Introduction to

Conformal Geometry

Lecture 2

Weyl Curvature Tensor

Characterization of Conformal Flatness

(M^n, g) smooth Riemannian manifold, $n \geq 2$

Fundamental objects: Levi-Civita connection ∇ and Riemann curvature tensor R_{ijkl}

Notation: $\nabla_k v_j = v_{j,k} = \partial_k v_j - \Gamma^i_{jk} v_i$

Curvature tensor defined by commuting covariant derivatives:

$$v_{j,kl} - v_{j,lk} = R^i{}_{jkl} v_i \qquad \text{(index raised using } g^{ij}\text{)}$$

Theorem: There exist local coordinates x^i so that $g = \sum (dx^i)^2$ if and only if $R = 0$.

There are many proofs. See Vol. 2 of Spivak's *A Comprehensive Introduction to Differential Geometry*. In Riemann's proof, $R = 0$ arises as the integrability condition for the application of Frobenius' Theorem to an overdetermined system of pde's.

Next suppose given a conformal class $[g]$ of metrics

$\widehat{g} \sim g$ if $\widehat{g} = e^{2\omega} g$, $\omega \in C^\infty(M)$.

Analogous question: given g, under what conditions do there exist local coordinates so that $g = e^{2\omega} \sum (dx^i)^2$? Such a metric is said to be locally conformally flat.

If $n = 2$, always true: existence of isothermal coordinates. So assume $n \geq 3$.

Certainly $R = 0$ suffices for existence of x^i, ω. But we'll see only need a piece of R to vanish. Decompose R into pieces: trace-free part and trace part.

(V, g) inner product space, $g \in S^2V^*$ metric

Simpler analogous decomposition for S^2V^*.

Any $s \in S^2V^*$ has a trace $\mathrm{tr}_g\, s = g^{ij}s_{ij} \in \mathbb{R}$.

Fact: any $s \in S^2V^*$ can be uniquely written

$$s = s_0 + \lambda g, \quad \lambda \in \mathbb{R}, \quad s_0 \in S^2V^*, \quad \mathrm{tr}_g\, s_0 = 0$$

Proof: Take trace: $\mathrm{tr}_g\, s = n\lambda$, so $\lambda = \frac{1}{n}\mathrm{tr}_g\, s$.

Then $s_0 = s - \frac{1}{n}(\mathrm{tr}_g\, s)g$ works. $\square$

So $S^2V^* = S^2_0V^* \oplus \mathbb{R}g$, where $S^2_0V^* = \{s : \mathrm{tr}_g\, s = 0\}$

Back to curvature tensors (linear algebra):

Def: $$\mathcal{R} = \Big\{ R \in \otimes^4V^* : R_{ijkl} = -R_{jikl} = -R_{ijlk} \text{ and } \underbrace{R_{ijkl} + R_{iklj} + R_{iljk} = 0}_{\text{same as } R_{i[jkl]} = 0, \text{ where}} \Big\}$$

$$6R_{i[jkl]} = R_{ijkl} + R_{iklj} + R_{iljk} - R_{ijlk} - R_{ikjl} - R_{ilkj}$$

Have $\text{tr} : \mathcal{R} \to S^2V^*$, $(\text{tr}\, R)_{ik} = Ric_{ik} = g^{jl}R_{ijkl}$.

Also $\text{tr}^2 R \in \mathbb{R}$, $\text{tr}^2 R = S = g^{ik}Ric_{ik}$

Definition: $\mathcal{W} = \{W \in \mathcal{R} : \text{tr}\, W = 0\}$

Need a way to embed $S^2V^* \hookrightarrow \mathcal{R}$ by "multiplying by g", analogous to $\mathbb{R} \hookrightarrow S^2V^*$ by $\lambda \to \lambda g$.

Given s, $t \in S^2V^*$, define $s \owedge t \in \mathcal{R}$ by:

$$(s \owedge t)_{ijkl} = s_{ik}t_{jl} - s_{jk}t_{il} - s_{il}t_{jk} + s_{jl}t_{ik}$$

Theorem: $\mathcal{R} = \mathcal{W} \oplus \left(S^2V^* \owedge g\right)$

Proof: Given $R \in \mathcal{R}$, want to find W, P so that

$$R_{ijkl} = W_{ijkl} + (P_{ik}g_{jl} - P_{jk}g_{il} - P_{il}g_{jk} + P_{jl}g_{ik})$$

Take tr: $Ric_{ik} = 0 + nP_{ik} - P_{ik} - P_{ik} + P_j{}^j g_{ik}$

$$= (n-2)P_{ik} + P_j{}^j g_{ik}$$

Again: $S = (n-2)P_j{}^j + nP_j{}^j = 2(n-1)P_j{}^j$, so

$P_j{}^j = \frac{S}{2(n-1)}$ and $P_{ik} = \frac{1}{n-2}\left(Ric_{ik} - \frac{S}{2(n-1)}g_{ik}\right)$ $\square$

Proposition: $\mathcal{W} = \{0\}$ if $n = 3$.

