The degree-diameter problem, and some Cayley graphs on cyclic groups

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Given a graph of diameter k, placing an upper bound d on the degree of a graph's vertices limits the number of vertices that the graph can possess. Indeed, straightforward calculation shows that for $d \geq 3$, such a graph can have order no more than the *Moore bound* $1 + d\frac{(d-1)^k - 1}{d-2} \approx d^k$.

For a given class of graphs and for some fixed diameter and fixed maximal degree, a natural question is thus to determine this maximal possible order exactly. This so-called *degree-diameter problem*, a basic problem in graph theory, is of interest both mathematically and for its applications to optimisation problems for networks (for example in telecommunications) with nodes of limited physical capacity. The focus of this project is the degree-diameter problem for *Cayley graphs*. These highly symmetrical graphs arise naturally from the study of finite groups. Group theory hence provides powerful tools for investigation of the properties of Cayley graphs.

For Cayley graphs on abelian groups, computational evidence for small values of d suggests that for most k and $d > k$ the maximal possible order is close to a Moore-like upper bound, approximately $\frac{d^k}{k!}$. However, the best known constructions valid for infinitely many values of d yield Cayley graphs of significantly smaller order. For the special case of Cayley graphs on cyclic groups, and of diameter two, computational evidence still suggests that the maximal possible order is close to this theoretical bound $1 + d + \frac{1}{2}d^2$. However, the previous best construction valid for infinitely many values of d produced graphs of order approximately $\frac{d^2}{4}$.

Here, inspired by the observation that both the additive and multiplicative groups of the field $\mathbb{Z}_p$ are cyclic, we outline a new construction for Cayley graphs on cyclic groups with diameter two and degree from an infinite set of positive integers d. Graphs in this family have order close to $\frac{d^2}{3}$.