol}(M, g)$ of the metric g contains fundamental geometric information. If the manifold is locally irreducible, as it can always be assumed locally by the well-known de Rham splitting theorem, requiring that $\mathrm{Hol}(M, g)$ is smaller that $\mathrm{SO}(n)$ has strong geometric implications at the level of the geometric structure supported by M (see [52] for an account of the classification of irreducible Riemannian holonomies). Nowadays, this has been extended to cover the classification of possible holonomy groups of torsion-free affine connections (see [9], [10], [44] for an overview).

From a somewhat different perspective, in the past years there has been increased interest in the theory of metric connections with totally skew-symmetric torsion on Riemannian manifolds. The holonomy group of such a connexion is once again an insightful object which can be used to describe the geometry in many situations. One general reason supporting this fact is that the de Rham splitting theorem fails to hold in this generalised setting and the counterexamples to this extent are more intricate than just Riemannian products. Additional motivation for the study of connections in the above mentioned class comes from string theory, where this requirement is part of various models (see [1], [54] for instance).

connexion or connection??

Actually, some of these directions go back to the work of A.Gray, in the seventies, in connection to the notion of weak holonomy [31], [3], aimed to generalise that of Riemannian holonomy. The study of nearly-Kähler manifolds initiated by Gray in [30], [32], [33] has been instrumental in the theory of geometric structures of non-integrable type. As it is well known, for almost Hermitian structures a classification has been first given by Gray and Hervella in [34], which proved to be quite suitable for generalisation.

Recently, general G-structures of non-integrable type have been studied in [55], [26], [17], [25], by taking into account the algebraic properties of the various torsion modules. Using results from holonomy theory they have indicated how to read off various intersections of G-modules interesting geometric properties when in presence, say, of a connection with totally skew-symmetric torsion. Aspects of such an approach are contained in the Gray–Hervella classification and its analogues for other classical groups, we shall comment on later. Note that a central place in [17] is occupied by connections with parallel skew-symmetric torsion, studied since then by many authors.

By combining these two points of view, let us return to the Gray–Hervella classification and recall that one of its subclasses, that of nearly-Kähler structures, has a priori parallel torsion for a connection with totally skew-symmetric torsion [40]. This suggests perhaps that geometric properties of the intrinsic torsion might by obtained in more general context.

In this chapter we will at first investigate some of the properties of almost Hermitian structures in the Gray–Hervella class $W_1 + W_3 + W_4$, also referred to – as we shall in what follows – as the class $\mathcal{G}_1$. It is best described [26] as the class of almost Hermitian structures admitting a metric and Hermitian connection with totally skew-symmetric torsion. Yet, this property is equivalent [26] with requiring the Nijenhuis tensor of the almost complex structure to be a 3-form when evaluated using the Riemannian metric of the structure. There are several subclasses of interest, for example the class $W_3 + W_4$ consisting of Hermitian structures, the class W_4 of locally conformally Kähler metrics and the class W_1 of nearly-Kähler structures, which will occupy us in the second half of the exposition. One general question which arises is how far the class $W_1 + W_3 + W_4$ is, for instance, from the subclass $W_3 + W_4$ of Hermitian structures.

The chapter is organised as follows. In Section 2 we present a number of algebraic facts related to representations of the unitary group and also some background material on algebraic curvature tensors. We also briefly review some basic facts from almost-

Hermitian geometry, including a short presentation of the Hermitian connections of relevance for what follows. At the end of Section 2 we introduce the notion of Hermitian Killing form with respect to a Hermitian connection with torsion. Although this is a straightforward variation on the Riemannian notion it will serve an explanatory rôle in the next section.

Section 3 is devoted to the study of the properties of the torsion tensor of an almost Hermitian structure in the class $\mathcal{G}_1$. More precisely we prove:

Theorem 1.1. *Let (M^{2m}, g, J) be an almost-Hermitian structure of type $\mathcal{G}_1$ and let D be the unique Hermitian connection whose torsion tensor T^D belongs to $\Lambda^3(M)$. Then*

$$D_X \psi^+ = -\frac{1}{2}(X \lrcorner \, dT^D)_{\lambda^3}$$

for all X in TM, where ψ^+ is proportional to the Nijenhuis form of J and the subscript indicated orthogonal projection on $\lambda^3(M)$, the bundle of real valued forms of type $(0, 3) + (3, 0)$.

For unexplained notation and the various numerical conventions we refer the reader to Section 2. In the language of that section, this can be rephrased to say that ψ^+ is a Hermitian Killing form of type $(3, 0) + (0, 3)$. Theorem 1.1 is extending to the wider class of $\mathcal{G}_1$-structures the well-known [40] parallelism of the Nijenhuis tensor of a nearly-Kähler structure. We also give necessary and sufficient conditions for the parallelism of the Nijenhuis tensor, in a $\mathcal{G}_1$-context and specialise Theorem 1.1 to the subclass of $W_1 + W_4$ manifolds which has been recently subject of attention [13], [16], [47] in dimension 6. Section 3 ends with the explicit computation of the most relevant components of the curvature tensor of the connection D above, in a ready to use form.

Section 4 contains a survey of available classification results in nearly-Kähler geometry. Although this is based on [48], [49] we have adopted a slightly different approach, which emphasises the rôle of the non-Riemannian holonomy system which actually governs the way various holonomy reductions are performed geometrically. We have outlined, step by step, the procedure reducing a holonomy system of nearly-Kähler type to an embedded irreducible Riemannian holonomy system, which is the key argument in the classification result which is stated below.

Theorem 1.2 ([49]). *Let (M^{2m}, g, J) be a complete SNK-manifold. Then M is, up to finite cover, a Riemannian product whose factors belong to the following classes:*

  (i) *homogeneous SNK-manifolds;*

 (ii) *twistor spaces over positive quaternionic Kähler manifolds;*

(iii) *6-dimensional SNK-manifolds.*

We have attempted to make the text as self-contained as possible, for the convenience of the reader.

2 Almost Hermitian geometry

An almost Hermitian structure (g, J) on a manifold M consists in a Riemannian metric g on M and a compatible almost complex structure J. That is, J is an endomorphism of the tangent bundle to M such that $J^2 = -1_{TM}$ and

$$g(JX, JY) = g(X, Y)$$

for all X, Y in TM. It follows that the dimension of M is even, to be denoted by $2m$ in the subsequent. Any almost Hermitian structure comes equipped with its so-called Kähler form $\omega = g(J\cdot, \cdot)$ in Λ^2. Since ω is everywhere non-degenerate the manifold M is naturally oriented by the top degree form ω^m.

We shall give now a short account on some of the $U(m)$-modules of relevance for our study. Let $\Lambda^\star$ denote the space of differential p-forms on M, to be assumed real-valued, unless stated otherwise.

We consider the operator $\mathcal{J} \colon \Lambda^p \to \Lambda^p$ acting on a p-form α by

$$(\mathcal{J}\alpha)(X_1, \ldots, X_p) = \sum_{k=1}^{p} \alpha(X_1, \ldots, JX_k, \ldots, X_p)$$

for all $X_1, \ldots, X_p$ in TM. For future use let us note that alternatively

$$\mathcal{J}\alpha = \sum_{i=1}^{2m} e_i^\flat \wedge (Je_i \lrcorner \alpha)$$

for all α in $\Lambda^\star$, where $\{e_i, \ 1 \le i \le 2m\}$ is some local orthonormal frame on M. The almost complex structure J can also be extended to $\Lambda^\star$ by setting

$$(J\alpha)(X_1, \ldots, X_p) = \alpha(JX_1, \ldots, JX_p)$$

for all $X_i, 1 \le i \le p$ in TM.

Let us recall that the musical isomorphism identifying vectors and 1-forms is given by $X \in TM \mapsto X^\flat = g(X, \cdot) \in \Lambda^1$. Note that in our present conventions one has $JX^\flat = -(JX)^\flat$ whenever X belongs to TM.

Now $\mathcal{J}$ acts as a derivation on $\Lambda^\star$ and gives the complex bi-grading of the exterior algebra in the following sense. Let $\lambda^{p,q}$ be given as the $-(p-q)^2$-eigenspace of $\mathcal{J}^2$. One has an orthogonal, direct sum decomposition

$$\Lambda^s = \bigoplus_{p+q=s} \lambda^{p,q},$$

called the bidegree splitting of Λ^s, $0 \le s \le 2m$. Note that $\lambda^{p,q} = \lambda^{q,p}$. Of special importance in our discussion are the spaces $\lambda^p = \lambda^{p,0}$; forms α in λ^p are such that the assignment

$$(X_1, \ldots, X_p) \to \alpha(JX_1, X_2, \ldots, X_p)$$

is still an alternating form which equals $p^{-1} \mathcal{J} \alpha$. We now consider the operator $L \colon \Lambda^\star \to \Lambda^\star$ given as multiplication with the Kähler form ω, together with its adjoint, to be denoted by $L^\star$. Forms in

$$\Lambda_0^\star = \operatorname{Ker} L^\star$$

are called primitive and one has

$$[L^\star, L] = (m - p) 1_{\Lambda^p}$$

on the space of p-forms. Note that the inclusion $\lambda^p \subseteq \Lambda_0^p$ holds whenever $p \geq 0$.

Let $\lambda^p \otimes_1 \lambda^q$ be the space of tensors $Q \colon \lambda^p \to \lambda^q$ which satisfy

$$Q \mathbb{J} = -\mathbb{J} Q,$$

where $\mathbb{J} \colon \lambda^p \to \lambda^p$ is given by $\mathbb{J} = p^{-1} \mathcal{J}$. We also define $\lambda^p \otimes_2 \lambda^q$ as the space of tensors $Q \colon \lambda^p \to \lambda^q$ such that $Q \mathbb{J} = \mathbb{J} Q$. The following lemma is easy to verify.

Lemma 2.1. *Let $a \colon \lambda^p \otimes \lambda^q \to \Lambda^{p+q}$ be the total alternation map. Then:*

(i) *the image of the restriction of a to $\lambda^p \otimes_1 \lambda^q \to \Lambda^{p+q}$ is contained in $\lambda^{p,q}$;*

(ii) *the image of the restriction of a to $\lambda^p \otimes_2 \lambda^q \to \Lambda^{p+q}$ is contained in λ^{p+q};*

(iii) *the total alternation map $a \colon \lambda^p \otimes_1 \lambda^q \to \Lambda^{p+q}$ is injective for $p \neq q$;*

(iv) *the kernel of $a \colon \lambda^p \otimes \lambda^q \to \Lambda^{p+q}$ is contained in $\lambda^p \otimes_2 \lambda^q$.*

In this chapter most of our computations will involve forms of degree up to 4. For further use we recall that one has

$$\mathcal{J} = J$$

on $\lambda^{1,2}$ and also that

$$J_{\lambda^{2,2} \oplus \lambda^4} = 1_{\lambda^{2,2} \oplus \lambda^4}, \quad J_{\lambda^{1,3}} = -1_{\lambda^{1,3}}.$$

We will now briefly introduce a number of algebraic operators which play an important rôle in the study of connections with totally skew-symmetric torsion. If α belongs to Λ^2 and φ is in $\Lambda^\star$ we define the commutator [1]

$$[\alpha, \varphi] = \sum_{i=1}^{2m} (e_i \lrcorner \alpha) \wedge (e_i \lrcorner \varphi),$$

where $\{e_i, 1 \leq i \leq 2m\}$ is a local orthonormal frame in TM. Note that

$$[\omega, \varphi] = -\mathcal{J} \varphi$$

for all φ in $\Lambda^\star$. It is also useful to mention that

$$[\lambda^{1,1}, \lambda^3] \subseteq \lambda^3, \quad [\lambda^{1,1}, \lambda^{1,2}] \subseteq \lambda^{1,2}, \quad [\lambda^2, \lambda^3] \subseteq \lambda^{1,2}. \tag{2.1}$$

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[1]This is proportional to the commutator in the Clifford algebra bundle $\operatorname{Cl}(M)$ of M.

Lastly, for any couple of forms (φ_1, φ_2) in $\Lambda^p \times \Lambda^q$ we define their product $\varphi_1 \bullet \varphi_2$ in Λ^{p+q-2} by

$$\varphi_1 \bullet \varphi_2 = \sum_{k=1}^{2m} (e_k \lrcorner \varphi_1) \wedge (e_k \lrcorner \varphi_2),$$

for some local orthonormal frame $\{e_k, 1 \leq k \leq 2m\}$ on M. We end this section by listing a few properties of the product $\bullet$ for low degree forms, to be used in what follows.

Lemma 2.2. *The following hold:*

(i) $\varphi_1 \bullet \varphi_2$ *belongs to* $\lambda^{2,2} \oplus \lambda^{1,3}$ *for all* φ_1, φ_2 *in* $\lambda^{1,2}$;

(ii) $\varphi \bullet J\varphi$ *is in* $\lambda^{1,3}$ *for all* φ *in* $\lambda^{1,2}$;

(iii) $\varphi \bullet \psi$ *belongs to* $\lambda^{1,3} \oplus \lambda^4$ *whenever* φ *is in* $\lambda^{1,2}$ *and all* ψ *in* λ^3;

(iv) $\psi_1 \bullet \psi_2$ *belongs to* $\lambda^{2,2}$ *for all* ψ_1, ψ_2 *in* λ^3;

(v) *if* ψ *is in* λ^3 *we have that* $\psi \bullet J\psi = 0$.

The proofs consist in a simple verification which is left to the reader.

2.1 Curvature tensors in the almost Hermitian setting

We shall present here a number of basic facts concerning algebraic curvature tensors we shall need later on. Since this is intended to be mainly at an algebraic level, our context will be that of a given Hermitian vector space (V^{2m}, g, J). Let us recall that the Bianchi map $b_1 : \Lambda^2 \otimes \Lambda^2 \to \Lambda^1 \otimes \Lambda^3$ is defined by

$$(b_1 R)_x = \sum_{i=1}^{2m} e_i^\flat \wedge R(e_i, x)$$

for all R in $\Lambda^2 \otimes \Lambda^2$ and for all x in V. The space of algebraic curvature tensors on V is given by

$$\mathcal{K}(\mathfrak{so}(2m)) = (\Lambda^2 \otimes \Lambda^2) \cap \mathrm{Ker}(b_1)$$

and it is worth observing that $\mathcal{K}(\mathfrak{so}(2m)) = S^2(\Lambda^2) \cap \mathrm{Ker}(a)$. Restricting the group to the unitary one makes appear the space of Kähler curvature tensors given by

$$\mathcal{K}(\mathfrak{u}(m)) = (\lambda^{1,1} \otimes \lambda^{1,1}) \cap \mathrm{Ker}(b_1) = S^2(\lambda^{1,1}) \cap \mathrm{Ker}(a).$$

We shall briefly present some well-known facts related to the space of Kähler curvature tensors, with proofs given mostly for the sake of completeness. First of all, we have an inclusion of $\lambda^{2,2}$ into $S^2(\lambda^{1,1})$ given by

$$\Omega \mapsto \widehat{\Omega},$$

where $\widehat{\Omega}(x, y) = -\frac{1}{4}(\Omega(x, y) + \Omega(Jx, Jy))$ for all x, y in V. This is mainly due to the fact that forms of type $(2, 2)$ are J-invariant. A short computation yields

$$b_1(\widehat{\Omega}) = \Omega \tag{2.2}$$

for all Ω in $\lambda^{2,2}$. There is also an embedding of $\lambda^{1,3}$ into $\lambda^2 \otimes \lambda^{1,1}$ given by $\Omega \mapsto \check{\Omega}$, where

$$\check{\Omega}(x, y) = \Omega(x, y) - \Omega(Jx, Jy)$$

for all x, y in V. It is easily verified that

$$(b_1\check{\Omega})_x = Jx \lrcorner \mathcal{J}\Omega - 2x \lrcorner \Omega \tag{2.3}$$

for all x, y in V and for all Ω in $\lambda^{1,3}$.

Similarly to the well-known splitting $S^2(\Lambda^2) = \mathcal{K}(\mathfrak{so}(2m)) \oplus \Lambda^4$ we have:

Proposition 2.1. *There is an orthogonal, direct sum splitting*

$$S^2(\lambda^{1,1}) = \mathcal{K}(\mathfrak{u}(m)) \oplus \lambda^{2,2}.$$

Explicitly, any Q in $S^2(\lambda^{1,1})$ can be uniquely written as $Q = R + \widehat{\Omega}$ for some Kähler curvature tensor R and some Ω in $\lambda^{2,2}$.

Proof. Let Q belong to $S^2(\lambda^{1,1})$. It satisfies $Q(Jx, Jy) = Q(x, y)$ for all x, y in V and since Q belongs to $S^2(\Lambda^2)$ it splits as

$$Q = R + \Omega,$$

where R is in $\mathcal{K}(\mathfrak{so}(2m))$ and Ω belongs to Λ^4. In particular

$$R(Jx, Jy) + \Omega(Jx, Jy) = R(x, y) + \Omega(x, y) \tag{2.4}$$

for all x, y in V.