Proof: Choose basis so that $g_{ij} = \delta_{ij}$. Consider components W_{ijkl}. Two of $ijkl$ must be equal, say $i = k = 1$. Now $W_{1j1l} = 0$ unless j, $l \in \{2, 3\}$. $\operatorname{tr} W = 0 \implies W_{1j1l} = -W_{2j2l} - W_{3j3l}$. This vanishes unless $j = l$. So have left W_{1212}, W_{1313}, W_{2323}, and $W_{1212} = -W_{2323}$. Have 3 numbers, any two are negatives, so all must vanish. $\square$

Given g metric on M, take $R =$ curvature tensor of g and decompose. Then W is called the Weyl tensor of g and P the Schouten tensor of g.

How are W, P for g related to $\widehat{W}$, $\widehat{P}$ for $\widehat{g} = e^{2\omega} g$?

Calculate how Levi-Civita connection changes:

$$\widehat{\nabla}_k v_j = \nabla_k v_j - \omega_k v_j - \omega_j v_k + \omega^i v_i g_{jk}. \qquad (\omega_k = \nabla_k \omega)$$

Commute derivatives to see how R changes.

Iterating formula for $\widehat{\nabla}$ gives $\nabla^2 \omega$ terms and $(\nabla \omega)^2$ terms.

End up with:

$$\widehat{R} = e^{2\omega}\,(R + \Lambda \owedge g),$$

where $\Lambda_{ij} = -\omega_{ij} + \omega_i\omega_j - \frac{1}{2}\omega_k\omega^k g_{ij}$. $\Big(\omega_{ij} = \nabla_i\nabla_j\omega\Big)$

Decompose:

$$\widehat{W} + \widehat{P} \owedge \widehat{g} = e^{2\omega}\,[W + (P + \Lambda) \owedge g]$$

Conclusion: $\widehat{W} = e^{2\omega}W, \qquad \widehat{P} = P + \Lambda.$

W is conformally invariant up to the scale factor $e^{2\omega}$. In particular, if $W = 0$, then $\widehat{W} = 0$. Follows that if a metric is locally conformally flat, then its Weyl tensor must vanish. We'll see that the converse is true if $n \geq 4$.

Under conformal change, P transforms by adding Λ, which is expressed in terms of $\nabla^2\omega$ and $(\nabla\omega)^2$.

If $n = 3$, the condition $W = 0$ is automatically true. There is another tensor, the Cotton tensor C, which plays the role of W. Involves one more differentiation. C is also relevant for $n \geq 4$, as we will see.

First see how C arises in the context of the second Bianchi identity. Recall

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0$$

Contract on i, m. Get

$$R^i{}_{jkl,i} - Ric_{jl,k} + Ric_{jk,l} = 0, \text{ or}$$

$$R^i{}_{jkl,i} = Ric_{jl,k} - Ric_{jk,l}$$

Now write everything in terms of W and P via:

$$R = W + P \owedge g, \qquad Ric = (n-2)P + P_i{}^i g$$

Plug in, simplify, get

$$W^i{}_{jkl,i} = (3-n)\left(P_{jk,l} - P_{jl,k}\right)$$

Definition: $C_{jkl} = P_{jk,l} - P_{jl,k}$ Cotton tensor

Have $C_{jkl} = -C_{jlk}$ and $C_{[jkl]} = 0$.

Rewrite Bianchi formula:

$$W^i{}_{jkl,i} = (3-n)C_{jkl}.$$

This is vacuous if $n=3$. But if $n \geq 4$, shows that $C=0$ if $W=0$. In particular $C=0$ if $n \geq 4$ and g is locally conformally flat.

Suggests to see how C transforms conformally. Could use above identity for $n \geq 4$, but not for $n=3$.

Need to calculate $\widehat{\nabla}_l \widehat{P}_{jk}$. Recall $\widehat{P} = P + \Lambda$, or:

$\widehat{P}_{jk} = P_{jk} - \omega_{jk} + \omega_j\omega_k - \frac{1}{2}\omega^i\omega_i g_{jk}$. Differentiate:

$$\widehat{\nabla}_l \widehat{P}_{jk} = \nabla_l \widehat{P}_{jk} + \nabla\omega \cdot \widehat{P}$$

$$= \nabla_l P_{jk} - \omega_{jkl} + \nabla^2\omega \cdot \nabla\omega + \nabla\omega \cdot P + (\nabla\omega)^3$$

Recall that:

$$\widehat{C}_{jkl} = \widehat{\nabla}_l \widehat{P}_{jk} - \widehat{\nabla}_k \widehat{P}_{jl}$$

Obtain

$$\widehat{C}_{jkl} = C_{jkl} - \left(\omega_{jkl} - \omega_{jlk}\right) + \nabla^2\omega\cdot\nabla\omega + \nabla\omega\cdot P + (\nabla\omega)^3$$

But $\omega_{jkl} - \omega_{jlk} = R^i{}_{jkl}\omega_i$. Use $R = W + P \owedge g$. Get

$$\widehat{C}_{jkl} = C_{jkl} - W^i{}_{jkl}\omega_i + \left[\nabla^2\omega\cdot\nabla\omega + \nabla\omega\cdot P + (\nabla\omega)^3\right]$$

Turns out that $[\ldots] = 0$, so that

$$\widehat{C}_{jkl} = C_{jkl} - W^i{}_{jkl}\omega_i$$

If $n = 3$, have $W = 0$, so $\widehat{C} = C$. The Cotton tensor is conformally invariant when $n = 3$! So: if $n = 3$ and g is locally conformally flat, then $C = 0$. When $n \geq 4$, the condition $C = 0$ is not conformally invariant, but it is in the presence of $W = 0$ (which forces $C = 0$ too as we have seen).

