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On length and product of harmonic forms in Kähler geometry

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Abstract Motivated by understanding the limiting case of a certain systolic inequality we study compact Riemannian manifolds having all harmonic 1-forms of constant length. We give complete characterizations as far as Kähler and hyperbolic geometries are concerned. In the second part of the paper, we give algebraic and topological obstructions to the existence of a geometrically 2-formal Kähler metric, at the level of the second cohomology group. A strong interaction with almost Kähler geometry is to be noted. In complex dimension 3, we list all the possible values of the second Betti number of a geometrically 2-formal Kähler metric.

Keywords Harmonic forms · Kähler manifold · Almost Kähler structure

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1 Introduction

Let (M^n, g) be a compact oriented Riemannian manifold. We denote by $\Lambda^p(M)$, $0 \leq p \leq n$ the space of smooth, real valued, p-forms of M. The standard deRham complex

$$\cdots \to \Lambda^p(M) \xrightarrow{d} \Lambda^{p+1}(M) \to \cdots$$

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where d stands for the exterior derivative is then used to introduce the deRham cohomology groups, to be denoted by $H^p_{DR}(M)$. The topological information contained in these cohomology groups may be understood geometrically, using Hodge theory, by means of the isomorphisms

$$H^p_{DR}(M) \equiv \mathcal{H}^p(M, g), \ 0 \leq p \leq n \tag{1.1}$$

Here the space of *harmonic* p-forms of (M^n, g) is defined by

$$\mathcal{H}^p(M, g) = \{\alpha \in \Lambda^p(M) : \Delta\alpha = 0\}.$$

The Laplacian on forms is given by $\Delta = dd^\star + d^\star d$ where $d^\star$ is the formal adjoint of d with respect to the given metric and orientation of M.

In this paper we investigate various notions of "constancy" related to harmonic forms. The first one is introduced by the following

Definition 1.1 Let (M^n, g) be compact and oriented. It is said to satisfy the hypothesis (CL_p) for some $1 \leq p \leq n-1$ iff every harmonic p-form has pointwisely constant norm.

Manifolds satisfying hypothesis (CL_1) appear to be naturally related to a generalized systolic inequality. More precisely, for a compact, orientable Riemannian manifold (N^n, g) with non-vanishing first Betti number one defines the stable 1-systole $stsys_1(g)$ in terms of the stable norm (see [2,3] for details). Let $sys_{n-1}(g)$ be the infimum of the $(n-1)$-volumes of all nonseparating hypersurfaces in N. Then the following systolic inequality, previously established in [7] in the case when the first Betti number equals 1 holds (see [2]) :

$$stsys_1(g) \cdot sys_{n-1}(g) \leq \gamma'_{b_1(N)} \cdot vol(g). \tag{1.2}$$

Here $\gamma'_{b_1(N)}$ is the Bergé-Martinet constant for whose definition we send again the reader to [3]. The important point for us is that it was shown in [3] that if equality in (1.2) occurs then (N^n, g) satisfies the hypothesis (CL_1). Note that the converse is false, as flat tori always satisfy (CL_1) but saturate (1.2) iff they are dual-critical.

Riemannian manifolds (N^n, g) saturating (1.2) have strong geometric properties. It was proved in [3], Thm.1.2, that in this case (N^n, g) is the total space of a Riemannian submersion with minimal fibers to a flat torus, whose projection is actually the Albanese map. Therefore, in the special case when $b_1(N) = n-1$ it follows that the fibers of the Albanese map must be totally geodesic. Using Chern-Weil theory and an argument that reproduces in part that in section 6 of [3], we showed in [14] that the only possible topologies of manifolds N^n which admit a metric satisfying (CL_1) and have $b_1(N) = n-1$ are those of 2-step nilmanifolds with 1-dimensional kernel. Equivalently, the above class of manifolds is parametrized by couples (T, ω) where T is a flat $(n-1)$-torus and ω is a non zero, integral cohomology class on T.

For a compact oriented Riemannian manifold (M, g) we now set

$$H^\star_{DR}(M) = \bigoplus_{p\geq 0} H^p_{DR}(M) \text{ and } \mathcal{H}^\star(M, g) = \bigoplus_{p\geq 0} \mathcal{H}^p(M, g)$$

Whilst $H^\star(M)$ is a graded algebra, in general $\mathcal{H}^\star(M, g)$ is not an algebra with respect to the wedge product operation for there is no reason the isomorphism

(1.1) descends to the level of harmonic forms. Our next definition is related to this fact.

Definition 1.2 Let (M^n, g) be compact and oriented.

(i) The metric g is p-formal for some $1 \leq p \leq n-1$ iff the product of any harmonic p-forms remains harmonic.

(ii) The metric g is formal iff the product of any two harmonic forms remains harmonic.

Following [11] we also set

Definition 1.3 Let M^n be compact and oriented. M^n is geometrically formal iff it admits a formal Riemannian metric.

From a topological viewpoint, geometric formality implies that the rational homotopy type of the manifold is a formal consequence of the cohomology ring [16]. Basic examples are compact Riemannian symmetric spaces. In fact, in the recent [11], it was proved that in dimension 3 and 4 every geometrically formal manifold has the real cohomology algebra of a compact Riemannian symmetric space. In higher dimensions, there are very few general facts known about geometrically formal manifolds; for instance formal metrics satisfy hypothesis (CL_p) for all $1 \leq p \leq n-1$ [11]. By contrast, the class of (non necessarily invariant) metrics on nilmanifolds studied in [14] satisfy hypothesis (CL_p) whenever $1 \leq p \leq n-1$ but none of the p-formality hypothesis. Moreover, it is known that certain classes of homogeneous spaces fail to be geometrically formal for cohomological reasons [12].

In this note we place ourselves in the context of Kähler manifolds and we investigate geometric consequences of the constant length hypothesis and of geometric formality for low degree harmonic forms. Our paper is organized as follows. In section 2 we prove the following.

Theorem 1.1 *Let (M^{2n}, g, J) be a compact Kähler manifold. Then every harmonic 1-form of pointwisely constant length is parallel with respect to the Levi-Civita connection of g. In particular, if g satisfies the hypothesis (CL_1) then (M^{2n}, g, J) is locally the Riemannian (and biholomorphic) product of a compact, simply connected Kähler manifold and of a flat torus.*

Note that the result of Theorem 1.1 is no longer available if instead of having a Kähler structure we require only the presence of an almost Kähler one (see section 2 for an example). It also follows that a compact Kähler manifold which is locally irreducible and not flat never saturates the systolic inequality (1.2). Moreover in section 2 we remark that the length of a harmonic 1-form on a compact hyperbolic manifold cannot be constant; this is actually a consequence of a result in [10] and holds in fact for compact locally symmetric spaces of negative curvature [19]. We propose a different, very simple proof.

The rest of the paper is concerned with the study of obstructions to geometric 2-formality. Note however that every compact Kähler is *topologically* formal by results in [5]. In section 3, we show that harmonic 2-forms of a 2- formal Kähler manifold have a global spectral decomposition and constant eigenvalues. This is enforcing the opinion, already presented in [11] that geometric formality is weakening the notion of Riemannian holonomy reduction. Based upon this we are able to prove the following.