Since Q belongs to $\lambda^{1,1} \otimes \lambda^{1,1}$ we have that $(b_1 Q)x$ belongs to $\lambda^{1,2}$ for all x in V. But R is a curvature tensor, hence $(b_1 Q)x = -3x \lrcorner \Omega$, leading to $x \lrcorner \Omega$ in $\lambda^{1,2}$ for all x in V. It is then easy to see that Ω must be an element of $\lambda^{2,2}$. It follows by direct verification that R_Ω given by

$$R_\Omega(x, y) = \Omega(Jx, Jy) - \frac{1}{3}\Omega(x, y)$$

belongs to $\mathcal{K}(\mathfrak{so}(2m))$. Setting now $R_0 = R - \frac{3}{4} R_\Omega$ it is easy to see from (2.4) that R_0 belongs to $\mathcal{K}(\mathfrak{u}(m))$ and our claim follows after appropriate rescaling from

$$Q(x, y) = R_0(x, y) + \frac{3}{4}\big[\Omega(Jx, Jy) + \Omega(x, y)\big]$$

for all x, y in V. $\square$

Our next and last goal in this section is to have an explicit splitting of certain elements of $\Lambda^2 \otimes \lambda^{1,1}$ along $\Lambda^2 = \lambda^{1,1} \oplus \lambda^2$. More explicitly, we will look at this in

the special case of a tensor $\mathcal{R}$ in $\Lambda^2 \otimes \lambda^{1,1}$ such that

$$\mathcal{R}(x, y, z, u) - \mathcal{R}(z, u, x, y) = \gamma_x(y, z, u) - \gamma_y(x, z, u)$$
$$- \gamma_z(u, x, y) + \gamma_u(z, x, y) \tag{2.5}$$

holds for all x, y, z, u in V. Here γ belongs to $\Lambda^1 \otimes \Lambda^3$ and in what follows we shall use the notation

$$T = a(\gamma)$$

for the total alternation of γ. The tensor $\mathcal{R}$ is the algebraic model for the curvature tensor of a Hermitian connection with totally skew-symmetric torsion, which will be our object of study later on in the chapter.

Remark 2.1. A complete decomposition of $\mathcal{K}(\mathfrak{so}(2m))$ into irreducible components under the action of $U(m)$ has been given by Tricerri and Vanhecke in [56]. Further information concerning the splitting of $\Lambda^2 \otimes \Lambda^2$, again under the action of $U(m)$ is given in detail in [20]. While the material we shall present next can be equivalently derived from these references, it is given both for self-containedness and also as an illustration that one can directly proceed, in the case of a connexion with torsion, to directly split its curvature tensor without reference to the Riemann one. As a general observation we also note that the procedure involves only control of the orthogonal projections onto the relevant $U(m)$-submodules.

Recall that for any α in Λ^2 its orthogonal projection on $\lambda^{1,1}$ is given by

$$\alpha_{\lambda^{1,1}} = \frac{1}{2}(\alpha + J\alpha).$$

To obtain the decomposition of $\mathcal{R}$ we need the following preliminary lemma.

Lemma 2.3. *Let η in $\lambda^1 \otimes \lambda^{1,2}$ be given. We consider the tensors H_1, H_2 in $\Lambda^2 \otimes \lambda^{1,1}$ given by*

$$H_1(x, y) = (x \lrcorner \eta_y - y \lrcorner \eta_x)_{\lambda^{1,1}} \quad and \quad H_2(x, y) = H_2(Jx, Jy)$$

for all x, y in V. Then:

(i) $2(b_1 H_1)_x = 3\eta_x + J\eta_{Jx} + x \lrcorner a(\eta) - Jx \lrcorner a(J\eta)$;

(ii) $2(b_1 H_2)_x = \eta_x + 3J\eta_{Jx} + Jx \lrcorner a^c(\eta) + x \lrcorner a^c(J\eta)$

for all x in V. Here $J\eta$ in $\lambda^1 \otimes \lambda^{1,2}$ is defined by $(J\eta)_x = J\eta_x$ for all x in V, and the complex alternation map $a^c : \lambda^1 \otimes \lambda^{1,2} \to \Lambda^4$ is given by

$$a^c(\zeta) = \sum_{k=1}^{2m} e_k^\flat \wedge \eta_{Je_k}.$$

Proof. (i) Let $\{e_k, 1 \leq k \leq 2m\}$ be an orthonormal basis in V. Directly from its definition, the tensor H_1 is given by

$$2H_1(x, y) = x \lrcorner \eta_y - Jx \lrcorner J\eta_y - y \lrcorner \eta_x + Jy \lrcorner J\eta_x,$$

hence

$$2(b_1 H_1)_x = 4\eta_x - \sum_{k=1}^{2m} e_k^\flat \wedge (x \lrcorner \eta_{e_k}) + \sum_{k=1}^{2m} e_k^\flat \wedge (Jx \lrcorner J\eta_{e_k})$$

since $\mathcal{J} = J$ on $\lambda^{1,2}$. Now

$$x \lrcorner a(\eta) = \eta_x - \sum_{k=1}^{2m} e_k^\flat \wedge (x \lrcorner \eta_{e_k})$$

and

$$Jx \lrcorner a(J\eta) = J\eta_{Jx} - \sum_{k=1}^{2m} e_k^\flat \wedge (Jx \lrcorner J\eta_{e_k})$$

for all x in V and the claim follows immediately.

(ii) It is enough to use (i) when replacing η by $J\eta$.   □   "when" ok?

We split now

$$\gamma = \gamma^{1,2} + \gamma^3$$

along $\Lambda^1 \otimes \Lambda^3 = (\Lambda^1 \otimes \lambda^{1,2}) \oplus (\Lambda^1 \otimes \lambda^3)$ and also

$$T = T^{2,2} + T^{1,3} + T^4$$

along the bidegree decomposition of Λ^4 in order to be able to introduce some of the components of the tensor $\mathcal{R}$. These are the tensor $\mathcal{R}^a$ in $\lambda^{1,1} \otimes \lambda^{1,1}$ given by

$$\mathcal{R}^a(x, y) = (\gamma_x y - \gamma_y x)_{\lambda^{1,1}} + (\gamma_{Jx} Jy - \gamma_{Jy} Jx)_{\lambda^{1,1}} - \frac{1}{2}(T^{2,2}(x, y) + T^{2,2}(Jx, Jy))$$

for all x, y in V, and the tensor $\mathcal{R}^m$ in $\lambda^2 \otimes \lambda^{1,1}$ defined by

$$\mathcal{R}^m(x, y) = (\gamma_x y - \gamma_y x)_{\lambda^{1,1}} - (\gamma_{Jx} Jy - \gamma_{Jy} Jx)_{\lambda^{1,1}} - \frac{1}{2}(T^{1,3}(x, y) - T^{1,3}(Jx, Jy))$$

for all x, y in V. Here we use $\gamma_x y$ as a shorthand for $y \lrcorner \gamma_x$, whenever x, y are in V.

The promised decomposition result for $\mathcal{R}$ is achieved mainly by projection of (3.5) onto

$$\Lambda^2 \otimes \lambda^{1,1} = (\lambda^{1,1} \otimes \lambda^{1,1}) \oplus (\lambda^{1,1} \otimes \lambda^2)$$

while taking into account that $\mathcal{R}$ belongs to $\Lambda^2 \otimes \lambda^{1,1}$.

Theorem 2.1. *Let $\mathcal{R}$ belong to $\Lambda^2 \otimes \lambda^{1,1}$ such that (2.5) is satisfied. We have a decomposition*

$$\mathcal{R} = \mathcal{R}^K + \hat{\Omega} + \frac{1}{2}\mathcal{R}^a + \mathcal{R}^m,$$

where $\mathcal{R}^K$ belongs to $\mathcal{K}(\mathfrak{u}(m))$ and Ω is in $\lambda^{2,2}$. Moreover:

(i) *R^a belongs to $\Lambda^2(\lambda^{1,1})$ and satisfies the Bianchi identity*

$$(b_1 R^a)_x = -2(\gamma_x^{1,2} + J\gamma_{Jx}^{1,2}) - \frac{1}{2}x \lrcorner A_1 - \frac{1}{2}Jx \lrcorner A_2$$

for all x in V. The forms A_1 and A_2 are explicitly given by

$$A_1 = a(\gamma^{1,2}) + a^c(J\gamma^{1,2}) - 4T^{2,2},$$
$$A_2 = a^c(\gamma^{1,2}) - a(J\gamma^{1,2}).$$

(ii) *The Bianchi identity for R^m in $\lambda^2 \otimes \lambda^{1,1}$ is*

$$(b_1 R^m)_x = -(\gamma_x^{1,2} - J\gamma_{Jx}^{1,2}) - \frac{1}{2}x \lrcorner A_3 + \frac{1}{2}Jx \lrcorner A_4$$

for all x in V, where

$$A_3 = a(\gamma^{1,2}) - a^c(J\gamma^{1,2}) - 2T^{1,3},$$
$$A_4 = a(J\gamma^{1,2}) + a^c(\gamma^{1,2}) - \not{J}T^{1,3}.$$

Proof. We first show that R^a belongs to $\Lambda^2(\lambda^{1,1})$. Directly from the definition of the map a we have that the tensor

$$Q(x, y) := \gamma_x y - \gamma_y x - \frac{1}{2}T(x, y)$$

belongs to $\Lambda^2(\Lambda^2)$, so after projection on $\lambda^{1,1} \otimes \lambda^{1,1}$ it follows that

$$(\gamma_x y - \gamma_y x)_{\lambda^{1,1}} + (\gamma_{Jx}Jy - \gamma_{Jy}Jx)_{\lambda^{1,1}} - \frac{1}{2}(T(x, y) + T(Jx, Jy))_{\lambda^{1,1}}$$

is an element of $\Lambda^2(\lambda^{1,1})$. We conclude by recording that

$$(T(x, y) + T(Jx, Jy))_{\lambda^{1,1}} = T^{2,2}(x, y) + T^{2,2}(Jx, Jy)$$

since forms in $\lambda^{1,3}$ are J-anti-invariant whilst those in $\lambda^{2,2} \oplus \lambda^4$ are J-invariant and moreover any φ in λ^4 satisfies $\varphi(J\cdot, J\cdot, \cdot, \cdot) = -\varphi(\cdot, \cdot, \cdot, \cdot)$.

The next step is to notice that (2.5) actually says that $\mathcal{R} - Q$ belongs to $S^2(\Lambda^2)$ and to project again on $\lambda^{1,1} \otimes \lambda^{1,1}$. By the argument above it follows that

$$\mathcal{R}(x, y) + \mathcal{R}(Jx, Jy) - R^a(x, y)$$

is in $S^2(\lambda^{1,1})$, thus by Proposition 2.1 we can write

$$\mathcal{R}(x, y) + \mathcal{R}(Jx, Jy) - R^a(x, y) = 2R^K(x, y) + 2\hat{\Omega}(x, y) \qquad (2.6)$$

for all x, y in V, where Ω is in $\lambda^{2,2}$ and R^K belongs to $\mathcal{K}(\mathfrak{u}(m))$.

Now, rewriting (2.5) as

$$\mathcal{R}(x, y, z, u) - \mathcal{R}(z, u, x, y) = 2(\gamma_x(y, z, u) - \gamma_y(x, z, u)) - T(x, y, z, u)$$

we obtain after projection on $\lambda^2 \otimes \lambda^{1,1}$ that

$$\mathcal{R}(x,y) - \mathcal{R}(Jx, Jy) = 2(\gamma_x y - \gamma_y x)_{\lambda^{1,1}} - 2(\gamma_{Jx} Jy - \gamma_{Jy} Jx)_{\lambda^{1,1}}$$
$$- (T(x,y) - T(Jx, Jy))_{\lambda^{1,1}}$$

for all x, y in V. Since $\lambda^4 \oplus \lambda^{2,2}$ consists in J-invariant forms, it is easy to see that

$$(T(x,y) - T(Jx, Jy))_{\lambda^{1,1}} = T^{1,3}(x,y) - T^{1,3}(Jx, Jy),$$

in other words

$$\frac{1}{2}(\mathcal{R}(x,y) - \mathcal{R}(Jx, Jy)) = \mathcal{R}^m(x,y) \qquad (2.7)$$

for all x, y in V. The splitting of $\mathcal{R}$ follows now from (2.6) and (2.7) and from the fact that the component of γ on $\lambda^1 \otimes \lambda^3$ is not seen by the projection on $\lambda^{1,1}$.

To finish the proof, it remains only to prove the Bianchi identities for $\mathcal{R}^a$ and $\mathcal{R}^m$. Using Lemma 2.3 and (2.2) it is easy to get that

$$(b_1 R^a)_x = -2(\gamma_x^{1,2} + J\gamma_{Jx}^{1,2}) + 2x \lrcorner T^{2,2}$$
$$- \frac{1}{2}x \lrcorner (a(\gamma^{1,2}) + a^c(J\gamma^{1,2})) - \frac{1}{2}Jx \lrcorner (a^c(\gamma^{1,2}) - a(J\gamma^{1,2}))$$

for all x in V. The Bianchi identity for $\mathcal{R}^m$ is proved along the same lines and therefore left to the reader. $\qquad \square$

Remark 2.2. Underlying Theorem 2.1 are the following isomorphisms of $\mathfrak{u}(m)$-modules. The first is

$$b_1 \colon \Lambda^2(\lambda^{1,1}) \to (\lambda^1 \otimes_1 \lambda^{1,2}) \cap \mathrm{Ker}(a),$$

where

$$\lambda^1 \otimes_1 \lambda^{1,2} = \{\zeta \in \lambda^1 \otimes \lambda^{1,2} : \zeta_{Jx} = -J\zeta_x \text{ for all } x \in V\}.$$

The second is given by

$$b_1 \colon \lambda^2 \otimes \lambda^{1,1} \to \lambda^1 \otimes_2 \lambda^{1,2},$$

where $\lambda^1 \otimes_2 \lambda^{1,2} = \{\zeta \in \lambda^1 \otimes \lambda^{1,2} : \zeta_{Jx} = J\zeta_x \text{ for all } x \in V\}$.

This makes that in practice it is not necessary to work with the somewhat involved expressions for R^a, R^m but rather with their Bianchi contractions, which are tractable.

2.2 The Nijenhuis tensor and Hermitian connections

Let (M^{2m}, g, J) be an almost Hermitian manifold. Recall that the Nijenhuis tensor of the almost complex structure J is defined by

$$N_J(X,Y) = [X,Y] - [JX, JY] + J[JX, Y] + J[X, JY]$$

for all vector fields X, Y on M. It satisfies

$$N_J(X, Y) + N_J(Y, X) = 0,$$
$$N_J(JX, JY) = -N_J(X, Y),$$
$$N_J(JX, Y) = -JN_J(X, Y)$$

for all X, Y in TM. By evaluating N_J using the Riemannian metric g we can also form the tensor N^J defined by

$$N_X^J(Y, Z) = g(N_J(Y, Z), X)$$

for all X, Y, Z in TM, which therefore belongs to $\lambda^1 \otimes_2 \lambda^2$. When N_J vanishes identically, J is said to be integrable and gives M the structure of a complex manifold.

Denoting by ∇ the Levi-Civita connection attached to the Riemannian metric g we form the tensor ∇J in $\lambda^1 \otimes \lambda^2$. Then one can alternatively compute the Nijenhuis tensor as

$$N_J(X, Y) = -\big[(\nabla_{JX}J)Y - (\nabla_{JY}J)X\big] + J\big[(\nabla_X J)Y - (\nabla_Y J)X\big]. \tag{2.8}$$

It is now a good moment to recall that an almost Hermitian structure (g, J) is called Kähler if and only if $\nabla J = 0$, or equivalently $\nabla \omega = 0$. This implies the integrability of J by making use of (2.8) and also that ω is a symplectic form, in the sense that $d\omega = 0$ where d denotes the exterior derivative. It turns out that for an arbitrary almost complex structure (g, J), the tensors $d\omega$ in Λ^3 and N^J in $\lambda^1 \otimes_2 \lambda^2$ form a full set of obstructions to having (g, J) Kähler as the following general fact shows.

Proposition 2.2. *For any almost Hermitian structure (g, J) on M we have*

$$2\nabla_X \omega = -N_{JX}^J + X \lrcorner \, d\omega + JX \lrcorner \, Jd\omega \tag{2.9}$$

for all X in TM.

Proof. Although this is a standard fact we give the proof for self-containedness. From the definition of the exterior derivative we have

$$d\omega(X, Y, Z) = (\nabla_X \omega)(Y, Z) - (\nabla_Y \omega)(X, Z) + (\nabla_Z \omega)(X, Y)$$

for all X, Y, Z in TM. Since $\nabla_X \omega$ is in λ^2 for all X in TM we obtain

$$d\omega(X, Y, Z) - d\omega(JX, JY, Z)$$
$$= (\nabla_X \omega)(Y, Z) - (\nabla_Y \omega)(X, Z)$$
$$\quad - (\nabla_{JX}\omega)(JY, Z) + (\nabla_{JY}\omega)(JX, Z) + 2(\nabla_Z \omega)(X, Y)$$

for all X, Y, Z in TM. This can be rewritten by using (2.8) as

$$d\omega(X, Y, Z) - d\omega(JX, JY, Z) = \langle N_J(X, Y), JZ \rangle + 2(\nabla_Z \omega)(X, Y)$$

whenever X, Y, Z belong to TM, and the claim follows. $\square$

Therefore $d\omega = 0$ and $N_J = 0$ result in $\nabla J = 0$, in other words in (g, J) being Kähler.