Main Theorem: g metric on M

$n \geq 4$. g is locally conformally flat $\iff W_g = 0$

$n = 3$. g is locally conformally flat $\iff C_g = 0$

Proof. Note first that hypotheses imply $W = 0$ **and** $C = 0$ for all n. Try to find $\omega \in C^\infty(M)$ so that $\widehat{g} = e^{2\omega} g$ has $\widehat{P} = 0$. Gives $\widehat{R} = 0$; then use Riemann's criterion for isometric to Euclidean.

Recall $\widehat{P} = P + \Lambda$. So $\widehat{P} = 0$ is $-\Lambda = P$, or

$$\omega_{jk} - \omega_j \omega_k + \tfrac{1}{2}\omega^i \omega_i g_{jk} = P_{jk}.$$

Unknown: single scalar function ω.

$n(n+1)/2$ equations. Very overdetermined.

Recall: Frobenius' Theorem produces solutions of overdetermined systems, if an integrability condition is satisfied.

Frobenius' Theorem. Let $n \geq 2$, $N \geq 1$, $x \in \mathbb{R}^n$.
Unknowns: $u^\alpha(x)$, $1 \leq \alpha \leq N$.
Given smooth functions $F_k^\alpha(x,u)$, consider system:

$$\partial_k u^\alpha(x) = F_k^\alpha(x, u(x)), \qquad 1 \leq k \leq n, \quad 1 \leq \alpha \leq N.$$

Integrability condition: comes from $\partial^2_{kl} u^\alpha = \partial^2_{lk} u^\alpha$

Chain rule gives $\partial^2_{kl} u^\alpha = \partial_l F_k^\alpha + \partial_\beta F_k^\alpha \cdot \partial_l u^\beta$

So require the integrability condition:

$$\partial_l F_k^\alpha + F_l^\beta \partial_\beta F_k^\alpha = \partial_k F_l^\alpha + F_k^\beta \partial_\beta F_l^\alpha.$$

Frobenius' Theorem: if this condition holds identically in (x, u), then the overdetermined system has a solution $u^\alpha(x)$, and $u^\alpha(x_0)$ can be prescribed arbitrarily.

Apply to

$\omega_{jk} - \omega_j\omega_k + \frac{1}{2}\omega^i\omega_i g_{jk} = P_{jk}$. Write as

$$\partial^2_{jk}\omega = \Gamma^i_{jk}\omega_i + \omega_j\omega_k - \tfrac{1}{2}\omega^i\omega_i g_{jk} + P_{jk}.$$

First forget that $\omega_j = \partial_j\omega$. Try to find n functions u_j so that

$$\partial_k u_j = \Gamma^i_{jk}u_i + u_j u_k - \tfrac{1}{2}u^i u_i g_{jk} + P_{jk} \equiv F_{jk}(x, u).$$

If we have u_j, then certainly $\partial_k u_j = \partial_j u_k$, so Poincaré Lemma (special case of Frobenius) implies $u_j = \partial_j\omega$ for some ω, and we are done.

Use Frobenius with $N = n$. Check integrability.

Equation is same as

$$\nabla_k u_j = u_j u_k - \tfrac{1}{2} u^i u_i g_{jk} + P_{jk}.$$

Can calculate integrability condition commuting $\nabla_l \nabla_k$ instead of $\partial_l \partial_k$, using $\nabla_l \nabla_k u_j - \nabla_k \nabla_l u_j = R^i{}_{jkl} u_i$. Can do it directly (or note that this is essentially the same calculation we did before in calculating $\widehat{\nabla}_l \widehat{P}_{jk} - \widehat{\nabla}_k \widehat{P}_{jl}$).

Directly:

$$\nabla_l \nabla_k u_j = u_{j,l} u_k + u_{k,l} u_j - u^i{}_{,l} u_i g_{jk} + P_{jk,l}$$

$$= P_{jk,l} + P \cdot u + u^3.$$

Skew on k, l:

$$R^i{}_{jkl} u_i = C_{jkl} + P \cdot u + u^3.$$

Substitute $R = W + P \owedge g$. Turns out the u^3 terms vanish and the $(P \owedge g) \cdot u$ term on LHS cancels the $P \cdot u$ term on RHS. Thus the integrability condition is precisely

$$C_{jkl} - W^i{}_{jkl} u_i = 0.$$

This is satisfied because $W = 0$ and $C = 0$. $\square$