Theorem 1.2 *Let (M^{2n}, g, J) be a compact Kähler manifold and assume that the metric g is 2-formal. Then :*

(i) the space $\mathcal{H}^{1,1}$ of J-invariant harmonic forms is spanned by almost Kähler forms compatible with the metric g.

(ii) the space $\mathcal{H}^2_-$ of J-anti-invariant harmonic two-forms consists only in parallel forms.

By contrast, recall that simply connected, irreducible compact Hermitian symmetric space have second Betti number equal to 1. As an application of Theorems 1.1 and 1.2 we show in section 4 that geometrically formal Kähler manifold having a maximal Betti number are flat. Further consequences of Theorem 1.2 are investigated under various curvature assumptions in section 4. For example, we prove that locally irreducible 8-dimensional hyperkähler manifolds cannot be geometrically formal. In section 5 we study the case of geometrically formal Kähler manifolds of complex dimension 3. We are able to give the possible values of the second Betti number in this situation together with more precisions concerning the algebraic structure of the second cohomology group. We prove :

Theorem 1.3 *Let (M^6, g, J) be a geometrically formal Kähler manifold. If the metric g is locally irreducible then $b_1(M) = b_2^-(M) = 0$. Moreover one has $b_2(M) \leq 3$ and $\mathcal{H}^{1,1}$ is spanned by mutually commuting almost Kähler structures.*

As a final remark, we mention that finding further, first order obstructions to geometric formality in the Kähler case relies on understanding the algebraic structure of the space of harmonic p-forms, $p \geq 3$.

2 The length of harmonic 1-forms

This section will be devoted to the investigation of geometric issues of the existence of a harmonic 1-form of constant length on a compact Kähler manifold. In particular our discussion will lead to the proof of Theorem 1.1. Before proceeding we need to recall some basic material related to a particular class of foliations.

Let (M, g) be a Riemannian manifold equipped with a smooth foliation $\mathcal{F}$ and let us denote by $\mathcal{V}$ the integrable distribution on M induced by $\mathcal{V}$. We consider the splitting

$$TM = \mathcal{V} \oplus H \tag{2.1}$$

where H is the orthogonal complement of $\mathcal{V}$. From now on we will denote by V, W vector fields in $\mathcal{V}$ and by X, Y, Z etc. vector fields in H. Let ∇ be the Levi-Civita connection of the metric g. Recall that H is *totally geodesic* iff $\nabla_X Y$ belongs to H. Foliations $\mathcal{F}$ satisfying this condition -to be assumed, unless otherwise stated, in the rest of our preliminaries- shall be termed *transversally totally geodesic*. Then we note that $\mathcal{F}$ is a particular kind of *Riemannian foliation*, meaning that

$$(\mathcal{L}_V g)(X, Y) = 0.$$

We will present below some basic notions related to this class of foliations, following closely [4,17]. To begin with, let $\overline{\nabla}$ be the orthogonal projection of ∇ onto the splitting (2.1). Then it is easy to verify that $\overline{\nabla}$ defines a metric connection (with torsion) preserving the distributions $\mathcal{V}$ and H.

An important object in our study will be the O'Neill tensor T defined by (see [17], p. 49)

$$T_E F = (\nabla_{E_{\mathcal{V}}} F_{\mathcal{V}})_H + (\nabla_{E_{\mathcal{V}}} F_H)_{\mathcal{V}}$$

whenever E, F belong to TM; here the subscript denotes orthogonal projection on the subspace. It follows that T vanishes on $H \times H$ and $H \times \mathcal{V}$, it is symmetric on $\mathcal{V} \times \mathcal{V}$ (since $\mathcal{V}$ is integrable) and furthermore we have $< T_V X, W >= - < X, T_V W >$.

Based on these definitions it is easy to check that the connections ∇ and $\overline{\nabla}$ are related to the tensor T by :

$$\begin{aligned} &\nabla_X Y = \overline{\nabla}_X Y && \nabla_X V = \overline{\nabla}_X V \\ &\nabla_V W = \overline{\nabla}_V W + T_V W && \nabla_V X = \overline{\nabla}_V X + T_V X. \end{aligned}$$

The last notion needed for our purposes is related to the curvature $\overline{R}$ of the connection $\overline{\nabla}$ defined by $\overline{R}(E, F) = \overline{\nabla}_{[E,F]} - \overline{\nabla}_E \overline{\nabla}_F + \overline{\nabla}_F \overline{\nabla}_E$ for all vector fields E and F of M. Then the *transversal Ricci tensor* $Ric^H : H \to H$ is given by

$$< Ric^H X, Y >= \sum_{e_i \in H} \overline{R}(X, e_i, Y, e_i)$$

for an arbitrary local orthonormal basis $\{e_i\}$ of H.
We consider now a compact Kähler manifold (M^{2n}, g, J) admitting a harmonic 1-form α of pointwisely constant length. Let ζ be the vector field dual to α and consider the distribution H spanned by ζ and $J\zeta$. Moreover, let $\mathcal{V}$ be the orthogonal complement of H in TM. Our starting point toward the proof of Theorem 1.1 is the following

Lemma 2.1 *The distribution $\mathcal{V}$ is integrable and the distribution H is totally geodesic. Moreover we have $Ric^H = 0$.*

Proof If β is a 1-form on M we let J act on β by $(J\beta)X = \beta(JX)$ for all X in TM. Since M is compact we know that $J\alpha$ must be closed. Together with the closedeness of α this leads to the integrability of $\mathcal{V}$. By construction, we have that the splitting $TM = \mathcal{V} \oplus H$ is J-invariant and moreover on a compact Kähler manifold any harmonic 1-form γ is holomorphic, that is

$$\nabla_{JX}\gamma = J\nabla_X \gamma \tag{2.2}$$

whenever X belongs to TM, where ∇ is the Levi-Civita of the metric g. Therefore H is a holomorphic distribution and we use results in [20], to conclude that H is totally geodesic. Furthermore H is actually transversally flat; this follows easily from (2.2) and the fact that α has constant length and leads to the last assertion of our Lemma. □

But the geometry of the foliations satisfying the conditions in Lemma 2.1 can be completely ruled out. In fact we shall prove the slightly more general

Proposition 2.1 *Let (M^{2n}, g, J) be a compact Kähler manifold, supporting a foliation with complex leaves which is transversally totally geodesic and with non-negative transverse Ricci curvature. Then M is locally a Riemannian (and Kähler) product.*