We shall call a linear connection on the tangent bundle to M Hermitian if it respects both the metric and the almost complex structure. In the framework of almost Hermitian geometry two connections play a distinguished rôle. The first is called the first canonical Hermitian connection and it is defined by

$$\bar{\nabla}_X = \nabla_X + \eta_X,$$

where η in $\lambda^1 \otimes \lambda^2$ is given by $\eta_X = \frac{1}{2}(\nabla_X J)J$. The tensor η is called the intrinsic torsion tensor of the $U(m)$-structure on M induced by (g, J). The connection $\bar{\nabla}$ is minimal [28], in the sense of minimising the norm within the space of almost Hermitian connections.

To introduce the second Hermitian connection we need some preliminaries. First of all let us decompose the 3-form $d\omega = d^{1,2}\omega + d^3\omega$ along the splitting $\Lambda^3 = \lambda^{1,2} \oplus \lambda^3$. Then:

Lemma 2.4. *Let* (M^{2m}, g, J) *be almost Hermitian. Then* $a(N^J) = 4Jd^3\omega$.

Proof. Using (2.9) we obtain

$$d\omega = a(\nabla\omega) = -\sum_{i=1}^{2m} e_i^\flat \wedge N_{Je_i}^J + 3d\omega + \mathcal{J}(Jd\omega),$$

where $\{e_i, 1 \le i \le 2n\}$ is some local orthonormal frame. Since N^J belongs to $\lambda^1 \otimes_2 \lambda^2$ it is easy to see that $\mathcal{J}(a(N^J)) = 3\sum_{i=1}^{2m} e_i^\flat \wedge N_{Je_i}^J$ and the claim follows by taking into account that $\mathcal{J} = J$ on $\lambda^{1,2}$. □

Since $\lambda^3 \subseteq \lambda^1 \otimes_2 \lambda^3$ it follows that the Nijenhuis tensor splits as

$$N_X^J = \hat{N}_X^J + \frac{4}{3}X \lrcorner Jd^3\omega$$

for all X in TM, where the tensor $\hat{N}^J$ belongs to the irreducible $U(m)$-module

$$W_2 = (\lambda^1 \otimes_2 \lambda^2) \cap \text{Ker}(a).$$

Proposition 2.3. *Let* (M^{2m}, g, J) *be almost Hermitian. The linear connection* D *defined by* $D_X = \nabla_X + \zeta_X$, *where*

$$2\zeta_X = X \lrcorner Jd^{1,2}\omega - \frac{1}{3}X \lrcorner Jd^3\omega + \frac{1}{2}\hat{N}_X^J,$$

is Hermitian.

Proof. That D is metric is clear since ζ_X is a two form for all X in TM. To see that $D\omega = 0$ it is enough to show that $\nabla_X\omega + [\zeta_X, \omega] = 0$ or equivalently

$$\nabla_X\omega + \mathcal{J}\zeta_X = 0$$

for all X in TM. But

$$2 \mathcal{J} \zeta_X = \mathcal{J}(X \lrcorner J d^{1,2} \omega) - \frac{1}{3} \mathcal{J}(X \lrcorner J d^3 \omega) + \frac{1}{2} \mathcal{J} \hat{N}_X^J$$

$$= -X \lrcorner d^{1,2} \omega - JX \lrcorner J d^{1,2} \omega - \frac{2}{3} X \lrcorner d^3 \omega + \hat{N}_{JX}^J$$

after taking into account that $\hat{N}^J$ belongs to $\lambda^1 \otimes_2 \lambda^2$. Therefore, after taking into account (2.9),

$$2 \nabla_X \omega + 2 \mathcal{J} \zeta_X = -\left[\hat{N}_{JX}^J + \frac{4}{3} JX \lrcorner J d^3 \omega \right] + X \lrcorner d\omega + JX \lrcorner J d\omega$$

$$- X \lrcorner d^{1,2} \omega - JX \lrcorner J d^{1,2} \omega - \frac{2}{3} X \lrcorner d^3 \omega + \hat{N}_{JX}^J = 0$$

after a straightforward computation, and the result follows. $\qquad\square$

It follows from Proposition 2.2 and Lemma 2.4 that the intrinsic torsion tensor of the almost Hermitian structure (g, J) is completely determined by

$$(d\omega, \hat{N}^J) \text{ in } \Lambda^3 \oplus W_2.$$

Since $\Lambda^3 \oplus W_2$ has four irreducible components under the action of $\mathrm{U}(m)$ the Gray–Hervella classification [34] singles out 16-classes of almost Hermitian manifolds. Similar classification results are available for the groups G_2 [22], [16] and $\mathrm{Spin}(7)$ [21] as well as for quaternionic structures [41] and $\mathrm{SU}(3)$-structures on 6-dimensional manifolds [14]. The case of $\mathrm{Spin}(9)$-structures on 16-dimensional manifolds has been equally treated in [24]. For $\mathrm{PSU}(3)$-structures on 8-dimensional manifolds and $\mathrm{SO}(3)$-structures in dimension 5 the decomposition of the intrinsic torsion tensor has been studied in [37], [58] and [8].

2.3 Admissible totally skew-symmetric torsion

In this section we shall start to specialise our discussion to a particular class of almost Hermitian manifolds, to be characterized in terms of the torsion tensor of the Hermitian connection D. We recall that the torsion tensor of a linear connexion on M, e.g. D, is the tensor T^D in $\Lambda^2 \otimes \Lambda^1$ given by

$$T^D(X, Y) = D_X Y - D_Y X - [X, Y]$$

for all vector fields X, Y on M. In the case of the connection $\overline{\nabla}$, the torsion will be denoted simply by T.

The following result of Friedrich and Ivanov clarifies in which circumstances an almost Hermitian structure admits a Hermitian connection with totally skew-symmetric torsion.

Theorem 2.2 ([26]). *Let (M^{2m}, g, J) be almost Hermitian. There exists a Hermitian connection with torsion in Λ^3 if and only if N^J is a 3-form. If the latter holds, the connection is unique and equals D.*

Almost Hermitian manifolds with totally skew-symmetric Nijenhuis tensor form the so-called Gray–Hervella class $W_1 + W_3 + W_4$ and are usually called $\mathcal{G}_1$ almost Hermitian structures. As it follows from the discussion above an almost Hermitian structure (g, J) belongs to the class $\mathcal{G}_1$ if and only if $\hat{N}^J = 0$. Because of its uniqueness, the connection D in Theorem 2.2 will be referred to as the characteristic connection of the $\mathcal{G}_1$-manifold (M^{2m}, g, J).

In the remaining part of this section we will work on a given $\mathcal{G}_1$-manifold (M^{2m}, g, J) and we will derive a number of facts to be used later on. We have

$$N^J = \frac{4}{3} J d^3 \omega,$$

$$2\zeta = T^D = J d^{1,2} \omega - \frac{1}{3} J d^3 \omega.$$

It is also worth noting that (2.9) is then updated to

$$2\nabla_X \omega = X \lrcorner d^{1,2} \omega + JX \lrcorner J d^{1,2} \omega + \frac{2}{3} X \lrcorner d^3 \omega$$

for all X in TM. Under a shorter and perhaps more tractable form this reads

$$\nabla_X \omega = X \lrcorner t + JX \lrcorner Jt + X \lrcorner \psi^+ \tag{2.10}$$

for all X in TM, where the 3-forms t in $\lambda^{1,2}$ and ψ^+ in λ^3 are given by

$$t = \frac{1}{2} d^{1,2} \omega, \quad \psi^+ = \frac{1}{3} d^3 \omega.$$

In the same spirit we can also re-express $d\omega$ and the torsion form as

$$d\omega = 2t + 3\psi^+ \tag{2.11}$$

and

$$T^D = 2Jt + \psi^-, \tag{2.12}$$

where the 3-form ψ^- in λ^3 is given by $\psi^- = \psi^+(J\cdot, \cdot, \cdot)$.

Remark 2.3. (i) When $m = 2$ any almost Hermitian structure of class $\mathcal{G}_1$ is automatically Hermitian, that is $N_J = 0$. This is due to the vanishing of λ^3 in dimension 4. In what follows we shall therefore assume that $m \geq 3$.

(ii) From (2.10) it is easy to see that almost Hermitian structures in the class $\mathcal{G}_1$ are alternatively described as those satisfying $(\nabla_{JX} J)JX = -(\nabla_X J)X$ for all X in TM.

When J is integrable it is easy to see that $T^D = 2Jd\omega$ belongs to Λ^3 (actually $\lambda^{1,2}$ after taking into account Lemma 2.4). In this case the connection D is referred to as the Bismut connection [7].

Almost quaternionic or almost hyperhermitian structures admitting a structure preserving connection with totally skew-symmetric torsion have been introduced and given various characterisations in [38], [29], [39], [43]. The resulting geometries are known under the names QKT (quaternion-Kähler with torsion) and HKT (hyperkähler with torsion).

2.4 Hermitian Killing forms

In this section, briefly recalling the definition of Killing forms in Riemannian geometry we shall present a variation of that notion, more suitable to the almost Hermitian setting. This is done to prepare the ground to present, after proving the results in the next section, the special relationship binding together almost Hermitian structure of type $\mathcal{G}_1$ and this kind of generalised Killing form.

Let (M^{2m}, g, J) be almost Hermitian in the class $\mathcal{G}_1$. Using the metric connection D we can form the associated exterior differential d_D. It is given as in the case of the usual exterior derivative d by

$$d_D = \sum_{i=1}^{2m} e_i^\flat \wedge D_{e_i}.$$

Given that D is a Hermitian connexion it is easy to see that

$$[\mathcal{J}, d_D] = (-1)^p J d_D J \tag{2.13}$$

holds on Λ^p. Its dual reads

$$[\mathcal{J}, d_D^\star] = (-1)^p J d_D^\star J \tag{2.14}$$

on Λ^p, where $d_D^\star \colon \Lambda^\star \to \Lambda^\star$ is the formal adjoint of d_D. Note that $d_D^\star$ is computed at a point m of M by

$$d_D^\star = -\sum_{i=1}^{2m} e_i \lrcorner D_{e_i},$$

where $\{e_i, 1 \le i \le 2m\}$ is a local frame around m, geodesic at m w.r.t. the connection D. A straightforward implication of (2.13) is that

$$d_D \varphi \in \lambda^{p+1,q} \oplus \lambda^{p,q+1}$$

for all φ in $\lambda^{p,q}$. Denoting by $\varphi_{\lambda^{p,q}}$ the orthogonal projection of φ in $\Lambda^\star$ on $\lambda^{p,q}$ we split

$$d_D = \partial_D + \bar{\partial}_D,$$

where $\partial_D \varphi = (d_D \varphi)_{\lambda^{p+1,q}}$ and $\bar{\partial}_D \varphi = (d_D \varphi)_{\lambda^{p,q+1}}$ for all φ in $\lambda^{p,q}$.

Lastly, we mention that the Kähler identities

$$[d_D, L^\star] = (-1)^p J d_D^\star J \quad \text{and} \quad [d_D^\star, L] = (-1)^{p+1} J d_D J \tag{2.15}$$

hold whenever φ belongs to Λ^p. We can now make the following

Definition 2.1. A form φ in $\lambda^{p,q}$ is a Hermitian Killing form if and only if

$$D_X \varphi = (X \lrcorner A)_{\lambda^{p,q}}$$

for some form A in $\lambda^{p,q+1} \oplus \lambda^{p+1,q}$ and for all X in TM.

This is much in analogy with the concept of twistor form from Riemannian geometry, see [53] for details. We shall just recall that that a differential form φ in $\Lambda^p(N)$, where (N^n, h) is some Riemannian manifold, is called a twistor form or conformal Killing form if

$$\nabla_X \varphi = \frac{1}{p+1} X \lrcorner d\varphi + \frac{1}{n-p+1} X^\flat \wedge d^\star \varphi$$

holds, for all X in TM. Moreover φ is called a Killing form if it also coclosed, that is $d^\star \varphi = 0$.

Remark 2.4. Hermitian Killing 1-forms are dual to Killing vector fields for the metric g. This is essentially due to the fact that D has totally skew-symmetric torsion.

In the rest of this section we shall make a number of elementary observations on Hermitian Killing forms in $\lambda^\star$, the most relevant case for our aims.

Proposition 2.4. *Let φ in λ^p, $p \geq 2$ be a Hermitian Killing form. The following hold:*
  (i) $D_X \varphi = (X \lrcorner A)_{\lambda^p}$ *for all X in TM, where $A = \partial \varphi + \frac{1}{p+1} \bar{\partial} \varphi$;*
 (ii) *$\mathcal{J}\varphi$ is a Hermitian Killing form.*

Proof. (i) From the definition we have

$$D_X \varphi = (X \lrcorner A)_{\lambda^p}$$

for all X in TM, where A is in $\lambda^{p,1} \oplus \lambda^{p+1}$. We split $A = B + C$ where B and C belong to $\lambda^{p,1}$ and λ^{p+1} respectively. Since for all X in TM we have

$$(X \lrcorner B)_{\lambda^p} = \frac{1}{2(p-1)}(JX \lrcorner \mathcal{J}B + (p-1)X \lrcorner B),$$
$$(X \lrcorner C)_{\lambda^p} = X \lrcorner C,$$

we find, after taking the appropriate contractions that

$$d_D \varphi = B + (p+1)C$$

and the claim follows.

(ii) After applying $\mathcal{J}$ to the Hermitian Killing equation satisfied by φ, it is enough to notice that

$$\mathcal{J}(X \lrcorner B)_{\lambda^p} = \frac{p}{p-1}(X \lrcorner \mathcal{J}B)_{\lambda^p}$$

holds, together with

$$\mathcal{J}(X \lrcorner C) = \frac{p}{p+1} X \lrcorner \mathcal{J}C$$

for all X in TM.   □

Note that, unlike Riemannian Killing forms, Hermitian Killing ones need not be necessarily coclosed. This can be easily verified by means of (2.14) and (2.15). The following identity is useful in order to better understand the notion of Hermitian Killing form.

Lemma 2.5. *Let φ in λ^p, $p \geq 1$ be given. Then:*

$$D_{JX}\varphi - D_X \mathbb{J}\varphi = -2(X \lrcorner \partial_D \mathbb{J}\varphi)_{\lambda^p}$$

for all X in TM.

Proof. Let us consider the tensor γ in $\lambda^1 \otimes_1 \lambda^p$ given by

$$\gamma_X = D_{JX}\varphi - D_X \mathbb{J}\varphi$$

for all X in TM. For notational convenience we define $B = \partial_D \mathbb{J}\varphi$ in $\lambda^{1,p}$ and note, as in the proof of Proposition 2.4 that $X \in TM \mapsto (X \lrcorner B)_{\lambda^p}$ belongs to $\lambda^1 \otimes_1 \lambda^p$ as well. Now a straightforward calculation shows that

$$a(X \in TM \mapsto (X \lrcorner B)_{\lambda^p}) = B.$$

At the same time one has

$$a(\gamma) = -2\partial_D \mathbb{J}\varphi$$

hence $\gamma_X = -2(X \lrcorner \partial_D \mathbb{J}\varphi)_{\lambda^p}$ for all X in TM by making use of Lemma 2.1 (iii) when $p \neq 1$. When $p = 1$ the claim follows by a simple direct verification which is left to the reader.   □

Let us now give an equivalent characterisation of Hermitian Killing forms.

Proposition 2.5. *The following hold:*

(i) *a form φ in λ^p, $p \geq 2$ is a Hermitian Killing form if and only if the component of $D\varphi$ on $\lambda^1 \otimes_2 \lambda^p$ is determined by $\bar{\partial}_D \mathbb{J}\varphi$, that is*

$$D_{JX}\varphi + D_X \mathbb{J}\varphi = \frac{2}{p+1} X \lrcorner \bar{\partial}_D \mathbb{J}\varphi$$

   for all X in TM;

(ii) *a form φ in λ^m is a Hermitian Killing form if and only if*

$$|\varphi| D\varphi = \frac{1}{2} d\,|\varphi|^2 \otimes \varphi + \frac{1}{2} Jd\,|\varphi|^2 \otimes \mathbb{J}\varphi.$$

Proof. (i) We have

$$D_X \mathbb{J}\varphi = \frac{1}{2}(D_{JX}\varphi + D_X\mathbb{J}\varphi) - \frac{1}{2}(D_{JX}\varphi - D_X\mathbb{J}\varphi)$$

$$= (X \lrcorner \partial_D\varphi)_{\lambda^3} + \frac{1}{2}(D_{JX}\varphi + D_X\mathbb{J}\varphi)$$

for all X in TM, and the claim follows immediately from (i) in Proposition 2.4.

(ii) If φ is in λ^m, we have that $\bar{\partial}_D(\mathbb{J}\varphi) = 0$ since $\lambda^{m+1} = 0$. From (i) we know that φ is a Hermitian Killing form if and only if

$$D_{JX}\varphi + D_X\mathbb{J}\varphi = 0 \qquad\qquad (2.16)$$

holds for any X in TM. Let U be the open subset of M given by $U = \{m \in M : \varphi_m \neq 0\}$ and let Z be the complement of U in M. We claim that (2.16) holds iff it holds on U. Indeed (2.16) holds trivially on $\mathrm{int}(Z)$ and $U \cup \mathrm{int}(Z)$ is dense in M.

