Department of Mathematics

Maths 260 Assignment 1

August 1, 2010 Due: 4pm, Tuesday August 10th, 2010

- Put your completed assignment in the appropriate box on the ground floor of the Maths/Physics building **before** 4pm on the date due. Late assignments or assignments placed in the wrong box will not be marked.
- Your assignment must be accompanied by a blue Mathematics Department coversheet. Copies of the coversheet are available from a box in the basement.
- You must show all your working.

1. This question is about the differential equation

$$\frac{dy}{dt} = \frac{2te^{-3t}}{y}.$$

(a) Find a one-parameter family of solutions to the differential equation (i.e., a formula for solutions, with one arbitrary constant in the formula).

(b) Are there any solutions to the differential equation that are missing from the set of solutions you found in (a)? Explain your answer.

(c) Find a solution to the differential equation that satisfies the initial condition $y(0) = -2$.

(d) Find a solution to the differential equation that satisfies the initial condition $y(0) = 2$.

(e) Use the Matlab function *analyzer* to draw three different solutions to the differential equation, including the solutions you found in parts (c) and (d). Make sure your solutions are easily distinguishable in the analyzer plot. Draw all three graphs on the same picture, print your picture, and hand it in with your assignment.
Note: One or more of your solutions may not be defined for all t but *analyzer* may still plot something for the values of t for which the solution is not defined. Make sure you indicate on your Matlab plots which parts (if any) are Matlab plotting errors.

2. (a) Consider the differential equation

$$\frac{dx}{dt} = x^3 - 3x - t.$$

i. Use *dfield* to plot and print two copies of the direction field for this differential equation. Use the ranges $t \in [-3, 3]$ and $x \in [-3, 3]$.

ii. Use one copy of the direction field to sketch at least four representative solutions to the differential equation, including the solution that satisfies the initial condition $x(-2) = 0$. Plot your solutions going forward and backward in time.

iii. Use the other copy of the direction field to show what would be obtained if Euler's method with stepsize $h = 1$ was used to compute an approximation at final time $t = 1$ to the solution that satisfies the initial condition $x(-2) = 0$. You do not need to do any calculations to do this part of the question; just use the information on the direction field.

(b) Use Improved Euler's method with stepsize $h = 1$ to compute an approximate value of the solution to the initial value problem

$$\frac{dx}{dt} = x^2 - t, \; x(0) = -2$$

at final time $t = 3$. Show all your working.

3. A numerical method was used to find an approximation to the solution of a certain initial value problem. Several different step sizes were used and the following results were obtained.

stepsize	approximate solution at $t = 2$
0.1	0.0567102
0.05	0.0048399
0.025	0.0004687
0.0125	0.0000440

(a) Use these results to estimate the value of the solution to the IVP at $t = 2$. Give an answer that is accurate to 5 decimal places.

(b) Estimate the errors in the numerical approximation obtained using stepsizes of 0.05 and 0.025.

(c) Hence calculate the effective order of the method at stepsize $h = 0.025$.

(d) Which numerical method do you think might have been used to get these results? Give a reason for your answer.

4. This question is about the differential equation

$$\frac{dy}{dt} = 3t(y-1)^{\frac{1}{3}}.$$

(a) Use the Existence and Uniqueness Theorems to show that there is a unique solution satisfying the differential equation and the initial condition $y(0) = 3$. You do not need to find this solution.

(b) Use substitution to show that both $y_1(t) = 1$ and $y_2(t) = 1 + t^3$ satisfy the differential equation and the initial condition $y(0) = 1$.

(c) Does you answer to (b) contradict the Uniqueness Theorem? Give a reason for your answer.

5. Challenge question This question is harder, and is worth 10% of the total assignment mark. Only attempt this question when you are happy with your answers to the other questions.

In Lecture 6, a model for a student loan was discussed. Assuming that the interest rate on the loan is 6.8%, that no repayments are made for the first two years of the loan, and that after two years the loan is paid off at $3000 a year, the following differential equation model was derived:

$$\frac{dL}{dt} = \begin{cases} 0.068L, & 0 \le t \le 2 \\ 0.068L - 3, & t > 2, \end{cases}$$

with $L(0) = 20$. Here $L(t)$ is a variable representing the size of the loan at time t. L is measured in tens of thousands of dollars (so $L = 20$ means the loan is for $20,000), and t is measured in years.

(a) Use the Matlab function *dfield* to plot a numerical solution to this IVP. Hence work out how long it takes to pay back the loan.
Hints:

- The differential equation can be rewritten as

$$\frac{dL}{dt} = 0.068L - 3\mathrm{H}(t-2)$$

  where

$$\mathrm{H}(t-2) = \begin{cases} 0, & t < 2, \\ 1, & t > 2. \end{cases}$$

  H is known as the Heaviside function. You can use Help within Matlab to find out how to use the Heaviside function in *dfield*.

- Make sure your step size is small enough to give you an accurate answer. How do you know if the stepsize is small enough?

(b) How long does it take to pay back the loan if the interest rate is 5.5% instead of 6.8% per year?

(c) What rate of repayment is necessary (instead of \$3000 per year) if the interest rate is 6.8% and the loan is to be paid back in ten years, i.e., by the time $t = 10$?

(d) How long does it take to repay the loan if the interest rate is 6.8% for the first two years and then changes to 7.2%?

Be sure to include relevant printouts from *dfield* as evidence of the answers you give for parts (a)–(d).