Proof Let $\mathcal{V}$ be the distribution tangent to the leaves of the foliation and H its orthogonal complement. The splitting $TM = \mathcal{V} \oplus H$ is then orthogonal and J-invariant. Let ∇ be the Levi-Civita connection of the metric g. Because (g, J) is Kähler the connection $\overline{\nabla}$ is Hermitian, i.e $\overline{\nabla} J = 0$ and also the O'Neill tensor T of the foliation satisfies $[T_V, J] = 0$. Since H is a totally geodesic distribution we have [4] :

$$< (\overline{\nabla}_X T)(V, W), Y >=< (\overline{\nabla}_Y T)(V, W), X > .$$

It follows that $(\overline{\nabla}_{JX} T)(JV, W) = (\overline{\nabla}_X T)(V, W)$. Derivating in the direction of e_i (here $\{e_i\}$ is an arbitrary local orthonormal basis in H) we get :

$$(\nabla_H^{\star} \nabla_H)T = \frac{1}{2} J \sum_{e_i \in H} \overline{R}(e_i, Je_i).T$$

where ∇_H denotes derivation with respect to $\overline{\nabla}$, in the direction of H. The tensor $\overline{R}$, the curvature tensor of the connection $\overline{\nabla}$, acts on T by

$$(\overline{R}(X, Y).T)(V, W) = \overline{R}(X, Y)[T_V W] - T_{\overline{R}(X,Y)V} W - T_V \overline{R}(X, Y)W.$$

To compute this last term we note that

$$\overline{R}(X, Y, V, W) = R(X, Y, V, W) =< T_W X, T_V Y > - < T_V X, T_W Y >$$

(see [17]). After a short computation this yields

$$\frac{1}{2} J \sum_{e_i \in H} (\overline{R}(e_i, Je_i).T)(V, W) = -(Ric^H (T_V W) + T_{SV} W + T_V SW)$$

where the symmetric endomorphism $S : \mathcal{V} \to \mathcal{V}$ is given by $< SV, W >= \sum_{e_i \in H} < T_V e_i, T_W e_i >$. Taking the scalar product with T implies by means of the positivity of the transversal Ricci curvature that $< (\nabla_H^{\star} \nabla_H)T, T > \leq -2|S|^2$. The vanishing of T (and hence the ∇-parallelism of the splitting $TM = \mathcal{V} \oplus H$) follows now simply by integration over M followed by a positivity argument. □

Remark 2.1

(i) Proposition 2.1 actually holds when relaxing the hypothesis on the foliation to transversal integrability. Since the proof is more involved and not directly related to our present investigations we chose not to present it here.

(ii) A result similar to Proposition 2.1 was proved in [1], under the assumption that Ric has constant positive eigenvalues on $\mathcal{V}$ and H, by making use of Sekigawa's integral formula.

The following example shows that the splitting result in Theorem 1.1 is intimately related to the presence of a Kähler structure and cannot hold in presence of an almost Kähler, non-Kähler, structure.

Example 2.1 Let (N, g) be a 3-dimensional geometrically formal manifold. Many non-symmetric examples are known to exist [11], and in particular we must have $b_1(N) = 1$. Let α be the harmonic 1-form of N whose length equals 1 and let $M = \mathbb{S}^1 \times N$ be endowed with the product metric. It is a simple verification that $\omega = dt \wedge \alpha + \star\alpha$ defines a compatible symplectic form giving M the structure of a compact almost Kähler manifold. But $b_1(M) = 2$ and in general N is a locally irreducible Riemannian manifold.

Higher dimensional examples, endowed with non-homogeneous Riemannian metrics can be obtained by taking the product with $\mathbb{S}^1$ of the class of nilmanifolds studied in [14].

Combining Proposition 2.1 with Lemma 2.1 we are lead directly to the proof of Theorem 1.1. As a direct consequence we obtain :

Corollary 2.1 *Let (M^{2n}, g, J) be a locally irreducible Kähler manifold with $b_1(N) > 0$. Then inequality (1.2) is always strict.*

We equally have :

Corollary 2.2 *Let (M^{2n}, g, J) be a compact Kähler manifold. If g is 1-formal then any harmonic 1-form is parallel for the Levi-Civita connection of g.*

Proof Let α be a harmonic 1-form, with dual vector field ζ. Because α and $J\alpha$ are co-closed a simple computation shows that the vector field dual to $d^\star(\alpha \wedge J\alpha)$ equals $[\zeta, J\zeta]$. Since g is 1-formal it follows that $[\zeta, J\zeta] = 0$ and since α is holomorphic (i.e. it satisfies (2.2)) we arrive at $\nabla_\zeta \zeta = 0$. The closedeness of α implies that α is of constant length hence the proof is completed by applying Theorem 1.1. $\square$

The Riemannian submersion technique used above can be also used to disqualify some other locally symmetric spaces from having harmonic 1-forms of constant length. Also the following proposition happens to provide the answer to an open question in [3] as well as it provides a serious obstruction to the geometric formality of compact hyperbolic manifolds. Note that the result below follows in fact directly from the more general result in [10], asserting the non-existence of Riemannian submersions from compact hyperbolic manifolds. We give the (different) proof mainly because of its simplicity.

Proposition 2.2 *Let (M^{2n}, g), $n \geq 1$ be a compact manifold with constant negative sectional curvature. Then any harmonic 1-form of constant length vanishes.*

Proof Let us suppose that (M, g) admits a non-vanishing harmonic 1-form α of constant length. Let ζ be the vector field dual to α and assume, for simplicity, that

ζ is of unit length. We define now H to be the 1-dimensional distribution spanned by ζ and let $\mathcal{V}$ be its orthogonal complement in TM.

Let ∇ be the Levi-Civita connection of the metric g. The compactness of M imply that α is closed and co-closed. These two equations imply easily (see [3] for a related discussion) that $\nabla_{\zeta}\zeta = 0$ (H is totally geodesic) and furthermore that $\mathcal{V}$ is integrable and also minimal. For the minimality of $\mathcal{V}$ will be quite important for us we note that it is equivalent with α being coclosed. In other words, the splitting $TM = \mathcal{V} \oplus H$ defines a transversally totally geodesic foliation (hence Riemannian) on M and we shall use O'Neill's structure equations for such an object.

Since H is 1-dimensional the O'Neill tensor T can be written as

$$T_V W = < SV, W > \zeta$$

where $S : \mathcal{V} \to \mathcal{V}$ is a symmetric and traceless tensor (because of the integrability and minimality of $\mathcal{V}$). If R denotes the curvature tensor of the Levi-Civita connection, we recall that the following equation holds (see [17]):

$$R(\zeta, V, \zeta, W) = - < (\overline{\nabla}_{\zeta} T)(V, W), \zeta > + < T_V \zeta, T_W \zeta > \tag{2.3}$$

whenever V, W belong to $\mathcal{V}$, where $< T_V \zeta, W > = - < \zeta, T_V W >$. Taking into account that, after a suitable renormalization, we can assume that

$$-R(X, Y, Z, U) = < X, U >< Y, Z > - < X, Z >< Y, U >$$

for all X, Y, Z, U in TM, equation (2.3) can further rewritten as

$$< V, W > = - < (\overline{\nabla}_{\zeta} S)V, W > + < SV, SW > \tag{2.4}$$

for all V, W in $\mathcal{V}$. Starting from $Tr(S) = 0$, an elementary manipulation of (2.4) yields by induction $Tr(S^{2k+1}) = 0$ and $Tr(S^{2k}) = n - 1$ for all natural k. But the last relation implies immediately that $S^2 = 1_H$ and because S is traceless we find that $n - 1$ is even, a contradiction. □

Corollary 2.3 *Let (M^{2n}, g) be compact with constant negative sectional curvature. The following hold :*

(i) the inequality (1.2) is a strict one.