Suppose now that φ in λ^m is a Hermitian Killing form. Then $\varphi, \mathbb{J}\varphi$ is a basis of $\lambda^m_{|U}$, hence

$$D\varphi = a \otimes \varphi + b \otimes \mathbb{J}\varphi$$

for a couple of 1-forms a, b on U. The first is determined by $|\varphi|a = \frac{1}{2}d|\varphi|^2$. On the other hand the fact that φ is a Hermitian form implies that $b = Ja$, hence our claim is proved on U and by the density argument above on M. The converse statement is also clear from the previous observations. $\qquad\square$

An immediate consequence of (ii) in the proposition above is that a Hermitian Killing form in λ^m is *parallel* with respect to the connection D as soon as it has constant length. It is now a good moment to provide some examples of Hermitian Killing forms.

Proposition 2.6. *Let (M^{2m}, g, J) be a Kähler manifold and let ζ_k, $k = 1, 2$ be holomorphic Killing vector fields, that is*

$$L_{\zeta_k}g = 0 \quad and \quad L_{\zeta_k}J = 0 \quad for \ k = 1, 2.$$

The form $\varphi = (\zeta_1^\flat \wedge \zeta_2^\flat)_{\lambda^2}$ is a Hermitian Killing form.

Proof. The Killing equation yields

$$\nabla_X \zeta_k^\flat = \frac{1}{2}X \lrcorner d\zeta_k^\flat$$

for all X in TM and $k = 1, 2$. After a few manipulations we get

$$\nabla_X(\zeta_1^\flat \wedge \zeta_2^\flat) = \frac{1}{2}(X \lrcorner d\zeta_1^\flat) \wedge \zeta_2^\flat + \frac{1}{2}\zeta_1^\flat \wedge (X \lrcorner d\zeta_2^\flat)$$

$$= \frac{1}{2}X \lrcorner d(\zeta_1^\flat \wedge \zeta_2^\flat) - \frac{1}{2}\langle\zeta_2, X\rangle d\zeta_1^\flat + \frac{1}{2}\langle\zeta_1, X\rangle d\zeta_2^\flat$$

for all X in TM. Since our given vector fields are also holomorphic, the forms $d\zeta_k^\flat$, $k = 1, 2$ belong to $\lambda^{1,1}$ and the claim is proved by projecting onto λ^2 while using the Definition 2.1. We note that $\bar{\partial}\varphi = 0$ because one has, as is well known, $d(J\zeta_k^\flat) = 0$ for $k = 1, 2$. $\qquad\square$

We continue by giving an exterior algebra characterisation of Hermitian Killing forms, in the context of a $\mathscr{G}_1$-manifold (M^{2m}, g, J).

Proposition 2.7. *Let $p \geq 3$ be odd. Then:*

(i) *if φ in λ^p is a Hermitian Killing form, then*

$$d_D^\star(\varphi \wedge \mathbb{J}\varphi) = (d_D^\star\varphi) \wedge \mathbb{J}\varphi - \varphi \wedge d_D^\star\mathbb{J}\varphi - \frac{2}{p+1}\varphi \bullet \bar{\partial}_D\mathbb{J}\varphi;$$

(ii) *if φ in λ^p satisfies*

$$d_D^\star(\varphi \wedge \mathbb{J}\varphi) = (d_D^\star\varphi) \wedge \mathbb{J}\varphi - \varphi \wedge d_D^\star\mathbb{J}\varphi - \frac{2}{p+1}\varphi \bullet \bar{\partial}_D\mathbb{J}\varphi$$

and it is everywhere nondegenerate, then φ is a Hermitian Killing form.

Proof. We have

$$e_i \lrcorner D_{e_i}(\varphi \wedge \mathbb{J}\varphi) = e_i \lrcorner (D_{e_i}\varphi \wedge \mathbb{J}\varphi + \varphi \wedge D_{e_i}\mathbb{J}\varphi)$$
$$= (e_i \lrcorner D_{e_i}\varphi) \wedge \mathbb{J}\varphi - D_{e_i}\varphi \wedge (e_i \lrcorner \mathbb{J}\varphi)$$
$$+ (e_i \lrcorner \varphi) \wedge D_{e_i}\mathbb{J}\varphi - \varphi \wedge (e_i \lrcorner D_{e_i}\mathbb{J}\varphi).$$

After summation, we get

$$-d_D^\star(\varphi \wedge \mathbb{J}\varphi) = -(d_D^\star\varphi) \wedge \mathbb{J}\varphi + \varphi \wedge d_D^\star\mathbb{J}\varphi$$
$$- \sum_{k=1}^{2m} D_{e_i}\varphi \wedge (e_i \lrcorner \mathbb{J}\varphi) + \sum_{k=1}^{2m} (e_i \lrcorner \varphi) \wedge D_{e_i}\mathbb{J}\varphi.$$

But

$$\sum_{k=1}^{2m} D_{e_i}\varphi \wedge (e_i \lrcorner \mathbb{J}\varphi) = \sum_{k=1}^{2m} D_{e_i}\varphi \wedge (Je_i \lrcorner \varphi)$$
$$= -\sum_{k=1}^{2m} D_{Je_i}\varphi \wedge (e_i \lrcorner \varphi) = -\sum_{k=1}^{2m} (e_i \lrcorner \varphi) \wedge D_{Je_i}\varphi$$

whence

$$-d_D^\star(\varphi \wedge \mathbb{J}\varphi) = -(d_D^\star\varphi) \wedge \mathbb{J}\varphi + \varphi \wedge d_D^\star\mathbb{J}\varphi$$
$$+ \sum_{k=1}^{2m} e_i \lrcorner \varphi \wedge (D_{Je_i}\varphi + D_{e_i}\mathbb{J}\varphi). \qquad (2.17)$$

(i) If φ is a Hermitian Killing form one uses (2.17) and (i) in Proposition 2.5 to obtain the conclusion.

(ii) In this case, using (2.17) again we have that

$$\sum_{i=1}^{2m}(e_i \lrcorner \varphi) \wedge \gamma_{e_i} = 0,$$

where γ in $\lambda^1 \otimes_2 \lambda^p$ is given by

$$\gamma_X = D_{JX}\varphi + D_X \mathbb{J}\varphi - \frac{2}{p+1}X \lrcorner \bar{\partial}_D \mathbb{J}\varphi$$

for all X in TM. We define now $\hat{\gamma}$ in $\lambda^{p-1} \otimes \lambda^p$ by

$$\hat{\gamma}(X_1, \ldots, X_{p-1}) = \gamma_{X_1 \lrcorner \cdots \lrcorner X_{p-1} \lrcorner \varphi}$$

whenever $X_k, 1 \le k \le p-1$ belong to TM. It is easy to verify that $\hat{\gamma}$ is in $\lambda^{p-1}\otimes_1\lambda^p$, and since $a(\hat{\gamma}) = 0$ Lemma 2.1 (iii) yields $\hat{\gamma} = 0$. Because φ is everywhere non-degenerate we obtain that γ vanishes and we conclude by Proposition 2.5 (i).   □

To end this section let us present another way of characterizing Hermitian Killing forms, this time under the form of a product rule with respect to the exterior differential d_D.

Proposition 2.8. *Let $p \ge 3$ be odd. We have:*

(i) *any Hermitian Killing form φ in λ^p satisfies*

$$-d_D(\varphi \bullet \varphi) = d_D\varphi \bullet \varphi + d_D\mathbb{J}\varphi \bullet \mathbb{J}\varphi - \frac{2}{p+1}J(\varphi \bullet \bar{\partial}_D \mathbb{J}\varphi);$$

(ii) *if φ in λ^p satisfies*

$$-d_D(\varphi \bullet \varphi) = d_D\varphi \bullet \varphi + d_D\mathbb{J}\varphi \bullet \mathbb{J}\varphi - \frac{2}{p+1}J(\varphi \bullet \bar{\partial}_D \mathbb{J}\varphi)$$

and it is nowhere degenerate, then φ is a Hermitian Killing form.

Proof. We will mainly use Proposition 2.7 and the Kähler identities. Indeed, let us observe that

$$L^\star(\varphi \wedge \mathbb{J}\varphi) = -\varphi \bullet \varphi.$$

Using (2.15) it follows that

$$d_D L^\star(\varphi \wedge \mathbb{J}\varphi) = L^\star d_D(\varphi \wedge \mathbb{J}\varphi) + (Jd_D^\star J)(\varphi \wedge \mathbb{J}\varphi)$$

hence

$$-d_D(\varphi \bullet \varphi) = L^\star d_D(\varphi \wedge \mathbb{J}\varphi) + Jd_D^\star(\varphi \wedge \mathbb{J}\varphi).$$

It is easy to see that

$$L^\star(d_D\varphi \wedge \mathbb{J}\varphi) = (L^\star d_D\varphi) \wedge \mathbb{J}\varphi + d_D\varphi \bullet \varphi,$$
$$-L^\star(\varphi \wedge d_D\mathbb{J}\varphi) = -(L^\star d_D\mathbb{J}\varphi) \wedge \varphi + d_D\mathbb{J}\varphi \bullet \mathbb{J}\varphi,$$

hence we get further

$$-d_D(\varphi \bullet \varphi) = (L^\star d_D \varphi) \wedge \mathbb{J}\varphi + d_D\varphi \bullet \varphi$$
$$- (L^\star d_D \mathbb{J}\varphi) \wedge \varphi + d_D \mathbb{J}\varphi \bullet \mathbb{J}\varphi + Jd_D^\star(\varphi \wedge \mathbb{J}\varphi).$$

Suppose now that φ is a Hermitian Killing form. Then, by using (i) in Proposition 2.7 we get

$$d_D^\star(\varphi \wedge \mathbb{J}\varphi) = d_D^\star\varphi \wedge \mathbb{J}\varphi - \varphi \wedge d_D^\star(\mathbb{J}\varphi) - \frac{2}{p+1}\varphi \bullet \bar{\partial}_D(\mathbb{J}\varphi)$$

while the second part of the Kähler identities (2.15) provides us with

$$-L^\star d_D(\mathbb{J}\varphi) = d_D^\star\varphi,$$
$$L^\star d_D\varphi = d_D^\star \mathbb{J}\varphi.$$

The claim in (i) follows now by a simple computation. The converse statement in (ii) is proved by using methods similar to those employed for (ii) in Proposition 2.7, and it is therefore left to the reader.   □

3 The torsion within the $\mathscr{G}_1$ class

3.1 $\mathscr{G}_1$-structures

In the following (M^{2m}, g, J) will be almost-Hermitian, in the class $\mathscr{G}_1$. The connection D acts on any form φ in $\Lambda^\star$ according to

$$D_X\varphi = \nabla_X\varphi + \frac{1}{2}[X \lrcorner T^D, \varphi]$$

for all X in TM. Based on this fact it is straightforward to check that the differential d_D is related to d by

$$d_D\varphi = d\varphi - \sum_{k=1}^{2m}(e_k \lrcorner T^D) \wedge (e_k \lrcorner \varphi)$$

for all φ in $\Lambda^\star$. This yields, in the particular case of 3-forms the following comparison formula

$$d_D\varphi = d\varphi - T^D \bullet \varphi \qquad (3.1)$$

for all φ in Λ^3.

Let now R be the curvature tensor of the metric g with the convention that $R(X,Y) = -\nabla^2_{X,Y} + \nabla^2_{Y,X}$ for all X, Y in TM. In the same time we consider the curvature tensor R^D of the connection D where the same convention applies. We shall now recall that the first Bianchi identity for the connection D takes the following form.

Lemma 3.1. *For all X in TM,*

$$b_1(R^D)_X = D_X T^D + \frac{1}{2} X \lrcorner\, dT^D.$$

Proof. It is easy to see, directly from the definition, that the curvature tensors of the Levi-Civita connection and of the connection D are related by

$$R^D(X,Y) = R(X,Y) - \frac{1}{2}\big[Y \lrcorner\, D_X T^D - X \lrcorner\, D_Y T^D\big] + \frac{1}{4}\varepsilon^{T^D}(X,Y) \quad (3.2)$$

for all X, Y in TM. Here we have set

$$\varepsilon^{T^D}(X,Y) = [T_X^D, T_Y^D] - 2T_{T_X^D Y}^D.$$

Since R satisfies the algebraic Bianchi identity and

$$b_1(\varepsilon^{T^D})_X = 2X \lrcorner\, (T^D \bullet T^D),$$

the assertion follows eventually by setting $X = e_k$, taking the wedge product with e_k in (3.2) and summing over $1 \le k \le 2m$. Note that in the process one also uses the comparison formula (3.1). $\qquad\square$

Let us gather now some information on the differentials of the components of the torsion form T^D.

Proposition 3.1. *Let (M^{2m}, g, J) belong to the class $\mathcal{G}_1$. The following hold:*

(i) $\partial_D t = 0$;

(ii) $3\partial_D \psi^+ + 2\bar{\partial}_D t = -4(t \bullet Jt) - 4(Jt \bullet \psi^+)_{\lambda^{1,3}}$;

(iii) $\bar{\partial}_D \psi^+ = -\frac{8}{3}(Jt \bullet \psi^+)_{\lambda^4}$.

Proof. Since $d\omega = 2t + 3\psi^+$ is closed it follows that

$$2dt + 3d\psi^+ = 0.$$

Rewritten by means of the connection D this identity yields, when also using the comparison formula (3.1),

$$2(d_D t + T^D \bullet t) + 3(d_D \psi^+ + T^D \bullet \psi^+) = 0.$$

We now take into account that $T^D = 2Jt + \psi^-$ to arrive, after expansion of the product and use of Lemma 2.2 (v) at

$$2d_D t + 4Jt \bullet t + 2\psi^- \bullet t + 3d_D \psi^+ + 6Jt \bullet \psi^+ = 0.$$

An elementary computation yields $J(\psi^- \bullet t) = Jt \bullet \psi^+$, in particular

$$(\psi^- \bullet t)_{\lambda^4} = (Jt \bullet \psi^+)_{\lambda^4},$$
$$(\psi^- \bullet t)_{\lambda^{1,3}} = -(Jt \bullet \psi^+)_{\lambda^{1,3}}.$$

It suffices thus to use Lemma 2.2 to obtain the proof of the claims after identifying the various components in the equation above along the bidegree decomposition of Λ^4.

□

We are now ready to prove the main result of this section.

Theorem 3.1. *Let (M^{2m}, g, J) be almost Hermitian, of type $\mathcal{G}_1$. Then ψ^- is a Hermitian Killing form, that is*

$$D_X \psi^- = \left[X \lrcorner (\partial_D \psi^- + \frac{1}{4} \bar{\partial}_D \psi^-) \right]_{\lambda^3}$$

for all X in TM.

Proof. Since D is a Hermitian connection we have $[R^D(X, Y), \omega] = 0$, in other words

$$\sum_{i=1}^{2m} R^D(X, Y)e_i \wedge (Je_i)^\flat = 0$$

for all X, Y in TM. Setting $X = e_k$ and taking the exterior product with e_k we find

$$\sum_{i,k=1}^{2m} e_k^\flat \wedge R^D(e_k, Y)e_i \wedge (Je_i)^\flat = 0.$$

Obviously, this is equivalent with

$$\sum_{i,k=1}^{2m} e_i \lrcorner (e_k^\flat \wedge R^D(e_k, Y)) \wedge (Je_i)^\flat = \sum_{i=1}^{2m} R^D(e_i, Y) \wedge (Je_i)^\flat$$

or further, by using Lemma 3.1,

$$-\mathcal{J}(D_X T^D + \frac{1}{2} X \lrcorner d_D T^D) = \sum_{i=1}^{2m} R^D(e_i, Y) \wedge (Je_i)^\flat$$

for all Y in TM. Since D is Hermitian, we have that $R^D(X, Y)$ is in $\lambda^{1,1}$, hence the right-hand side above is in $\lambda^{1,2}$ given that $\Lambda^1 \wedge \lambda^{1,1} \subseteq \lambda^{1,2}$. Thus

$$\left(D_X T^D + \frac{1}{2} X \lrcorner d T^D \right)_{\lambda^3} = 0,$$

and it follows that

$$D_X \psi^- = -\frac{1}{2}(X \lrcorner d T^D)_{\lambda^3} \tag{3.3}$$

for all X in TM. Therefore, ψ^- is a Hermitian Killing form and the claim follows by using Proposition 2.4.

□

We have the following immediate consequence of the above result.

Theorem 3.2. *Let* (M^{2m}, g, J) *have totally skew-symmetric Nijenhuis tensor. Then*

$$DN_J = 0 \quad \text{if and only if} \quad dT^D \text{ belongs to } \lambda^{2,2}.$$

In particular N_J *is parallel if* T^D *is a closed 3-form.*

Proof. This is a direct consequence of (3.3) and of the properties of the projection on λ^3. $\qquad\square$

Remark 3.1. It also follows from (3.3) when combined with Theorem 3.1 that dT^D belongs to $\lambda^{2,2}$ if and only if $d_D\psi^+ = 0$.

In dimension 6, the fact that ψ^+ is a Hermitian Killing form is described by (ii) in Proposition 2.5. As observed in [13], it amounts then to the local parallelism of ψ^+ w.r.t. the characteristic connection, after performing a conformal transformation in order to normalise the length of ψ^+ to a constant. In dimension 6 structures of type $\mathcal{G}_1$ with $DT^D = 0$ have been classified in [4]. Classification results are also available [2] under the same assumption in the Hermitian case, provided that the holonomy of D is contained in $S^1 \times U(m-1)$.

3.2 The class $W_1 + W_4$

$W_1 + W_4$ structure In this section we shall record some of the additional features of the geometry of almost Hermitian manifolds in the Gray–Hervella class $W_1 + W_4$. It can be described as the subclass of $\mathcal{G}_1$ having the property that t has no component on Λ_0^3, or alternatively

$$t = \theta \wedge \omega. \tag{3.4}$$

Here θ is a 1-form on M, called the Lee form of the almost-Hermitian structure (g, J). It can be recovered directly from the Kähler form of (g, J) by

$$(m-1)J\theta = d^\star\omega,$$

in particular we have that $d^\star(J\theta) = 0$.

Proposition 3.2. *Let* $(M^{2m}, g, J), m \geq 3$ *be almost-Hermitian in the class* $W_1 + W_4$. *The following hold:*

(i) $\partial_D\theta = 0$;

(ii) $-\partial_D\psi^+ = 2(J\theta \wedge \psi^- + \theta \wedge \psi^+) + \frac{2}{3}(\bar{\partial}_D\theta - 2\zeta \lrcorner \psi^-) \wedge \omega$;

(iii) $\frac{1}{4}\bar{\partial}_D\psi^+ = \theta \wedge \psi^+ - J\theta \wedge \psi^-$.

Proof. All claims follows from Proposition 3.1, applied to our present situation, that is $t = \theta \wedge \omega$. Therefore, (i) is immediate from (i) in the previously cited proposition.

Through direct computation we find

$$(J\theta \wedge \omega) \bullet \psi^+ = -(\zeta \lrcorner \psi^-) \wedge \omega + 3J\theta \wedge \psi^-$$
$$(J\theta \wedge \omega) \bullet (\theta \wedge \omega) = 0.$$

Our last two assertions follow now from Proposition (3.1), after projection of the formulae above onto $\Lambda^4 = \lambda^{2,2} \oplus \lambda^{1,3} \oplus \lambda^4$. $\qquad\square$

In dimension 6 it has been shown in [13], [16] that an almost Hermitian manifold in the class $W_1 + W_4$ has closed Lee form, i.e., $d\theta = 0$. Theorem 3.1 combined with Proposition 3.2 seems a good starting point to investigate up to what extent structures of $W_1 + W_4$ are closed in arbitrary dimensions but we shall not pursue this direction here.

3.3 The $\mathfrak{u}(m)$-decomposition of curvature

In this section we shall investigate, for further use, the splitting of the curvature tensor of the Hermitian connection D. Our main goal is to identify explicitly the "non-Kähler" part of the tensor R^D and to give it an explicit expression in terms of the torsion form T^D of the characteristic connection. Note that for almost quaternionic-Hermitian or G_2-structures (not necessarily with skew-symmetric torsion) similar results have been obtained in [42], [15]. We will mainly use that

$$R^D(X, Y, Z, U) - R^D(Z, U, X, Y)$$
$$= -\frac{1}{2}\big[(D_X T^D)_Y(Z, U) - (D_Y T^D)_X(Z, U)\big] \qquad (3.5)$$
$$+ \frac{1}{2}\big[(D_Z T^D)_U(X, Y) - (D_U T^D)_Z(X, Y)\big]$$

holds for all X, Y, Z, U in TM, as it easily follows from the difference formula (3.2), after verifying that ε^{T^D} belongs to $S^2(\Lambda^2)$.

Theorem 3.3. *Let (M^{2m}, g, J) be almost Hermitian of type $\mathscr{G}_1$. We have*

$$R^D = R^K + \widehat{\Omega} + \frac{1}{2}R^a + R^m,$$

where R^K belongs to $\mathcal{K}(\mathfrak{u}(m))$. Moreover:

(i) *R^a belongs to $\Lambda^2(\lambda^{1,1})$ and satisfies the Bianchi identity*

$$\frac{1}{2}(b_1 R^a)_X = D_X(Jt) - D_{JX}t - \frac{1}{2}X \lrcorner \partial_D(Jt)$$

for all X in TM.

(ii) *The Bianchi identity for R^m in $\lambda^2 \otimes \lambda^{1,1}$ is*

$$(b_1 R^m)_X = D_X(Jt) + D_{JX}t + \frac{1}{4}(JX \lrcorner \mathcal{J}\partial_D\psi^- - 2X \lrcorner \partial_D\psi^-)$$

for all X in TM.

(iii) Ω *in* $\lambda^{2,2}$ *is given by*

$$\Omega = \frac{3}{2}\partial_D(Jt) + 2(Jt \bullet Jt)_{\lambda^{2,2}} + \frac{1}{2}\psi^- \bullet \psi^-.$$

Proof. This is a direct application of Theorem 2.1 which can be used, as it follows from (3.5), for the tensor $\mathcal{R} = R^D$ and

$$\gamma = -\frac{1}{2}DT^D = -D(Jt) - \frac{1}{2}D\psi^-.$$

Then we have $\gamma^{1,2} = -D(Jt)$ and $\gamma^3 = -\frac{1}{2}D\psi^-$, and in order to prove our claims we only need to determine the various antisymmetrisations of $\gamma^{1,2}$. Also note that in this context

$$T = -\frac{1}{2}d_D T^D = -d_D(Jt) - \frac{1}{2}d_D\psi^-,$$

in particular $T^{2,2} = -\partial_D(Jt)$.

   (i) Now, it is easy to see that

$$a(\gamma^{1,2}) = -d_D(Jt),$$
$$a(J\gamma^{1,2}) = d_Dt,$$
$$a^c(\gamma^{1,2}) = -Jd_Dt,$$
$$a^c(J\gamma^{1,2}) = -Jd_DJt.$$

Hence,

$$A_1 = -d_D(Jt) - Jd_D(Jt) - 4T^{2,2} = -d_D(Jt) - Jd_D(Jt) + 4\partial_D(Jt)$$
$$= -2\partial_D(Jt) + 4\partial_D(Jt) = 2\partial_D(Jt)$$

and

$$A_2 = -Jd_Dt - d_Dt = -2\partial_Dt = 0$$

by (i) of Proposition 3.1.

   (ii) Since $T^{1,3} = -\bar{\partial}_D(Jt) - \frac{1}{2}\partial_D\psi^-$ we have

$$A_3 = -d_D(Jt) + Jd_DJt + 2\bar{\partial}_D(Jt) + \partial_D\psi^- = \partial_D\psi^-$$

and

$$A_4 = d_Dt - Jd_Dt + \mathcal{J}\bar{\partial}_D(Jt) + \frac{1}{2}\mathcal{J}\partial_D\psi^- = \frac{1}{2}\mathcal{J}\partial_D\psi^-$$

after making use of (2.13). The claim in (ii) now follows.

   (iii) We apply the Bianchi operator to

$$R^D = R^K + \widehat{\Omega} + R^a + R^m$$

and find after making use of (i), (ii) and of (2.2) that

$$b_1(R^D)_X = X \lrcorner \Omega + 2D_X(Jt) - \frac{1}{2}X \lrcorner \partial_D(Jt) + \frac{1}{4}(JX \lrcorner \mathcal{J}\partial_D\psi^- - 2X \lrcorner \partial_D\psi^-)$$

for all X in TM. Plugging into this the Bianchi identity for D in Lemma 3.1, combined with the fact that ψ^- is a Hermitian Killing form as asserted in Theorem 3.1 yields further

$$\frac{1}{2}(X \lrcorner\, dT^D)_{\lambda^{1,2}} = X \lrcorner \left[\Omega - \frac{1}{2}\partial_D(Jt)\right] + \frac{1}{4}(JX \lrcorner\, \mathcal{J}\partial_D\psi^- - 2X \lrcorner\, \partial_D\psi^-)$$

$$= X \lrcorner \left[\Omega - \frac{1}{2}\partial_D(Jt)\right] - (X \lrcorner\, \partial_D\psi^+)_{\lambda^{1,2}}$$

for all X in TM. After identifying components on $\lambda^1 \otimes_1 \lambda^{1,2}$ and $\lambda^1 \otimes_2 \lambda^{1,2}$ respectively (see also Remark 2.1 for the definition of these spaces) we find that

$$\frac{1}{2}(dT^D)_{\lambda^{2,2}} = \Omega - \frac{1}{2}\partial_D(Jt).$$

Now $dT^D = d_D T^D + T^D \bullet T^D$ by the comparison formula (3.1) thus by projection on $\lambda^{2,2}$ we find

$$(dT^D)_{\lambda^{2,2}} = 2\partial_D(Jt) + (T^D \bullet T^D)_{\lambda^{2,2}}$$

$$= 2\partial_D(Jt) + 4(Jt \bullet Jt)_{\lambda^{2,2}} + \psi^- \bullet \psi^-$$

after expansion of the product and use of Lemma 2.2 (iii). The claim in (iii) follows now immediately.   □

Remark 3.2. The curvature decomposition in Theorem 3.3 can be still refined, given that the U(m)-modules

$$\mathcal{K}(\mathfrak{u}(m)), \quad \lambda^{2,2}, \quad \Lambda^2(\lambda^{1,1}) \quad \text{and} \quad \lambda^2 \otimes \lambda^{1,1}$$

are not irreducible. Although this is an algebraically simple procedure, the computations at the level of the derivative DT^D of the torsion form become somewhat involved and will not be presented here. We just illustrate the situation in the simpler case of $W_1 + W_4$ below.

Theorem 3.3 has various applications, a class of which consists in giving torsion interpretation of curvature conditions imposed on the tensors R^D or R. As an example in this direction we have the following:

Proposition 3.3. *Let (M^{2m}, g, J) belong to the class $\mathcal{G}_1$. The curvature tensor R^D is Hermitian, that is*

$$R^D(JX, JY) = R^D(X, Y)$$

for all X, Y in TM, if and only if

$$D_{JX}t + D_X(Jt) = \frac{2}{3}(X \lrcorner\, \bar{\partial}_D(Jt))_{\lambda^{1,2}}$$

for all X in TM and

$$2\bar{\partial}_D(Jt) = 3\partial_D\psi^-.$$

Proof. By Theorem 3.3 the tensor R^D is Hermitian if and only if $R^m = 0$ which in turn happens if and only if $b_1 R^m = 0$. Therefore, by (ii) in Theorem 3.3 the curvature of D is Hermitian if and only if

$$D_X(Jt) + D_{JX}t + \frac{1}{4}(JX \lrcorner \mathscr{J}\partial_D \psi^- - 2X \lrcorner \partial_D \psi^-) = 0$$

for all X in TM. Taking the alternation above we find

$$2\bar{\partial}_D(Jt) - 3\partial_D \psi^- = 0.$$

We conclude by recalling that

$$(X \lrcorner A)_{\lambda^{1,2}} = \frac{1}{4}(2X \lrcorner A - JX \lrcorner \mathscr{J}A) \tag{3.6}$$

for all X in TM, where A belongs to $\lambda^{1,3}$. $\qquad\square$

Remark 3.3. In the case when the curvature tensor of the connection D is Hermitian the fact that $2\bar{\partial}_D(Jt) = 3\partial_D \psi^-$ combined with (ii) in Proposition 3.1 further yields

$$\bar{\partial}_D(Jt) = -(t \bullet Jt) - (Jt \bullet \psi^+)_{\lambda^{1,3}}.$$

For the subclass $W_1 \oplus W_4 \subseteq \mathscr{G}_1$ more information on the curvature tensor of the connection D is available and it will actually turn out that the components R^a and R^m have simple algebraic expressions. We define

$$S^{2,-}(M) = \{S \in S^2(M) : SJ + JS = 0\}.$$

This is embedded in $\lambda^2 \otimes \lambda^{1,1}$ via $S \mapsto \mathring{S}$ where

$$\mathring{S}(X,Y) = \frac{1}{2}((SJX)^\flat \wedge Y^\flat + X^\flat \wedge (SJY)^\flat + (SX)^\flat \wedge (JY)^\flat + (JX)^\flat \wedge (SY)^\flat)$$

for all X, Y in TM. One verifies that

$$(b_1\mathring{S})_X = (SX)^\flat \wedge \omega \tag{3.7}$$

for all X in TM. We also have an embedding $\lambda^{1,3} \hookrightarrow \lambda^2 \otimes \lambda^{1,1}$ given by $\Omega \mapsto \tilde{\Omega}$ where we define

$$\tilde{\Omega}(X,Y) = \frac{1}{4}(\Omega(JX, JY) - \Omega(X,Y))$$

for all X, Y in TM. Elementary considerations ensure that this is well defined and subject to

$$(b_1\tilde{\Omega})_X = (X \lrcorner \Omega)_{\lambda^{1,2}} \tag{3.8}$$

whenever X belongs to TM. As a last piece of notation let the symmetrized action of D on 1-forms be defined by

$$(\mathring{D}\alpha)(X,Y) = \frac{1}{2}((D_X\alpha)Y + (D_Y\alpha)X)$$

whenever α is in λ^1 and X, Y belong to TM.

Proposition 3.4. *Let (M^{2m}, g, J) belong to the class $W_1 + W_4$. If θ denotes the Lee form of (g, J) the following hold:*

(i) $\Omega = \left[\frac{3}{2}\partial_D(J\theta) + 2|\theta|^2\omega - 4\theta \wedge J\theta\right] \wedge \omega + \frac{1}{2}\psi^- \bullet \psi^-$;

(ii) $R^a = \partial_D(J\theta) \otimes \omega - \omega \otimes \partial_D(J\theta)$;

(iii) $R^m = -\bar{\partial}_D(J\theta) \otimes \omega + \mathring{S}_\theta - \widetilde{\partial_D\psi^-}$, *where S_θ in $S^{2,-}(M)$ is defined by*
$S_\theta = (1 - J)\mathring{D}(J\theta)$.

Proof. Recall that for the class $W_1 + W_4$ we have that $t = \theta \wedge \omega$. Taking this into account we will apply now Theorem 3.3.

(i) follows from (iii) in Theorem 3.3 when observing that

$$(Jt \bullet Jt)_{\lambda^{2,2}} = |\theta|^2\omega \wedge \omega - 2\omega \wedge \theta \wedge J\theta.$$

(ii) First of all we note that

$$D_X(J\theta) - D_{JX}\theta = X \lrcorner \partial_D(J\theta)$$

for all X in TM. The quickest way to see this is to observe that the tensor q in $\lambda^1 \otimes \lambda^1$ defined by $q(X) = D_X(J\theta) - D_{JX}\theta - X \lrcorner \partial_D(J\theta)$ for all X in TM is actually in $\lambda^1 \otimes_1 \lambda^1$ and satisfies $a(q) = a^c(q) = 0$, by using also that $\partial_D\theta = 0$ (cf. Proposition 3.2 (i)). Our claim now follows by observing that (iii) in Lemma 2.1 continues to hold when $p = q = 1$.

Using now (i) in Theorem 3.3 we get

$$(b_1 R^a)_X = (X \lrcorner \partial_D(J\theta)) \wedge \omega - \partial_D(J\theta) \wedge (X \lrcorner \omega)$$

for all X in TM.

Now for any φ in $\lambda^{1,1}$ we consider the element $\omega \otimes \varphi - \varphi \otimes \omega$ in $\Lambda^2(\lambda^{1,1})$, which is explicitly given by

$$(\omega \otimes \varphi - \varphi \otimes \omega)(X, Y) = \omega(X, Y)\varphi - \varphi(X, Y)\omega$$

for all X, Y in TM. A straightforward computation following the definitions yields

$$b_1(\partial_D(J\theta) \otimes \omega - \omega \otimes \partial_D(J\theta))_X = (X \lrcorner \partial_D(J\theta)) \wedge \omega - \partial_D(J\theta) \wedge (X \lrcorner \omega)$$

for all X in TM, and we conclude by using the injectivity of the Bianchi map $b_1 \colon \Lambda^2(\Lambda^2) \to \Lambda^1 \otimes \Lambda^3$.

(iii) By (iii) in Theorem 3.3 we have

$$(b_1 R^m)_X = \left[D_X(J\theta) + D_{JX}\theta\right] \wedge \omega - (X \lrcorner \partial_D\psi^-)_{\lambda^{1,2}}$$

after also making use of (3.6). Since

$$D_X(J\theta) + D_{JX}\theta = X \lrcorner \bar{\partial}_D(J\theta) + (S_\theta X)^\flat$$

it follows by means of (3.7), (3.8), that $R^m + \bar{\partial}_D(J\theta) \otimes \omega - \mathring{S}_\theta + \widetilde{\partial_D\psi^-}$ belongs to $\mathrm{Ker}(b_1) \cap (\lambda^2 \otimes \lambda^{1,1})$. It therefore vanishes and the proof is finished. $\square$

Finally, we remark that R^m can be given a more detailed expression by taking into account (ii) of Proposition 3.2.

4 Nearly-Kähler geometry

We shall begin here our survey of nearly-Kähler geometry. This is the class of almost Hermitian structures introduced by the following.

Definition 4.1. Let (M^{2m}, g, J) be an almost Hermitian manifold. It is called nearly-Kähler (NK for short) if and only if

$$(\nabla_X J)X = 0$$

whenever X belongs to TM.