(ii) if g is a formal metric we must have $b_1(M) = 0$.

We finish this section by pointing out the important fact that both results of Proposition 2.2 and Corollary 2.3 hold in the more general context of compact locally symmetric spaces of negative (sectional) curvature in virtue of results in [19].

3 Algebraic obstructions

In this section we shall examine some elementary algebraic obstructions to the existence of a 2-formal Kähler metric. We begin by a brief review of some facts of Kähler geometry, of relevance for our purposes.

For any compact Kähler manifold (M^{2n}, g, J) we can consider the decomposition

$$\Lambda^2(M) = \Lambda^{1,1}(M) \oplus \Lambda^2_-(M)$$

where $\Lambda^2_-(M) = \{\alpha : J\alpha = -\alpha\}$. Here J acts on a two form α by $(J\alpha)(X, Y) = \alpha(JX, JY)$ whenever X, Y belong to TM. The non-standard notation is motivated by the fact that we are working with real-valued differential forms.

We have a further decomposition

$$\Lambda^{1,1}(M) = \Lambda_0^{1,1}(M) \oplus C^\infty(M).\omega$$

where $\omega = g(J\cdot, \cdot)$ is the Kähler form of (g, J) and $\Lambda_0^{1,1}(M)$ is the sub-bundle of $\Lambda^{1,1}(M)$ consisting of primitive forms. We denote now by $\mathcal{H}^p$, $p \geq 0$ the space of harmonic p-forms with respect to the metric g. The previous decompositions have analogues at the level of harmonic forms

$$\mathcal{H}^2 = \mathcal{H}^{1,1} \oplus \mathcal{H}^2_- \tag{3.1}$$

and

$$\mathcal{H}^{1,1} = \mathcal{H}_0^{1,1} \oplus \mathbb{R}.\omega \tag{3.2}$$

with the obvious notational conventions. Moreover, we will denote by $h^{1,1}$ the dimension of $\mathcal{H}^{1,1}$ and by b_2^- that of $\mathcal{H}^2_-$.

Let $\mathcal{S}$ the space of symmetric and J-invariant endomorphisms S of TM having the property that $< SJ\cdot, \cdot >$ belongs to $\mathcal{H}^{1,1}$. Clearly, $\mathcal{S}$ and $\mathcal{H}^{1,1}$ are isomorphic and its worthwhile to note that all elements of $\mathcal{S}$ have constant trace, in virtue of (3.2). In the same way we define the space $\mathcal{A}$ as the space of skew-symmetric, J-anti-commuting endomorphisms of TM which are associated to an element of $\mathcal{H}^2_-$. Note the important fact that $J \cdot \mathcal{A} \subseteq \mathcal{A}$.

Another aspect of Kähler geometry, of particular significance for us, is that the operator L defined as exterior multiplication with the Kähler form preserves the space of the harmonic forms of the manifold. This is a consequence of the fact that L commutes with the Laplacian acting on forms (see [6]). Since the Laplacian is a self-adjoint operator it also follows that $L^\star$, the adjoint of L, preserves the space of harmonic forms.

We now give a first set of elementary algebraic obstructions to the presence of a 2-formal Kähler metric. If A and B are endomorphisms of some vector bundle over a manifold we will denote by $\{A, B\} = AB + BA$ their anti-commutator. The whole discussion in this section will be based on the lemma below.

Lemma 3.1 *Let (M^{2n}, g, J) be a Kähler manifold such that the metric g is 2-formal. The following hold :*

$$\{\mathcal{S}, \mathcal{S}\} \subseteq \mathcal{S}. \tag{3.3}$$

and

$$\{\mathcal{A}, \mathcal{A}\} \subseteq \mathcal{S}. \tag{3.4}$$

We also have

$$\{\mathcal{S}, \mathcal{A}\} \subseteq \mathcal{A}. \tag{3.5}$$

Proof Let us prove (3.3). Consider α and β in $\mathcal{H}^{1,1}$ with associated symmetrics S_1 and S_2. We fix $\{e_i, 1 \le i \le 2n\}$ be a local orthonormal basis and write :

$$L^{\star}(\alpha \wedge \beta) = \frac{1}{2}\sum_{i=1}^{2n} Je_i \lrcorner (e_i \lrcorner (\alpha \wedge \beta)).$$

But

$$\begin{aligned} Je_i \lrcorner (e_i \lrcorner (\alpha \wedge \beta)) &= Je_i \lrcorner ((e_i \lrcorner \alpha) \wedge \beta + \alpha \wedge (e_i \lrcorner \beta)) \\ &= (Je_i \lrcorner e_i \lrcorner \alpha) \cdot \beta - (e_i \lrcorner \alpha) \wedge (Je_i \lrcorner \beta) + (Je_i \lrcorner \alpha) \wedge (e_i \lrcorner \beta) \\ &\quad + \alpha \cdot (Je_i \lrcorner e_i \lrcorner \beta). \end{aligned}$$

Assuming the basis to be Hermitian we get $L^{\star}(\alpha \wedge \beta) = L^{\star}\alpha \cdot \beta + \alpha \cdot L^{\star}\beta - \gamma$ where

$$\gamma = \sum_{i=1}^{2n} (e_i \lrcorner \alpha) \wedge (Je_i \lrcorner \beta).$$

Now a short computation shows that $\gamma = < \{S_1, S_2\}J\cdot, \cdot >$ and since $L^{\star}(\alpha \wedge \beta)$ belongs to $\mathcal{H}^{1,1}$ whilst $L^{\star}\alpha, L^{\star}\beta$ are constants we get that γ is equally in $\mathcal{H}^{1,1}$ and the proof of (3.3) is finished. The proof of (3.4) and (3.5) are completely analogous and will be left to the reader. □

Corollary 3.1 *Let (M^{2n}, g, J) be a compact Kähler manifold with a 2-formal Riemannian metric. Then :*

(i) The length of any harmonic 2-form is constant over M;

(ii) If α in $\mathcal{H}_0^{1,1}$ has vanishing square, then α is necessarily 0.

Proof Indeed, if α belongs to $\mathcal{H}^{1,1}$ or to $\mathcal{H}^2_-$ then we saw that S^2 belongs to $\mathcal{S}$ and all elements of $\mathcal{S}$ have constant trace. But the trace of S^2 equals the squared norm of α. The other statement is straightforward. □

Proposition 3.1 *Suppose that (M^{2n}, g, J) is a compact Kähler manifold such that the metric g is 2-formal. Let α belong to $\mathcal{H}^{1,1}$ and let S in $\mathcal{S}$ be the associated symmetric endomorphism of TM. Then :*

(i) The eigenvalues of S are constant with eigenbundles of constant rank.