This can be easily rephrased to say that an almost Hermitian structure (g, J) is nearly-Kähler if and only if its Kähler form is subject to

$$\nabla_X \omega = \frac{1}{3} X \lrcorner \, d\omega \tag{4.1}$$

for all X in TM. In other words, the Kähler form of any NK-structure is a Killing form and conversely U. Semmelmann [53] shows that any almost Hermitian structure with this property must be nearly-Kähler. Therefore, nearly-Kähler structures belong to the class W_1 in the sense that the 3-form t in Proposition 2.2 vanishes identically.

From now on we shall work on a given nearly-Kähler manifold (M^{2m}, g, J). For many of the properties of an NK-structure are best expressed by means of its first canonical Hermitian connection, it is important to note then the coincidence of the canonical connection and D, that is

$$D = \bar{\nabla}.$$

For this reason, the torsion tensor T^D will be denoted simply by T in what follows. It is given by

$$T^D = \psi^-$$

and hence belongs to λ^3. Therefore, in dimension 4 NK-structures are Kähler and we shall assume from now on that $m \geq 3$. The Nijenhuis tensor of the almost complex J is computed from (2.8) by

$$N^J = -4\psi^-.$$

As an immediate consequence one infers that the almost complex structure of an NK-manifold is integrable if and only if the structure is actually a Kähler one. This observation motivates the following

Definition 4.2. Let (M^{2m}, g, J) be an NK-structure. It is called strict iff $\nabla_X J = 0$ implies that $X = 0$ for all X in TM.

Therefore the non-degeneracy of any of the forms ψ^+ and ψ^- is equivalent with the strictness of the NK-structure (g, J). An important property of NK-structures is contained in the following result.

Theorem 4.1. *Let (M^{2m}, g, J) be a nearly-Kähler manifold. Then*

$$\bar{\nabla}\psi^\pm = 0$$

in other words the Nijenhuis tensor of J is parallel w.r.t. the canonical Hermitian connection.

This has been proved first by Kirichenko [40] and a short proof can be found in [5]. It also follows from our Theorem 3.1, which is unifying this type of property in the class $\mathcal{G}_1$.

Let us introduce now the symmetric tensor r in $S^2 M$ given by

$$\langle rX, Y \rangle = \langle X \lrcorner \psi^+, Y \lrcorner \psi^+ \rangle$$

for all X, Y in TM. It is easily seen to be J-invariant and if (M^{2m}, g, J) is strict then r is non-degenerate. Moreover, from Theorem 4.1 it also follows that

$$\bar{\nabla}r = 0.$$

Corollary 4.1 ([48]). *Any NK-manifold is locally the product of a Kähler manifold and a strict nearly-Kähler one.*

Proof. Consider the $\bar{\nabla}$-parallel distribution $\mathcal{V} := \{V \in TM : \psi_V^+ = 0\}$. Since the torsion of the connection $\bar{\nabla}$ vanishes in direction of $\mathcal{V}$ the latter must be parallel for the Levi-Civita and the result follows by using the de Rham splitting theorem.   □

It follows that locally and also globally if our original manifold is simply connected we can restrict to the study of strict nearly-Kähler structures (SNK for short). Note however that in dimension 6, any NK-structure which is not Kähler must be strict.

The parallelism of the torsion tensor w.r.t. $\bar{\nabla}$ is eventually reflected in the properties its curvature tensor. Indeed

Proposition 4.1 ([30], [33]). *The following hold:*

(i) $\bar{R}(X, Y, Z, U) = \bar{R}(Z, U, X, Y)$ *for all X, Y, Z, U in TM;*

(ii) $\bar{R}(JX, JY) = \bar{R}(X, Y)$ *for all X, Y in TM;*

(iii) $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = [\psi_X^-, \psi_Y^-]Z - \psi_{\psi_X^- Y}^- Z$ *for all X, Y, Z in TM.*

Proof. (i) is immediate from (3.5) and the parallelism of the torsion, whilst (ii) follows from (i) and the fact that $\bar{R}$ belongs to $\Lambda^2 \otimes \lambda^{1,1}$. The last claim follows for instance from Lemma 3.1   □

Remark 4.1. Proposition 4.1 continues to hold when the nearly-Kähler metric is allowed to have signature. This has been exploited in [18] to classify NK-structures compatible with a flat pseudo-Riemannian metric.

To establish first order properties of SNK-structures we will have a look at the Ricci tensor of such metrics. The Hermitian Ricci tensor of (g, J) is defined by

$$\langle \overline{\mathrm{Ric}} X, Y \rangle = \sum_{i=1}^{2m} \bar{R}(X, e_i, Y, e_i)$$

for all X, Y in TM, where $\{e_i, 1 \le i \le 2m\}$ is some local orthonormal frame. By making use of Proposition 4.1 we find that $\overline{\mathrm{Ric}}$ is actually symmetric and J-invariant. The Hermitian Ricci tensor is related to the usual Riemannian one by

$$\overline{\mathrm{Ric}} = \mathrm{Ric} - \frac{3}{4} r \qquad (4.2)$$

as implied by the general curvature comparison formula (3.2).

Theorem 4.2 ([48]). *Let (M^{2m}, g, J) be a strict nearly-Kähler manifold. The following hold:*

  (i) *the Ricci tensor of the metric g is parallel w.r.t. $\bar{\nabla}$, that is $\bar{\nabla} \mathrm{Ric} = 0$;*

 (ii) Ric *is positive definite.*

Proof. The proof of both relies on the explicit computation of the Ricci tensor of the metric g. From the parallelism of ψ^+, after derivation and use of the Ricci identity for the connection with torsion $\bar{\nabla}$ we find

$$[\bar{R}(X, Y), \psi^+] = 0 \qquad (4.3)$$

for all X, Y in TM. In a local orthonormal frame $\{e_i, 1 \le i \le 2m\}$ this reads

$$\sum_{i=1}^{2m} \bar{R}(X, Y) e_i \wedge \psi^+_{e_i} = 0$$

for all X, Y in TM. We now set $Y = e_k$, take the interior product with e_k above to find, after summation over $1 \le k \le 2m$ and some straightforward manipulations that

$$\psi^+_{\overline{\mathrm{Ric}} X} = \sum_{1 \le k, i \le 2m} \bar{R}(X, e_k) e_i \wedge \psi^+_{e_i, e_k}.$$

Since ψ^+ is a form, the sum in the right-hand side equals

$$\frac{1}{2} \sum_{1 \le k, i \le 2m} (\bar{R}(X, e_k) e_i - \bar{R}(X, e_i) e_k) \wedge \psi^+_{e_i, e_k}$$

$$= \frac{1}{2} \sum_{1 \le k, i \le 2m} ([\psi^+_X, \psi^+_{e_k}] e_i - \psi^+_{\psi^+_X e_k} e_i) \wedge \psi^+_{e_i, e_k} + \frac{1}{2} \sum_{1 \le k, i \le 2m} \bar{R}(e_i, e_k) X \wedge \psi^+_{e_i, e_k}$$

after using the Bianchi identity for $\bar{R}$. Now the last sum vanishes since ψ^+ is J-anti-invariant in the first arguments whereas $\bar{R}$ is J-invariant and our frame can be chosen to be Hermitian. After neglecting terms which are J-invariant in e_i, e_k in the algebraic sum above we end up with

$$\psi^+_{\overline{\mathrm{Ric}}X} = \frac{1}{2} \sum_{1 \leq k,i \leq 2m} \psi^+_X \psi^+_{e_k} e_i \wedge \psi^+_{e_i, e_k}$$

for all X in TM. Using now the definition of the tensor r a straightforward manipulation yields

$$\psi^+_{\overline{\mathrm{Ric}}X} = -\frac{1}{2} \sum_{k=1}^{2m} \psi^+_X e_k \wedge (re_k)^\flat \tag{4.4}$$

for all X in TM. By derivation and using the parallelism of ψ^+ it follows that

$$\psi^+_{(\bar{\nabla}_X \overline{\mathrm{Ric}})Y} = 0$$

for all X, Y in TM. (i) follows now from the fact that ψ^+ is nondegenerate and the comparison fact in (4.2). For the claim in (ii) we refer the reader to [48]. $\square$

Using Myer's theorem it follows from the above that complete SNK-manifolds must be compact with finite fundamental group. Therefore, in the compact case one can restrict attention, up to a finite cover, to simply connected nearly-Kähler manifolds.

Another important object is the first Chern form of the almost Hermitian structure (g, J) defined by

$$8\pi \gamma_1(X, Y) = \sum_{i=1}^{2m} \bar{R}(X, Y, e_i, Je_i)$$

for all X, Y in TM. Since $\bar{R}$ is J-invariant we have that γ_1 belongs to $\lambda^{1,1}$ so one can write $4\pi \gamma_1 = \langle CJ\cdot, \cdot \rangle$ for some C in $S^2(TM)$ such that $CJ = JC$. A straightforward computation using the first Bianchi identity yields the relation

$$C = \overline{\mathrm{Ric}} - r,$$

hence C must be parallel w.r.t. the canonical connection, that is

$$\bar{\nabla} C = 0$$

by means of Theorem 4.2. Since γ_1 is a closed form, as it follows from the second Bianchi identity for $\bar{\nabla}$, this results in the algebraic obstruction

$$-C(\psi^+_X Y) = \psi^+_X CY + \psi^+_{CX} Y \tag{4.5}$$

for all X, Y in TM. Note that this can be given a direct algebraic proof by observing that $[\gamma_1, \psi^+] = 0$, as it follows from (4.3) when taking into account that $\bar{R}$ belongs to $S^2(\Lambda^2)$.

4.1 The irreducible case

To obtain classification results for strict nearly-Kähler manifolds we shall start from examining the holonomy representation of the canonical Hermitian connection. At a point x of M where (M^{2m}, g, J) is some SNK-manifold this is the representation

$$\mathrm{Hol}_x(\bar{\nabla}) \colon T_x M \to T_x M$$

obtained by parallel transport w.r.t. $\bar{\nabla}$ along loops about x. The holonomy representation is Hermitian, for $\bar{\nabla}$ is a Hermitian connection and this gives rise to two different notions of irreducibility as indicated by the following well-known result from representation theory.

Proposition 4.2. *Let (V^{2m}, g, J) be a Hermitian vector space and let (G, V) be a Hermitian representation of some group G. If we write $V^{\mathbb{C}}$ for the complex vector space obtain from V by setting $iv = Jv$ for all v in V, the following cases can occur:*

(i) *(G, V) is irreducible.*

(ii) *$(G, V^{\mathbb{C}})$ is irreducible but not (G, V). In this case V splits orthogonally as $V = L \oplus JL$ for some G-invariant subspace L of V.*

(iii) *$(G, V^{\mathbb{C}})$ is reducible.*

In this section we shall deal with the instances when the holonomy representation of $\bar{\nabla}$ is irreducible in the sense of (i) or (ii) in the Proposition 4.2. The first is actually covered by the following powerful result of R. Cleyton and A. Swann.

Theorem 4.3 ([17]). *Let (N^n, g) be Riemannian such that there exists a metric connection D such that the following hold:*

(i) *the torsion tensor T of D belongs to Λ^3;*

(ii) *$DT = 0$ and T does not vanish identically.*

If the holonomy representation of D is irreducible then D is an Ambrose–Singer connection in the sense that $DR^D = 0$ where R^D denotes the curvature tensor of the connection D, provided that $n \neq 6, 7$.

The two exceptions in the result above correspond actually to nearly-parallel G_2-structures in dimensions 7 (see [27] for an account) and SNK-structures in dimension 6.

In the situation in the theorem above (N^n, g) is a locally homogeneous space (see [57]) for more details). Theorem 4.3 is proved by making use of general structure results on Berger algebras and formal curvature tensors spaces, for irreducible representations of compact Lie algebras (see also [44] for the non-compact case). We can now state the following.

Theorem 4.4. *Let (M^{2m}, g, J) be a strict nearly-Kähler manifold. Then either:*

(i) *$\bar{\nabla}$ is an Ambrose–Singer connection; or*

(ii) $m = 3$; *or*

(iii) *the holonomy representation of $\overline{\nabla}$ is reducible over $\mathbb{C}$.*

Proof. By making use of Theorem 4.3 we see that the case (i) in Proposition 4.2 corresponds to (i) in our statement. To finish the proof let us suppose that at some point x of M we have an orthogonal splitting

$$T_x M = L_x \oplus J L_x$$

for some $\mathrm{Hol}_x(\overline{\nabla})$-invariant subspace of $T_x M$. Using parallel transport L_x extends to a $\overline{\nabla}$-parallel distribution L of TM such that

$$TM = L \oplus JL.$$

It follows that $\overline{R}(L, JL) = 0$ by also using Proposition 4.1 (ii). Since $\overline{R}$ belongs to $S^2(\lambda^{1,1})$, this means that $\overline{R}$ is an algebraic expression in ψ^+, which can be explicitly determined from the first Bianchi identity (see [49] for details). It follows that that $\overline{\nabla}\, \overline{R} = 0$ and then $\overline{\nabla}$ is an Ambrose–Singer connection. $\qquad\square$

5 When the holonomy is reducible

In this section we present classification results for SNK-structures in the case when the holonomy representation of $\overline{\nabla}$ is complex reducible. We begin by setting up some terminology which is aimed to gain some understanding concerning the relation between the algebraic properties of the torsion form of an SNK-structure and the geometry of the underlying Riemannian manifold.

5.1 Nearly-Kähler holonomy systems

We start by the following definition which extracts the more peculiar facts from NK-geometry which relate to the holonomy of the canonical Hermitian connection.

Definition 5.1. A strict nearly-Kähler holonomy system $(V^{2m}, g, J, \psi^+, R)$ is constituted of the following data:

(i) a Hermitian vector space (V^{2m}, g, J);

(ii) a nondegenerate 3-form ψ^+ in λ^3;

(iii) a tensor R in $\lambda^{1,1} \otimes \lambda^{1,1}$ of the form

$$R = R^K + \widehat{\Omega},$$

where R^K in $\mathcal{K}(\mathfrak{u}(m))$ is such that

$$[R(x, y), \psi^+] = 0 \tag{5.1}$$

holds for any x, y in V. Moreover the form Ω in $\lambda^{2,2}$ must be given by $\Omega = \frac{1}{2}\psi^+ \bullet \psi^+$.

Geometrically, the assumption of having the form ψ^+ nondegenerate amounts to working on an SNK-manifold (see also Definition 4.2 and Corollary 4.1). In what follows we shall work on a given strict nearly-Kähler holonomy system to be denoted by $(V^{2m}, g, J, \psi^+, R)$. As with Riemannian holonomy systems, the liaison with the holonomy algebra of the canonical connection of a geometric SNK-structure is through the Lie subalgebra

$$\mathfrak{h} := \mathrm{Lie}\{R(x, y) : x, y \in V\}$$

of $\lambda^{1,1}$.

Definition 5.2. An SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ is complex reducible if the representation $(\mathfrak{h}, V^{\mathbb{C}})$ is reducible.

Another object of relevance here is the isotropy algebra $\mathfrak{g}$ of the form ψ^+ defined by

$$\mathfrak{g} = \{\alpha \in \Lambda^2 : [\alpha, \psi^+] = 0\}.$$

It is easy to see, starting from (2.1) and then using an invariance argument together with the non-degeneracy of ψ^+ that

$$\mathfrak{g} \subseteq \lambda^{1,1}.$$

Moreover, the condition (5.1) in Definition 5.1 reads $\mathfrak{h} \subseteq \mathfrak{g}$.

We are interested here in the structure of complex invariant subspaces of the metric representation $(\mathfrak{h}, V)$. The following definition singles out three main classes of subspaces of relevance for our situation.

Definition 5.3. A proper, J-invariant subspace $\mathcal{V}$ of V is said to be (w.r.t. ψ^+):
  (i) isotropic if $\psi^+(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$;
 (ii) null if $\psi^+(\mathcal{V}, \mathcal{V}) = 0$;
(iii) special if it is null and $\psi^+(H, H) = \mathcal{V}$, where H is the orthogonal complement of $\mathcal{V}$ in V.

Remark 5.1. Any 2-dimensional, J-invariant subspace of V is null w.r.t. ψ^+. Moreover, in dimension 6, any two dimensional J-invariant subspace is special w.r.t. ψ^+. However, we are interested here in isotropic or special subspaces which are invariant w.r.t. a particular Lie group or Lie algebra.

A useful criterion to prove that a subspace is special is the following.

Lemma 5.1. *Let (V^{2m}, g, J) be a Hermitian vector space and let ψ^+ be non-degenerate in λ^3. If $\mathcal{V} \subseteq V$ is null w.r.t. ψ^+ and such that $\psi^+(H, H) \subseteq \mathcal{V}$, then:*

(i) $\mathcal{V}$ *is special w.r.t.* ψ^+;

(ii) $\psi^+(\mathcal{V}, H) = H$.