(ii) If $\lambda_i, 1 \le i \le p$ are (the pairwise distinct)

eigenvalues of S we have an orthogonal and J-invariant decomposition

$$TM = \bigoplus_{j=1}^{p} E_i$$

where E_i is the eigenspace of S corresponding to λ_i. Furthermore, for all $1 \le i \le p$ the distributions E_i and $\hat{E}_i = \bigoplus_{j=1, j \ne i}^{p} E_i$ are integrable.

Proof (1) From (3.3) we deduce that S^k belongs to $\mathcal{S}$ for all k in $\mathbb{N}$. As $\mathcal{S}$ is finite dimensional there exists P in $\mathbb{R}[X]$ such that $P(S) = 0$. Since S is symmetric, P can be supposed to have only real roots and again by the symmetry of S we can moreover assume that all these roots are simple. Let λ_i, $1 \le i \le p$ be these (pairwise distinct) roots and let m_i be the dimension of the corresponding eigenbundle. To see that m_i, $1 \le i \le p$ are constant over M we use the fact that S^k belongs to $\mathcal{S}$ for all k in $\mathbb{N}$ in order to deduce that $Tr(S^k) = c_k$ for some constant c_k and for all natural k. In other words

$$\sum_{i=1}^{p} m_i \lambda_i^k = c_k$$

for all k in $\mathbb{N}$. Solving this Vandermonde system leads to the constancy of the functions m_i, $1 \le i \le p$.

(2) Let ω^i be the orthogonal projection of the Kähler form ω on E_i, $1 \le i \le p$. Then $\alpha = \sum_{i=1}^{p} \lambda_i \omega^i$ and moreover, by (3.3) we obtain that

$$\sum_{i=1}^{p} \lambda_i^k \omega^i$$

belongs to $\mathcal{H}^{1,1}$ for all natural k. We assume now that the eigenvalues of S are ordered by $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_p|$. We divide by $|\lambda_p|^k$ and make $k \to \infty$. It follows that ω_p belongs to $\mathcal{H}^{1,1}$. By induction the same holds for ω^i, $2 \le i \le p$. If $\lambda_1 \neq 0$, the form ω^1 is trivially in $\mathcal{H}^{1,1}$ and if $\lambda_1 = 0$ the same is true since $\omega = \sum\limits_{i=1}^{2n} \omega^i$.

Therefore ω^i, $1 \le i \le p$ are all closed. Fix $1 \le i \le p$ and consider the decomposition $TM = E_i \oplus \hat{E}_i$. Let X, Y be in E_i and V in $\hat{E}_i$. A straightforward computation yields to $(\nabla_X \omega^i)(Y, V) = - < \nabla_X Y, JV >$, $(\nabla_Y \omega^i)(X, V) = - < \nabla_Y X, JV >$ and $(\nabla_V \omega^i)(X, Y) = 0$. Now the closedeness of ω^i ensures the integrability of E_i. That of $\hat{E}_i$ is proved in a similar way, by computing $(d\omega^i)(V, W, X)$ with W in $\hat{E}_i$. □

As an immediate consequence of Proposition 3.1 we obtain :

Corollary 3.2 *Let (M^{2n}, g, J) be a compact Kähler manifold such that g is 2-formal. Then any element of $\mathcal{H}^{1,1}$ can be uniquely written as a linear combination of g-compatible symplectic forms.*

Proof Proposition 3.1 actually says that any 2-form α in $\mathcal{H}^{1,1}$ can be written as $\alpha = \sum_{j=1}^{p} \lambda_i \omega^i$ where ω^i denotes the projection of the Kähler form ω on E_i, $1 \le i \le p$. Moreover the forms ω^i belong to $\mathcal{H}^{1,1}$ for all $1 \le i \le p$. Now, for $1 \le k \le p$ we define an almost complex structure J_k on TM by setting

$$J_k = J \text{ on } \hat{E}_k, \text{ and } J_k = -J \text{ on } E_k.$$

An easy consequence of the integrability of *both* E_k and $\hat{E}_k$ is that (g, J_k) are almost Kähler structures (i.e. the corresponding Kähler forms $\Omega_k = g(J_k\cdot, \cdot)$ are closed) commuting with J, for any $1 \le k \le p$. Note that J_k is integrable, i.e. (g, J_k) is a Kähler structure iff E_k is parallel with respect to the Levi-Civita connection. To finish the proof of the Corollary it suffices to note that

$$\omega^k = \frac{1}{2}\omega - \frac{1}{2}\Omega_k$$

for all $1 \le k \le p$. □

Therefore the proof of part (i) of Theorem 1.2 is now complete.

4 More on Hodge numbers

The aim of this section is to provide some information about the Hodge numbers of a geometrically formal Kähler manifold. We begin with the following simple observation.

Proposition 4.1 *Let (M^{2n}, g, J) be a compact Kähler manifold such that the metric g is formal. Then $h^{0,n}(M) \le 1$ and equality holds iff g is a Ricci flat metric.*

Proof Because any harmonic form is of constant length the Hodge numbers of (M^{2n}, g, J) are bounded by the dimensions of their corresponding vector bundles hence $h^{0,n}(M) \le 1$. If equality holds, it follows that the canonical bundle of (M^{2n}, g, J) is trivialized by a harmonic $(0, n)$-form of constant length and this leads in the standard way to the vanishing of the Ricci tensor. □

We investigate now the structure of J-anti-invariant harmonic 2-forms.

Proposition 4.2 *Let (M^{2n}, g, J) be a compact Kähler manifold such that the metric g is 2-formal. Then :*

(i) Any non-zero element of $\mathcal{H}^2_-$ induces in a canonical way a local splitting of M as the Riemannian product of a compact Kähler manifold M_1 and a compact hyperkähler manifold M_2.

(ii) $\mathcal{H}^2_-$ consists only in parallel forms.

Proof (i) Let α be in $\mathcal{H}^2_-$ be non-zero and let A in $\mathcal{A}$ be its associated endomorphism. Then A^2 belongs to $\mathcal{S}$ by (3.4) and using proposition 3.1 we obtain a J-invariant and orthogonal decomposition

$$TM = \bigoplus_{i=1}^{p} E_i$$

where E_i are eigenspaces of A^2 for the (constant) eigenvalues $\mu_i \le 0, 1 \le i \le p$. Now using again (3.4) we get that A^{2k} belongs to $\mathcal{S}$ and further, by (3.5) that A^{2k+1} is in $\mathcal{A}$ for all k in $\mathbb{N}$. Let A_i, $1 \le i \le p$ be the orthogonal projections of A on E_i. An argument similar to the proof of Proposition 3.1, (i) shows that

A_i are in $\mathcal{A}$ for all $1 \leq i \leq p$. Equivalently, the forms α^i, associated to A_i are in $\mathcal{H}^2_-$ for all $1 \leq i \leq p$ and therefore have to be closed. We will show now that one can reduces to the case when A has no kernel. Indeed, let us assume that A has non-empty kernel, that is A^2 has a zero eigenvalue, say μ_1. Set $\mathcal{V} = E_1$ and $H = \hat{E}_1$. Then the endomorphism $F = \sum_{i=2}^{p} \frac{1}{\sqrt{-\mu_i}} A^i$, an element of $\mathcal{A}$, vanishes on $\mathcal{V}$ and defines an almost complex structure I on H, compatible with g and such that $IJ + JI = 0$. Since the 2-form form associated to F is closed we get :

$$< (\nabla_{U_1} F)U_2, U_3 > - < (\nabla_{U_2} F)U_1, U_3 > + < (\nabla_{U_3} F)U_1, U_2 >= 0 \quad (4.1)$$

for all U_j, $1 \leq j \leq 3$ in TM. Let $\overline{\nabla}$ be the metric connection leaving $\mathcal{V}$ and H parallel. Then $\nabla_X Y = \overline{\nabla}_X Y + A_X Y$ for all X, Y in H where the O'Neill-type tensor $A : H \times H \to \mathcal{V}$ is the obstruction to the distribution H to be totally geodesic. Taking $U_1 = X$, $U_2 = Y$ and $U_3 = V$ in (4.1) with X, Y in H and V in $\mathcal{V}$ we get :

$$< A_X IY - A_Y(IX), V > + < (\overline{\nabla}_V I)X, Y >= 0. \quad (4.2)$$

Since the connection $\overline{\nabla}$ is metric and $I^2 = -1$ on H it follows that $\langle (\overline{\nabla}_V I)IX, IY \rangle = -\langle (\overline{\nabla}_V I)X, Y \rangle$. Therefore, changing X in IX and Y in IY in (4.2) and summing the result with (4.2) we obtain $A_X IY - A_Y(IX) - A_{IX}(Y) + A_{IY} X = 0$. But A is symmetric as H is integrable (see Proposition 3.1, (ii)) hence

$$A_X(IY) = A_Y(IX) \quad (4.3)$$

for all X, Y in H. As (g, J) is Kähler and, by construction both $\mathcal{V}$ and H are J-invariant, we are lead to $A_X(JY) = JA_X Y$ for all X, Y in H. Taking this into account and replacing Y by JY in (4.3) yields after a standard manipulation the vanishing of A.

We showed that H is a totally geodesic distribution hence the foliation induced by $\mathcal{V}$ is a Riemannian one. Now, on any integral manifold of H, with respect to the induced metric, the triple $I, J, K = IJ$ induces a family of almost complex structures satisfying the quaternionic identities and with closed associated Kähler forms. Then a well known Lemma due to Hitchin [8] implies that the metric is hyperkähler and hence Ricci flat. It follows that the transversal Ricci curvature Ric^H of the Riemannian foliation induced by $\mathcal{V}$ vanishes and using Proposition 2.1 we obtain that $\mathcal{V}$ is also totally geodesic, hence the desired splitting.

(ii) It suffices to work on the compact, Ricci flat manifold M_2 where A has no kernel. Then any of the commuting almost Kähler structures induced by A^2 have to be Kähler by a theorem of Sekigawa [18]. This means that the spaces E_i are all parallel and on each of them A is a multiple of a Kähler structure (anti-commuting with J). This implies immediately the parallelism of α. □

In particular Theorem 1.2 is now completely proved. It can be used to refine, in the Kähler case, the Betti number estimates $b_p(N) \leq b_p(T^n)$ known to hold (see [12]) for an arbitrary geometrically formal manifold (N^n, h).

Corollary 4.1 *Let (M^{2n}, g, J) be Kähler such the metric g is formal. If (M^{2n}, g) is locally irreducible then $b_1(M) = 0$ and $b_{2p+1}(M) \leq C_{2n}^{2p+1} - 2n$ for all $p \geq 1$.*

Proof This is an immediate consequence of Theorem 1.1 and of the Lefschetz decomposition (see [6]) of the harmonic forms of a Kähler manifold . □

The previous Corollary can also be reformulated to say that if a geometrically formal Kähler manifold has a maximal Betti number of odd degree then the metric is a flat one. More generally we have :

Corollary 4.2 *Let* (M^{2n}, g, J) *be Kähler such that the metric* g *is formal. If there exists* $1 \le p \le 2n-1$ *such that* $b_p(M) = b_p(\mathbb{T}^{2n})$ *then* g *is flat metric.*

Proof By Corollary 4.1 it suffices to study the case $p = 2q$ and by Hodge duality we may also suppose that $p \le n$. Using the fact that harmonic forms of g are of constant length and the Lefschetz decomposition of a Kähler manifold, we are lead to $b_2(M) = b_2(\mathbb{T}^n)$ and further to $b_2^-(M) = b_2^-(\mathbb{T}^n)$. But in the case of the torus it is an algebraic fact that $\{\mathcal{A}, \mathcal{A}\} = \mathcal{S}$ hence in view of the parallelism of J-anti-invariant harmonic 2-forms $\mathcal{H}^{1,1}$ equally consists of parallel forms. We have therefore a framing of $\Lambda^2(M)$ by parallel two-forms and this implies in a standard way the desired result. □

The rest of the section will be consecrated to explore a number of consequences of Theorem 1.2 under various curvature assumptions. First of all we have :

Corollary 4.3 *Suppose that* (M^{2n}, g, J) *is a compact quotient of the complex hyperbolic space, endowed with its canonical Kähler metric. If* g *is a formal metric then* $b_1(M) = 0$ *and* $b_2(M) = 1$.

Proof Since (M^{2n}, g, J) is locally irreducible, it follows that $b_1(M) = 0$ by Theorem 1.1 and also that $b_2^-(M) = 0$ by Proposition 4.2. Now using Corollary 3.2 and the fact (see [13]) that on (M^{2n}, g, J) every orthogonal, J-commuting, almost Kähler structure has to be Kähler and therefore a multiple of J we are lead to $h^{1,1} = 1$. □

We investigate now the incidence of having constant scalar curvature on 2-formal Kähler metrics.

Theorem 4.1 *Let* (M^{2n}, g, J) *be a compact Kähler manifold. Assume that the metric* g *is* 2*-formal and locally irreducible. Then :*

(i) if the scalar curvature of g *is constant then the eigenvalues of* Ric *are constant (together with their multiplicities) over* M.

(ii) If the scalar curvature is constant and $Ric_g \ge 0$ *then* $h^{1,1} = 1$. *Moreover, under these assumptions* g *is an Einstein metric.*

Proof (i) If the scalar curvature is constant, the Ricci form is harmonic and the result follows by Proposition 3.1, (i).

(ii) In this case the Ricci tensor has only constant and non-negative eigenvalues, by (i). Using Proposition 3.1, (ii) we can always rescale, by an argument similar to Lemma 2.2, page 774, in [1], the metric along the eigenbundles of Ric in order to get a Kähler metric with 2 constant and non-negative eigenvalues. Then the splitting result of [1] asserts that every g-compatible almost Kähler structure, commuting with J is in fact Kähler. Therefore, by local irreducibility, $h^{1,1} = 1$ leading further, by (i) to the fact that g is Einstein. □

An immediate consequence of Proposition 4.2 and Theorem 4.1, (ii) is the following.