Proof. (i) First of all let us notice that

$$\psi^+(\mathcal{V}, H) \subseteq H$$

since $\psi^+(\mathcal{V}, H)$ is orthogonal to $\mathcal{V}$, as it follows from the fact that $\mathcal{V}$ is null. Let $\mathcal{V}_0 := \psi^+(H, H) \subseteq \mathcal{V}$ and let $\mathcal{V}_1$ be the orthogonal complement of $\mathcal{V}_0$ in $\mathcal{V}$. From the definition of $\mathcal{V}_1$ it follows that $\psi^+(H, \mathcal{V}_1)$ is orthogonal to H thus $\psi^+(H, \mathcal{V}_1) = 0$. But $\psi^+(\mathcal{V}, \mathcal{V}_1) = 0$ as well since $\mathcal{V}$ is null, in other words $\psi^+(\mathcal{V}_1, \cdot) = 0$ and we conclude that $\mathcal{V}_1 = 0$ using that ψ^+ is nondegenerate. This proves (i).

(ii) is proved by an argument similar to that in (i), which we leave to the reader. $\square$

We end this section with the following:

Definition 5.4. An SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ is said to split as

$$V = V_1 \oplus V_2$$

if V admits an $\mathfrak{h}$-invariant, orthogonal and J-invariant splitting $V = V_1 \oplus V_2$ such that

$$\psi^+ \text{ belongs to } \lambda^3(V_1) \oplus \lambda^3(V_2).$$

It is clear that if an SNK-holonomy system splits as $V = V_1 \oplus V_2$ then each of $(V_k, g_{|V_k}, J_{|V_k}, \psi^+_{|V_k})$, $k = 1, 2$ is again an SNK-holonomy system. Note that factors of dimension ≤ 4 are not permitted by this definition since in dimension 2 or 4 there are no non-zero holomorphic 3-forms. Also note that this type of decomposition of an SNK-holonomy system corresponds exactly to local products of SNK-manifolds.

5.2 The structure of the form ψ^+

We are now ready to have a look at the structure of the form ψ^+ when a complex holonomy reduction is given. The starting point of our approach to the classification problem of reducible SNK-holonomy systems is:

Proposition 5.1. *Let* $\mathcal{V}$ *be a proper, J-invariant subspace of* $(\mathfrak{h}, V)$. *If H denotes the orthogonal complement of* $\mathcal{V}$ *in V, the following hold:*

(i) $(\psi^+_x \circ \psi^+_v)w = 0$ *for all x in H and v, w in* $\mathcal{V}$;

(ii) $(\psi^+_x \circ \psi^+_y)z$ *belongs to H whenever x, y, z are in H;*

(iii) $(\psi^+_v \circ \psi^+_w)x$ *belongs to H for all x in H and v, w in* $\mathcal{V}$;

(iv) $(\psi^+_x \circ \psi^+_y)v$ *is in* $\mathcal{V}$, *for all x, y in H and all v in* $\mathcal{V}$.

Proof. Directly from the first Bianchi identity for R, combined with the fact that $R(\mathcal{V}, H) = 0$ we get

$$R(x, y, v, w) = \langle [\psi_v^+, \psi_w^+] x, y \rangle - \langle \psi_v^+ w, \psi_x^+, y \rangle \qquad (5.2)$$

for all x in H, y in V and v, w in $\mathcal{V}$. All claims are now easy consequences of the fact that R belongs to $S^2(\lambda^{1,1})$ and ψ^+ is in λ^3, see [49] for details. $\qquad \square$

We can show that any complex reducible SNK-holonomy system contains, up to products, a null invariant sub-space.

Proposition 5.2. *Let (V^{2m}, g, ψ^+, R) be a complex reducible SNK-holonomy system. Then V splits as*

$$V = V_1 \oplus V_2,$$

where V_2 contains a null invariant space.

Proof. The proof is completed in two steps we shall outline below.

Step 1: Existence of an isotropic invariant subspace.
   The reducibility of $(\mathfrak{h}, V)$ implies the existence of an invariant splitting

$$V = E \oplus F$$

which is moreover orthogonal and stable under J. Let F_0 be the subspace of F spanned by $\{(\psi_v^+ w)_F : v, w \text{ in } E\}$, where the subscript indicates orthogonal projection. Using Proposition 5.1 (i) we get

$$\psi_x^+ \psi_y^+ ((\psi_v^+ w)_F) + \psi_x^+ \psi_y^+ ((\psi_v^+ w)_E) = 0$$

for all x, y in F and whenever v, w are in E. But the first summand is in E by Proposition 5.1 (iv) while the second is in F by (ii) of the same proposition. therefore both summands vanish individually, and a positivity argument yields then

$$\psi^+(F, F_0) = 0.$$

In particular F_0 is null and from (5.1) and the $\mathfrak{h}$-invariance of E and F we also get that F_0 is $\mathfrak{h}$-invariant. Now $F_0 \neq V$ since $\psi^+ \neq 0$ and if $F_0 = 0$ we have that E is isotropic, that is $\psi^+(E, E) \subseteq E$.

Step 2: Existence of an invariant nullspace.
   Using Step 1, we can find a $\mathfrak{h}$-invariant subspace , say $\mathcal{V}$ in V, which is isotropic in the sense that $\psi^+(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$. Let us consider now the $\mathfrak{h}$-invariant tensor $r_1 : \mathcal{V} \to \mathcal{V}$ given by

$$\langle r_1 v, w \rangle = Tr_{\mathcal{V}}(\psi_v^+ \circ \psi_w^+)$$

for all v, w in $\mathcal{V}$. Obviously, this is symmetric and J-invariant. It follows that there is an orthogonal, $\mathfrak{h}$-invariant splitting

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1,$$

in the display, is this $T \cdot r$ or Tr for trace?

where $\mathcal{V}_0 = \mathrm{Ker}(r_1)$. Since $\mathcal{V}$ is isotropic, r_1 is given by $r_1 = -\sum_{k=1}^{d}(\psi_{v_k}^+)^2 v$ for all v in $\mathcal{V}$, where $d := \dim_{\mathbb{R}} \mathcal{V}$ and $\{v_k, 1 \le k \le d\}$ is an orthonormal basis in $\mathcal{V}$. Then Proposition 5.1 (i) implies that $\psi_x^+(r_1 v) = 0$ for all x in H and for all v in $\mathcal{V}$, in other words

$$\psi^+(H, \mathcal{V}_1) = 0.$$

But the definition of $\mathcal{V}_0$ yields that $\psi^+(\mathcal{V}, \mathcal{V}_0) = 0$, in particular $\mathcal{V}_0$ is null. We now form the $\mathfrak{h}$ and J-invariant subspace $H_1 = \mathcal{V}_0 \oplus H_0$ which is easily seen to satisfy that

$$\psi^+(H_1, H_1) \subseteq H_1, \ \psi^+(\mathcal{V}_1, \mathcal{V}_1) \subseteq \mathcal{V}_1, \ \psi^+(\mathcal{V}_1, H_1) = 0.$$

In other words $V = \mathcal{V}_1 \oplus H_1$ is a splitting of our holonomy system in the sense of Definition 5.4 and the result is proved.   □

This can be furthermore (see [49] for details) refined to:

Proposition 5.3. *Let* $(V^{2m}, g, J, \psi^+, R)$ *be an SNK-holonomy system. If it contains a proper invariant nullspace, it splits as*

$$V = V_1 \oplus V_2,$$

where V_2 contains a special invariant subspace.

Summarising the results obtained up to now we obtain, after an easy induction argument on the irreducible components of $(\mathfrak{h}, V)$:

Theorem 5.1. *Let* $(V^{2m}, g, J, \psi^+, R)$ *be a complex reducible SNK-holonomy system. Then V is product of SNK-holonomy systems belonging to one of the following classes:*
  (i) *irreducible SNK-holonomy systems;*
 (ii) *SNK-holonomy systems which contain a special invariant subspace.*

To advance with the classification of our holonomy systems we need therefore only to discuss the second class present in the theorem above. We first observe that given a special invariant subspace in some SNK-holonomy system one can explicitly determine the curvature along the special subspace.

Proposition 5.4. *Let* $(V^{2m}, g, J, \psi^+, R)$ *be an SNK-holonomy system containing a special invariant subspace $\mathcal{V}$. Then:*

$$R(\psi_x^+ y, v_1, v_2, v_3) = \langle [\psi_{v_1}^+, [\psi_{v_2}^+, \psi_{v_3}^+]] x, J y \rangle$$

whenever x, y are in $H = \mathcal{V}^{\perp}$ and for all v_k, $1 \le k \le 3$ in $\mathcal{V}$.

Actually, Proposition 5.4 singles out a second Lie algebra of relevance for us, obtained as follows. Consider the subspace $\mathfrak{p} \subseteq \lambda^2(H)$ given by

$$\mathfrak{p} := \{\psi_v^+ : v \text{ in } \mathcal{V}\}$$

together with the subspace $\mathfrak{q}$ of $\lambda^{1,1}(H)$ given by $\mathfrak{q} := [\mathfrak{p}, \mathfrak{p}]$.

Proposition 5.5. *The following hold:*

(i) $\mathfrak{p} \cap \mathfrak{q} = 0$;

(ii) $\mathfrak{r} = \mathfrak{p} \oplus \mathfrak{q}$ *is a Lie subalgebra of* $\Lambda^2(H)$.

Proof. From Proposition 5.4 we get

$$[\psi_{v_1}^+, [\psi_{v_2}^+, \psi_{v_3}^+]] = \psi_{R(v_1, v_2)v_3}^+$$

for all v_k, $1 \le k \le 3$ in $\mathcal{V}$. That is $\mathfrak{p}$ is a Lie triple system in the sense that $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \subseteq \mathfrak{p}$ hence $\mathfrak{p} + \mathfrak{q}$ is a Lie algebra (see [36] for details). To prove (i), let us pick z in $\mathfrak{p} \cap \mathfrak{q}$. Then $z = \psi_v^+$ for some v in $\mathcal{V}$ and moreover $[z, \mathfrak{p}] \subseteq \mathfrak{p}$. In particular $[\psi_v^+, \psi_{Jv}^+] = \psi_w^+$ for some w in $\mathcal{V}$, or equivalently $2(\psi_v^+)^2 = \psi_{Jw}^+$. But the left-hand side of this equality is symmetric whilst the right-hand is skew-symmetric which yields easily that $v = 0$. $\qquad\square$

The Lie algebra $\mathfrak{r}$ is best though of as the holonomy algebra of a symmetric space of compact type. In the realm of NK-geometry this will turn out to be precisely the case.

5.3 Reduction to a Riemannian holonomy system

We shall consider in what follows an SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ containing a special invariant sub-space $\mathcal{V}$, with orthogonal complement to be denoted by H.

At this moment need a finer notion of irreducibility for an SNK-holonomy system containing an $\mathfrak{h}$-invariant special subspace. Let us now define $R^H : \Lambda^2(H) \to \Lambda^2(H)$ by

$$R^H(x, y) = R(x, y) + \psi_{\psi_x^+ y}^+$$

for all x, y in H. Note this is well defined because $\psi^+(H, H) \subseteq \mathcal{V}$ and $\psi^+(\mathcal{V}, H) = H$ and also because H is $\mathfrak{h}$-invariant. Moreover, the Bianchi identity for R ensures that R^H is an algebraic curvature tensor on H, that is R^H belongs to $\mathcal{K}(\mathfrak{so}(H))$. The second Lie subalgebra of $\Lambda^2(H)$ of interest for us is

$$\mathfrak{h}^H = \mathrm{Lie}\{R^H(x, y) : x, y \text{ in } H\}.$$

Definition 5.5. $(H, g, \mathfrak{h}^H)$ is called the Riemannian holonomy system associated to $(V^{2m}, g, J, \psi^+, R)$. It is irreducible if the metric representation $(\mathfrak{h}^H, H)$ is irreducible.

The main result in this section is to prove that we can reduce, up to products in the sense of Definition 5.4 – hence of Riemannian type, the study of special SNK-holonomy to the case when the associated Riemannian holonomy system is irreducible.

Theorem 5.2. *Let $(V^{2m}, g, J, \psi^+, R)$ be an SNK-holonomy system containing an $\mathfrak{h}$-invariant subspace $\mathcal{V}$. Then V splits as*

$$V = V_1 \oplus \cdots \oplus V_q,$$

a product of special SNK-holonomy systems, such that each factor has irreducible associated Riemannian holonomy system.

Proof. Let $\mathcal{V}$ be a special $\mathfrak{h}$-invariant subspace and let us orthogonally decompose

$$H = H_1 \oplus H_2$$

in invariant subspaces for the action of $\mathfrak{h}^H$. As a straightforward consequence of having R in $S^2(\lambda^{1,1})$ we note that

$$R^H(\psi_v^+) = \frac{1}{2}\psi_{rv}^+ \tag{5.3}$$

on H, for all v in $\mathcal{V}$. Since $R^H(H, H, H_1, H_2) = 0$ by assumption and since r is invertible on $\mathcal{V}$ it follows that $\psi_v^+ H_1$ and H_2 are orthogonal, hence

$$\psi_v^+ H_k \subseteq H_k, \quad k = 1, 2 \tag{5.4}$$

given that $\psi_v^+ H \subseteq H$. It also follows that $\psi^+(H_1, H_2) = 0$. We will need now several preliminary steps.

Step 1: H_k are J-invariant for $k = 1, 2$.
    From (5.4) it follows that $r^{\mathcal{V}}(H_k) \subseteq H_k, k = 1, 2$ where $r^{\mathcal{V}} : H \to H$ is defined by

$$r^{\mathcal{V}}x = -\sum_{k=1}^d (\psi_{v_k}^+)^2 x$$

for all x in H, where $\{v_k, 1 \le k \le d = dim_{\mathbb{R}}\mathcal{V}\}$ is some orthonormal basis in $\mathcal{V}$. But $r^{\mathcal{V}}$ has no kernel, as an easy consequence of the fact that ψ^+ is non-degenerate and $\mathcal{V}$ is special. It follows that $r^{\mathcal{V}}_{|H_k} : H_k \to H_k$ is injective, hence surjective, that is

$$r^{\mathcal{V}}(H_k) = H_k$$

for $k = 1, 2$. Again from (5.4) we find that $r^{\mathcal{V}}(JH_k) \subseteq H_k$ hence after applying $(r^{\mathcal{V}})^{-1}$ (which as we have seen preserves H_k) we get that $JH_k \subseteq H_k$ for $k = 1, 2$.

Step 2: A first decomposition of V.
    Let $\mathcal{V}' := \mathcal{V} \oplus H_1$ and consider the orthogonal and J-invariant splitting

$$V = \mathcal{V}' \oplus H_2.$$

Since $R^H(H_1, H_2, H, H) = 0$ and also because $\psi^+(H_1, H_2) = 0$ we find that $R(H_1, H_2, H, H) = 0$. In other words we have $R(H, H)H_2 \subseteq H_2$ and also $R(H, \mathcal{V})H_2 = 0$ since $\mathcal{V}$ is $\mathfrak{h}$-invariant and R is symmetric. Now the Bianchi identity

for R, combined again with the $\mathfrak{h}$-invariance of $\mathcal{V}$ and its nullity gives

$$R(v, w)x = [\psi_v^+, \psi_w^+]x$$

for all v, w in $\mathcal{V}$ and for all x in H. Using again (5.4) we find that $R(\mathcal{V}, \mathcal{V})H_2 \subseteq H_2$ so H_2 is actually $\mathfrak{h}$-invariant.

Step 3: The decomposition of $\mathcal{V}$.

Thus we may apply Proposition 5.1 (i) to the splitting $V = V' \oplus H_2$ whence

$$\psi_x^+ \psi_v^+ w = 0$$

for all x in H_2 and whenever v, w belong to V'. Let us define $\mathcal{V}_k = \psi^+(H_k, H_k) \subseteq \mathcal{V}$ for $k = 1, 2$ and notice these are J-invariant. Then

$$\psi^+(H_2, \mathcal{V}^1) = 0$$

in particular $\mathcal{V}^1$ and $\mathcal{V}^2$ are orthogonal, after taking the scalar product with elements in H_2. Therefore

$$\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2$$

after using that $\psi^+(H, H) = \mathcal{V}$ and $\psi^+(H_1, H_2) = 0$.

Now since H_1 and H_2 play equal rôles, by repeating the arguments above we also get that $\psi^+(\mathcal{V}^2, H_1) = 0$.

Step 4: Proof of the theorem.

We consider the orthogonal and J-invariant splitting

$$V = V_1 \oplus V_2,$$

where $V_k := \mathcal{V}^k \oplus H_k, k = 1, 2$. From Step 3 it follows that ψ^+ belongs to

$$\lambda^3(V_1) \oplus \lambda^3(V_2)$$

and moreover that H_k are invariant under $\mathfrak{h}$ for $k = 1, 2$. Let us prove that, say $\mathcal{V}^1$, is $\mathfrak{h}$-invariant too. From (5.1) we get

$$R(x, y)(\psi_{x_1}^+ x_2) = \psi_{R(x,y)x_1}^+ x_2 + \psi_{x_1}^+ R(x, y)x_2$$

for all x, y in V and x_1, x_2 in H_1 and for all x, y in H. Since H is $\mathfrak{h}$-invariant and we have seen that $\psi^+(H, H_1) = \psi^+(H_1, H_1) = \mathcal{V}^1$, the invariance of of $\mathcal{V}^1$ follows and that of $\mathcal{V}^2$ is proved analogously. We conclude that V^k are invariant under $\mathfrak{h}$, for $k = 1, 2$.