Theorem 4.2 *Let (M^{2n}, g, J) be a compact Kähler manifold such that g is 2-formal and locally irreducible. Then either*

(i) $b_2^-(M) = 0$
or
(ii) $b_2^-(M) = 2$ *and (M, g) is a hyperkähler manifold. Moreover, in this case we must have $h^{1,1} = 1$ and thus $b_2(M) = 3$.*

Thus, a naturally arising question is to decide whether a hyperkähler metric can be geometrically formal. This seems quite unlikely from the perspective that known examples of compact hyperkähler manifolds (see [9] for an account) have second Betti number greater than 3. However, we were unable to prove that hyperkähler metrics cannot be geometrically formal, except in lows dimensions, as the following shows :

Proposition 4.3 *They are no geometrically formal and locally irreducible hyperkähler manifolds in dimensions* 4 *and* 8.

Proof In dimension 4 this was proven in [11]. To prove the statement in dimension 8 we need to recall some facts about the topology of hyperkähler manifolds. Thus, let Z^{4m} be a hyperKähler manifold. It was proven in [15] that the Betti numbers of Z satisfy the following remarkable relation

$$3P''(-1) = m(12m - 5)P(-1) \tag{4.4}$$

where $P(t) = \sum_{k=0}^{4m} b_k(Z)t^k$ is the Poincaré polynomial.

Suppose now that (M^8, g, I, J, K) is a hyperkähler manifold such that the metric g is formal. Then $b_1(M) = 0$ and $b_2(M) = 3$ hence after an easy computation (4.4) becomes

$$b_3(M) + b_4(M) = 76.$$

To obtain a contradiction we will produce estimates of the Betti numbers b_3 and b_4. We denote by $\omega_I, \omega_J, \omega_K$ the corresponding Kähler forms. Since $b_1(M) = 0$, any harmonic 3-form lives in the orthogonal complement of $\{\alpha \wedge \omega_I + \beta \wedge \omega_J + \gamma \wedge \omega_K : \alpha, \beta, \gamma \in \Lambda^1(M)\}$. Since any harmonic 3-form has constant length we get $b_3(M) \leq C_8^3 - 3 \cdot 8 = 32$. Let us denote by $\Lambda_o^2(M)$ the subbundle of $\Lambda^2(M)$ consisting of forms orthogonal to $\omega_I, \omega_J, \omega_K$. This bundle do not contain any harmonic form (since the second cohomology group is generated by the hyperkähler forms) and therefore any harmonic 4-form must be orthogonal to E, the subbundle of $\Lambda^4(M)$ generated by $\{\omega_I \wedge \alpha + \beta \wedge \omega_J + \gamma \wedge \omega_K : \alpha, \beta, \gamma \in \Lambda_o^2(M)\}$. Now, a simple computation shows that $\omega_I \wedge \alpha$ and $\omega_J \wedge \beta$ have to be orthogonal for all β in $\Lambda_o^2(M)$ and α in $\Lambda_K^{1,1}(M) \cap \Lambda_o^2(M)$. Therefore the rank of E is greater than $(C_8^2 - 3) + 15 = 40$ and this implies, again by the fact that harmonic 4-forms are of constant length, that $b_4(M) \leq C_8^4 - 40 = 30$. We found that $b_3(M) + b_4(M) \leq 62$, an obvious impossibility. □

5 6-dimensions

We consider in this section a 6-dimensional Kähler manifold (M^6, g, J) such that the metric g is formal and locally irreducible. Our aim is to give the possible values for the second Betti number and also to explore the algebraic structure of the second cohomology group. This will also lead to the proof of Theorem 1.3. We first prove:

Lemma 5.1 *Let (M^6, g, J) be a geometrically formal, Kähler manifold. If g is locally irreducible then :*

(i) $b_1(M) = b_2^-(M) = 0$.

(ii) we have either $Td(M) = 1$ or $Td(M) = 0$. If the last case occurs then g is Ricci flat and $b_2(M) = 1$.

Proof (i) Direct consequence of Theorem 1.1 and Proposition 4.2, (i).

(ii) As $h^{0,2}(M) = 0$ the use of Riemann-Roch tells us that $Td(M) = 1 - h^{0,3}$. Using Proposition 4.1 we get that either $h^{0,3}(M) = 0$ (and hence $Td(M) = 1$) or $h^{0,3}(M) = 1$ (thus $Td(M) = 0$) and g is Ricci flat. But if g is Ricci flat one uses Theorem 4.1, (ii) to get that $b_2(M) = 1$ and the proof is finished. □

We shall now study commutation rules inside $\mathcal{H}^{1,1}$. Our methods will be mainly topological and shall rely on the following :

Proposition 5.1 *Let (M^6, g, J) be a compact Kähler manifold. Suppose that we have an orthogonal and J-invariant decomposition $TM = \mathcal{V} \oplus H$ where $\mathcal{V}$ is of real rank 2. If I_1 and I_1 are almost complex structures on H which are compatible with g and such that $\{I_1, I_2\} = 0$ and $[I_k, J] = 0$, $k = 1, 2$ then $\chi(M)$ is divisible by 12.*

Proof We have $24Td(M, J) = c_1(M, J)c_2(M, J)$. Or $c_1(M, J) = c_1(\mathcal{V}) + c_1(H)$ and $c_2(M, J) = c_2(H) + c_1(\mathcal{V})c_1(H)$ where the bundles $\mathcal{V}$ and H are endowed with the complex structure induced by J. Then :

$$24Td(M, J) = c_1(\mathcal{V})c_2(H) + c_1(H)c_2(H) + c_1^2(\mathcal{V})c_1(H) + c_1(\mathcal{V})c_1^2(H). \tag{5.1}$$

Consider now the almost complex structure J_0 which equals $-J$ on $\mathcal{V}$ and J on H. Taking into account that J_0 is inducing the orientation opposite to that induced by J we get as before :

$$-24Td(M, J_0) = -c_1(\mathcal{V})c_2(H) + c_1(H)c_2(H) + c_1^2(\mathcal{V})c_1(H) - c_1(\mathcal{V})c_1^2(H).$$

Subtracting we obtain :

$$12(Td(M, J) + Td(M, J_0)) = c_1(\mathcal{V})(c_2(H) + c_1^2(H)).$$

Consider now the orthogonal involution $\sigma = JI_1$ of H. It can be used to obtain an orthogonal and J-invariant decomposition $H = H^+ \oplus H^-$ where $H^\pm$ are the ± 1-eigenspaces of σ. Since $\{\sigma, I_2\} = 0$ we have that $H^- = I_2H^+$ hence I_2 induces a complex isomorphism between H^+ and H^-. It follows that $c_1(H^+) = c_1(H^-)$ and therefore $c_1(H) = 2c_1(H^+)$ and $c_2(H) = c_1^2(H^+)$. We deduce that $c_1^2(H) = 4c_2(H)$ and further :

$$12(Td(M, J) + Td(M, J_0)) = 5c_1(\mathcal{V})c_2(H).$$

Now $\chi(M)=c_3(M, J)=c_1(\mathcal{V})c_2(H)$ and this leads to $12(Td(M, J)+Td(M, J_0)) = 5\chi(M)$, finishing the proof of the proposition. □

In order to prove Theorem 1.3 we need a number of preliminary results.