The claim follows now by induction on the number of irreducible components of $(\mathfrak{h}^H, H)$, provided we show that $\mathfrak{h}^H$ does not fix any vector in H. But this is implied by (5.3), for such a vector, say x_0 in H would satisfy then $\psi^+(\mathcal{V}, x_0) = 0$. Taking scalar products with elements in H this yields $\psi^+(x_0, H) = 0$ as $\psi^+(H, H) \subseteq \mathcal{V}$. This is to say that $\psi_{x_0}^+ = 0$ hence $x_0 = 0$ since ψ^+ is non-degenerate. This proves the absence of fixed points for the representation $(\mathfrak{h}^H, H)$ and the proof is finished. □

More properties of the non-Riemannian holonomy representation $(\mathfrak{h}, V)$ can now be formulated.

Proposition 5.6. *Let (V^{2m}, g, J, ψ^+) be a special SNK-holonomy system containing an invariant subspace $\mathcal{V}$ and such that the Riemannian holonomy system $(\mathfrak{h}^H, H)$ is irreducible. Then the representation $(\mathfrak{h}, \mathcal{V})$ is irreducible over $\mathbb{C}$.*

Proof. Let us suppose that $\mathcal{V}$ is not irreducible and split $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2$ as orthogonal sum of J-stable and $\mathfrak{h}$-invariant subspaces. Then $R(\psi_x^+ y, v, v_1, v_2) = 0$ for all x, y in H, v in $\mathcal{V}$ and for all v_k in $\mathcal{V}^k$, $k = 1, 2$. Using Proposition 5.4 this leads to

$$[\psi_v^+, [\psi_{v_1}^+, \psi_{v_2}^+]] = 0$$

whenever v belongs to $\mathcal{V}$ and v_1, v_2 are in $\mathcal{V}^1, \mathcal{V}^2$ respectively. An easy invariance argument which can be found in [49], page 492 yields

$$\psi_{v_1}^+ \psi_{v_2}^+ = 0 \tag{5.5}$$

for all v_k in $\mathcal{V}^k$, $k = 1, 2$. We form $H^k := \psi_{\mathcal{V}^k} H$, $k = 1, 2$ which are therefore orthogonal, J-invariant and such that $H = H^1 \oplus H^2$. The spaces H^k, $k = 1, 2$ are actually invariant under $\mathfrak{h}$ a fact which follows from (5.1) and the $\mathfrak{h}$-invariance of $\mathcal{V}^k$, $k = 1, 2$. Now by using (5.5) we obtain that

$$\psi_{\psi_x^+ y}^+ H^k \subseteq H^k$$

for all x, y in H, since $\psi_{\mathcal{V}^k}^+ H \subseteq H$ and $\psi^+(H, H) \subseteq \mathcal{V}$. This means, see also the definition of the tensor R^H, that H^k, $k = 1, 2$ are invariant under $\mathfrak{h}^H$ hence they cannot be both proper since $(\mathfrak{h}^H, H)$ is irreducible. If $H^1 = 0$ for instance, a routine argument leads to $\mathcal{V}^1 = 0$, a contradiction and the proof is finished. $\square$

5.4 Metric properties

Let (V, g, J, ψ^+, R) be an SNK-holonomy system with special invariant subspace $\mathcal{V}$ and such that the associated representation $(\mathfrak{h}^H, H)$ is irreducible. Let us define the tensors Ric, $\overline{\text{Ric}}$ and C exactly as in the beginning of this section, and note they are still subject to (4.4) and (4.5) as it follows from the corresponding proofs. Their invariance under $\mathfrak{h}$ is an easy consequence of (4.4). We shall prove here that one reduces the discussion to the case when C has only a few eigenvalues but skip most of the technical details. The eigenspaces of the tensor C will be used to construct invariant subspaces of the representation $(\mathfrak{h}, H)$ which is not necessarily irreducible, in contrast to $(\mathfrak{h}, V)$ which is irreducible by Proposition 5.6.

Proposition 5.7. *The tensor C has at most 3-eigenvalues.*

Proof. Directly from its definition C preserves the splitting $V = \mathcal{V} \oplus H$. Since C is $\mathfrak{h}$-invariant and $(\mathfrak{h}, \mathcal{V})$ is irreducible it follows that $C_{|\mathcal{V}} = \lambda 1_{\mathcal{V}}$ for some real number λ. Using (4.5) we find that the symmetric and J-invariant tensor $S : H \to H$, $S := C_{|H} + \frac{\lambda}{2}$ satisfies

$$S(\psi_v^+ x) = -\psi_v^+ S x,$$

hence

$$S^2(\psi_v^+ x) = \psi_v^+ S^2 x \qquad (5.6)$$

for all x in H and for all v in $\mathcal{V}$. Since C is $\mathfrak{h}$-invariant, so is S^2, and in fact the latter turns out to be $\mathfrak{h}^H$-invariant, after using (5.6). It follows that S^2 is a multiple of the identity and the claim follows. $\qquad\square$

Let us consider the tensor $r^{\mathcal{V}} \colon H \to H$ given by

$$r^{\mathcal{V}} = -\sum_{k=1}^{2d}(\psi_{v_k}^+)^2,$$

where $2d = \dim_{\mathbb{R}} \mathcal{V}$ and $\{v_k, 1 \le k \le 2d\}$ is some orthonormal basis in $\mathcal{V}$. It is easy to see that $r^{\mathcal{V}}$ is positive, J-commuting, symmetric and $\mathfrak{h}$-invariant and moreover that $r_{|H} = 2r^{\mathcal{V}}$.

When C has only one eigenvalue, then $C = 0$ and one can show [49] by using (4.4) that both Ric and r must actually by diagonal on V.

It remains to investigate the cases when C has two, respectively three eigenvalues which are described below.

Proposition 5.8. *The following hold:*

(i) *If C has exactly two eigenvalues then there exists $k > 0$ such that $r^{\mathcal{V}} = k1_H$ and moreover the eigenvalues together with the corresponding eigenspaces for the tensors r, C, Ric are given in the following table:*

Eigenvalue	r	Ric	C	Eigenspace
λ_1	$\frac{n-d}{d}k$	$\frac{n+7d}{4d}k$	$\frac{4(n-3d)}{d}k$	$\mathcal{V}$
λ_2	$2k$	$\frac{n+2d}{2d}k$	$-\frac{2(n-3d)}{d}k$	H

(ii) *If C has exactly 3-eigenvalues we have a $\mathfrak{h}$-invariant, orthogonal and J-invariant splitting*

$$V = \mathcal{V} \oplus H_1 \oplus H_2$$

such that $\psi^+(H_1, H_2) = \psi^+(H_2, H_2) = 0$ and $\psi^+(H_1, H_2) = \mathcal{V}$.

(iii) *In all cases, including when $C = 0$, we have*

$$\mathrm{Ric}^H = \mu g_{|H}$$

for some $\mu > 0$, where Ric^H denotes the Ricci contraction of R^H.

We refer the reader to [49] for details of the proof and also mention that in case (ii) above it is also possible to display the relations between the eigenvalues of all relevant symmetric endomorphisms.

5.5 The final classification

By Theorem 5.2 it is enough to consider an SNK-holonomy system $(V^{2m}, g, J, \psi^+, R)$ with a special invariant subspace $\mathcal{V}$ and such that the Riemannian holonomy system $(\mathfrak{h}^H, H)$ is irreducible, in the notations of the previous section. We shall combine here some well-known results for irreducible Riemannian holonomy systems and the information which is available in our present context. To progress in this direction we will first make clear the relationship between the Lie algebra $\mathfrak{r}$ and the Riemannian holonomy algebra $\mathfrak{h}^H$.

Proposition 5.9. *The following hold:*

(i) $\mathfrak{r}$ *is an ideal in* $\mathfrak{h}^H$;

(ii) $R^H = \frac{\nu}{2} 1_{\mathfrak{r}}$ *on* $\mathfrak{r}$ *for some* $\nu > 0$ *such that* $r_{|\mathcal{V}} = \nu 1_{\mathcal{V}}$.

Proof. (i) That $\mathfrak{p} \subseteq \mathfrak{h}^H$ follows from (5.3) hence $\mathfrak{q} = [\mathfrak{p}, \mathfrak{p}]$ is contained in $\mathfrak{h}^H$ as well. To see that $\mathfrak{r}$ is an ideal we use (5.1) to observe that

$$[R(x, y), \psi_v^+]$$

belongs to $\mathfrak{p}$ for all x, y in H and for all v in $\mathcal{V}$. The definition of R^H and that of $\mathfrak{r}$ now yields

$$[R^H(x, y), \psi_v^+]$$

in $\mathfrak{r}$ and it easy to conclude by using the Jacobi identity and Proposition 5.5.

(ii) First of all let us observe that r preserves $\mathcal{V}$, since the latter is special. The $\mathfrak{h}$-invariance of r, together with the irreducibility of $(\mathfrak{h}, \mathcal{V})$ implies the existence of a constant $\nu > 0$ such that $r_{|\mathcal{V}} = \nu 1_{\mathcal{V}}$. For the action of R^H on $\mathfrak{p}$ our claim follows now from (5.3). To prove it on $\mathfrak{q}$ we observe we proceed as follows. Given that R belongs to $S^2(\Lambda^2)$ we can alternatively rewrite (5.1) under the form

$$R(\psi_{x_1}^+ x_2, x_3) + R(x_2, \psi_{x_1}^+ x_3) = R(x_1, \psi_{x_2}^+ x_3)$$

whenever x_k are in $V, k = 1, 2, 3$. Using this for $x_1 = v, x_2 = \psi_w^+ e_i, x_3 = e_i$ where v, w are in $\mathcal{V}$ and $\{e_i\}$ is some orthonormal basis in H yields after summation and rearrangement of terms

$$\sum_i R(e_i, [\psi_v^+, \psi_w^+] e_i) = R(v, rw)$$

that is $R([\psi_v^+, \psi_w^+]) = \frac{\nu}{2} [\psi_v^+, \psi_w^+]$ on H, after also using (5.2). After a straightforward invariance argument this leads to $R^H([\psi_v^+, \psi_w^+]) = \frac{\nu}{2} [\psi_v^+, \psi_w^+]$ and the claim is proved.   □

Now we recall that for a given metric representation $(\mathfrak{l}, W)$ of some Lie algebra $\mathfrak{l}$ we have the Berger list of irreducible Riemannian holonomies, and proceed to the

final classification result, hence proving Theorem 1.1 in the introduction.

$\mathfrak{l}$	W	
$\mathfrak{so}(n)$	$\mathbb{R}^n$	
$\mathfrak{u}(m)$	$\mathbb{R}^{2m}$	
$\mathfrak{su}(m)$	$\mathbb{R}^{2m}$	
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R}^{4m}$	
$\mathfrak{sp}(m)$	$\mathbb{R}^{4m}$	
$\mathfrak{spin}(7)$	$\mathbb{R}^8$	
$\mathfrak{g}_2$	$\mathbb{R}^7$	

Theorem 5.3. *Let (M^{2m}, g, J) be a complete SNK-manifold. Then M is, up to finite cover, a Riemannian product whose factors belong to the following classes:*

  (i) *homogeneous SNK-manifolds;*

 (ii) *twistor spaces over positive quaternionic Kähler manifolds;*

(iii) *6-dimensional SNK-manifolds.*

Proof. First of all, because M must be compact with finite fundamental group we may assume up to a finite cover that it is simply connected. Pick now a point x in M and consider the holonomy representation $(\mathrm{Hol}_x(\overline{\nabla}), T_x M)$. If this is irreducible over $\mathbb{R}$ or complex irreducible but real reducible we conclude by Theorem 4.4.

If the holonomy representation of $\overline{\nabla}$ is reducible over $\mathbb{C}$ at x we consider the SNK-holonomy system $(T_x M, g_x, J_x, \psi_x^+, \overline{R}_x)$. By the theorem of Ambrose–Singer we have that $\mathfrak{h} \subseteq \mathfrak{hol}_x(\overline{\nabla})$. Also note that when *starting* the whole reduction procedure leading to Theorem 5.2 from some $\mathfrak{hol}_x(\overline{\nabla})$ complex invariant sub-space we end up with a splitting

$$T_x M = V_1 \oplus \cdots \oplus V_p$$

with the properties in Theorem 5.2 and which is furthermore $\mathfrak{hol}_x(\overline{\nabla})$-invariant together with the special sub-spaces contained in each factor. Using parallel transport w.r.t. to $\overline{\nabla}$ this extends to a $\overline{\nabla}$-parallel decomposition, which is actually ∇-parallel since the torsion of $\overline{\nabla}$ is split along the decomposition. The splitting theorem of de Rham applies and our study is reduced to that of SNK-manifolds such that at each point, we have an invariant special subspace having the property that the associated holonomy system is irreducible in the sense of Definition 5.5. Then we show (see [49]) that M is the total space of a Riemannian submersion

$$\pi : (M, g) \to (N^{2n}, h)$$

whose fibres are totally geodesic and w.r.t. the induced metric and almost complex structure are simply connected, irreducible and compact Hermitian symmetric spaces. The tensor R^H defined in Section 5.3 projects onto the Riemann curvature tensor of h making of $\mathfrak{h}^H$ a subalgebra of $\mathfrak{hol}(N, h)$ at each point. Using this, one shows that the Riemannian manifold (N^{2n}, h) is irreducible and Proposition 5.8 (iii) gives that h is Einstein of positive scalar curvature.

If the tensor C has three eigenvalues a short manipulation of the second Bianchi identity for $\bar{\nabla}$ combined with the algebraic structure of the splitting of TM given in (ii) of Proposition 5.8 gives that $\bar{\nabla}$ is an Ambrose–Singer connection.

Let us discuss now the case when C has two eigenvalues or it vanishes.

If (N^{2n}, h) is a symmetric space the relations of O'Neill (see [6]) for the Riemannian submersion $\pi \colon M \to N$ essentially say that the curvature tensor $\bar{R}$ is an explicit expression of ψ^+, R^H and the curvature of the fibre (which is a symmetric space) along the $\bar{\nabla}$-parallel splitting $TM = \mathcal{V} \oplus H$. This information results easily in having $\bar{\nabla}$ an Ambrose–Singer connection again.

Suppose now that (N^{2n}, h) is not a symmetric space. Then the holonomy representation of (N, h), at the Lie algebra level, corresponds to the representation $(\mathfrak{h}^H, H)$ and as it is well known (see [52]), must be one of the entries in the Berger list above. We shall mainly use now Proposition 5.9. The Lie algebras $\mathfrak{su}(m)$, $\mathfrak{sp}(m)$, $\mathfrak{spin}(7)$, $\mathfrak{g}_2$ are excluded because their curvature tensors are Ricci-flat. Supposing that $\mathfrak{h}^H = \mathfrak{u}(n)$, given that the ideal $\mathfrak{r}$ is at least two-dimensional we can only have $\mathfrak{r} = \mathfrak{su}(n), \mathfrak{u}(n)$. But using Proposition 5.9 combined with the fact that a curvature tensor of Kähler-Einstein type is completely determined by its restriction to $\mathfrak{su}(m)$ we find that (N^{2n}, h) is symmetric a contradiction. Similar arguments together with results from [5] enable us to conclude in the case when $\mathfrak{h}^H = \mathfrak{so}(2n)$ or when $\mathfrak{r} = \mathfrak{sp}(m)$. The only case remaining is when $\mathfrak{h}^H = \mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$ and $\mathfrak{r} = \mathfrak{sp}(1)$ which implies that $\mathcal{V}$ is of rank 2. Hence the holonomy group of $\bar{\nabla}$ is contained in $U(1) \times U(2m)$ and the fact that (M^{2m}, g, J) is a twistor space over a positive quaternionic-Kähler manifold has been proved in [48]. □

6 Concluding remarks

Due to considerations of space and time we have omitted to present some important facts related to nearly-Kähler geometry. The first aspect is related to the construction of examples, which is implicit in our present treatment. Given a quaternion-Kähler manifold (M^{4m}, g, Q) of positive scalar curvature the Salamon twistor construction [51] yields a Kähler manifold (Z, h, J) such that there is a Riemannian submersion with totally geodesic, complex fibres

$$S^2 \hookrightarrow (Z, h) \to (M, g).$$

Using this structure and a canonical variation of the metric h it has been shown [19] that Z admits an SNK-metric. Enlarging the context to that of Riemannian submersions with complex, totally geodesic fibres from Kähler manifolds one draws a similar conclusion [48] (see also [50] for some related facts). This observation can also be used to describe the homogeneous SNK-manifolds which appear in Theorem 5.3 as twistor spaces in the sense of [11] over symmetric spaces of compact type.

We have also omitted to discuss NK-structures in dimension 6, which have a rich geometry, although not yet fully understood. If (M^6, g, J) is a strict NK manifold the structure group automatically reduces to SU(3), hence $c_1(M, J) = 0$ [33]. Moreover the metric g must be Einstein [33], of positive scalar curvature. There is also a 1-1 correspondence between 6-dimensional manifolds admitting real Killing spinors and SNK-structures in dimension 6 [35].

In the compact case the only known examples are homogeneous, and various characterisations of these instances are available [12], [45]. However, compact examples with two conical singularities have been very recently constructed in [23] building on the fact that one can suitably rotate the SU(2)-structure of a Sasakian–Einstein 5-manifold and then extend it to a SU(3)-structure of NK-type through a generalised cone construction.

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