Lemma 5.2 *Let (M^6, g, J) be a geometrically formal Kähler manifold and let σ in $\mathcal{S}$ be an involution. Then :*

(i) *We have $\mathcal{S} = C_\sigma \oplus A_\sigma$ where we defined $C_\sigma = \{S \in \mathcal{S} : [S, \sigma] = 0\}$ and $A_\sigma = \{S \in \mathcal{S} : \{S, \sigma\} = 0\}$.*

(ii) *the dimension of A_σ is less or equal to* 1.

Proof (i) Let S be in $\mathcal{S}$ and decompose $S = S_1 + S_2$ where S_1 and S_2 are commuting resp. anti-commuting with σ. To prove the result it is enough to see that S_1 belongs to $\mathcal{S}$. But using (3.3) we obtain that $S\sigma + \sigma S$ is in $\mathcal{S}$ and again by (3.3), $\{\sigma, S\sigma + \sigma S\} = 2(S + \sigma S\sigma) = 4S_1$ belongs to $\mathcal{S}$ and the proof is finished.

(ii) Assume that g is not Ricci flat because otherwise $b_2(M) = 1$ (see Lemma 5.1, (ii)) and there is nothing to prove. Let $TM = \mathcal{V} \oplus H$ be the orthogonal and J-invariant decomposition of TM in the -1 and 1-eigenspaces of σ, and let us assume that $\mathcal{V}$ has real rank 2. Suppose that A_σ is non-empty and let S be a non-vanishing element of A_σ. Then $S(\mathcal{V}) \subseteq H$ and hence S defines an J-invariant isomorphism from $\mathcal{V}$ to its image H_1. If S' in A_σ is orthogonal to S then it defines a J-invariant isomorphism from $\mathcal{V}$ to H_2 ,the orthogonal complement of H_1 in H. In other words we have a decomposition $TM = \mathcal{V} \oplus H_1 \oplus H_2$ in J-isomorphic bundles. Let us denote by h the first Chern class of $\mathcal{V}$. Of course, $c_1(H_1) = c_1(H_2) = h$. From (5.1) we deduce easily that $24Td(M, J) = 9h^3$ and moreover $\chi(M) = c_3(M) = h^3$. But using Lemma 5.1, (ii) we infer that $Td(M) = 1$ and this leads to $24 = 9\chi(M)$ and since this equation has no integer solution we obtained a contradiction with the existence of S', hence finishing the proof of the lemma.

Remark 5.1 From the above proof, we see that Lemma 5.2 continues to hold for 2-formal metrics provided that the Todd genus $Td(M, J)$ is not divisible by 3.

Now we need another auxiliary result in order to get some precisions concerning the structure of harmonic 3-forms in some special cases. By contrast with the previous remark, from now on we will use the formality hypothesis in a crucial way.

Lemma 5.3 *(i) Let (M^6, g) be geometrically formal and let α be a harmonic 2-form. If L_α is the exterior multiplication with α then its adjoint, to be denoted by $L_\alpha^\star$ preserves the space of harmonic forms.*

(ii) Let (M^6, g, J) be an almost Kähler manifold, which is also geometrically formal. The for all primitive α in $\mathcal{H}^3$ we have that $J\alpha$ is also harmonic. Here J acts on a 3-form β by $(J\beta)(X, Y, Z) = \beta(JX, JY, JZ)$ whenever X, Y, Z are in TM.

Proof (i) Using the definition of the Hodge star operator we see that $L_{\alpha}^{\star}$ is up to a sign equal to the composition $\star L_{\alpha} \star$ and the result follows by the formality hypothesis.

(ii) It suffices to note, using for instance the definition of the Hodge-star operator, that $\star\alpha = J\alpha$ and the result follows. □

Our last preparatory result consists in giving a bound on the third Betti number in case that, for some involution σ of $\mathcal{S}$ the space A_σ is non-empty.

Lemma 5.4 *Let (M^6, g, J) be a geometrically formal Kähler manifold and let σ in $\mathcal{S}$ be an involution. If A_σ is 1-dimensional then :*

(i) $b_3(M) \leq 6$

(ii) $12|\chi(M)$.

Proof (i) Let $TM = \mathcal{V} \oplus H$ be the spectral decomposition of σ into $\pm$-eigenbundles and suppose furthermore that the real rank of $\mathcal{V}$ equals 2. It is easy to verify that we are then in the situation of Proposition 5.1, i.e. we have almost complex structures I_1, I_2 on H which are mutually anti-commuting and commuting with J. Let us consider now α a harmonic 3-form. If ω_1 is the orthogonal projection of ω on $\mathcal{V}$ we have that $L_{\omega_1}^{\star}\alpha = 0$ hence we can decompose $\alpha = \alpha_1 + \alpha_2$ where α_1 belongs to $\Lambda^1(\mathcal{V}) \otimes \Lambda^2(H)$ and α_2 is in $\Lambda^3(H)$. Then $J\alpha = J\alpha_1 + J\alpha_2$ belongs to $\mathcal{H}^3$ and also $J_1\alpha = J_1\alpha_1 + J_1\alpha_2 = -J\alpha_1 + J\alpha_2$ belongs to $\mathcal{H}^3$ where J_1 is the almost Kähler structure on M acting as $-J$ on $\mathcal{V}$ and as J on H. It follows that $J\alpha_2$ is in H and thus $\alpha_2 = -J(J\alpha_2)$ is a harmonic 3-form. If ω_2 is the orthogonal projection of ω on H then $L_{\omega_2}^{\star}\alpha_2 = 0$ meaning that α_2 vanishes.

We showed that any harmonic 3-form α belongs to $\Lambda^1(\mathcal{V}) \otimes \Lambda^2(H)$ and in order to prove the lemma, we have to take into account the hypothesis that A_σ is 1-dimensional. Look at α as a vector bundle morphism $\alpha : \Lambda^2(H) \to \mathcal{V}$. Since $L_{\omega_2}^{\star}\alpha = L_{\omega_{I_1}}^{\star}\alpha = L_{\omega_{I_2}}^{\star}\alpha = 0$ we find that α is in fact defined from $\Lambda_{-}^2(H) \oplus E$ to $\mathcal{V}$ where E is the orthogonal complement of the span of $\omega_{I_1}, \omega_{I_2}$ in $\Lambda_0^{1,1}(H)$. But all harmonic 3-forms are of constant length hence the third Betti number cannot exceed 6, the rank of the vector bundle $(\Lambda_{-}^2(H) \oplus E) \otimes \mathcal{V}$.

(ii) follows immediately from Proposition 5.1. □

We are now in position to give the

Proof of Theorem 1.3. By Lemma 5.1 and Corollary 3.2 we only need to prove the bound on $b_2(M)$. We can suppose that $b_2(M) \geq 2$, otherwise there is nothing to prove. It follows that $\mathcal{S}$ contains a non-trivial involution σ of TM whose ± 1-

eigenspaces will be denoted by $\mathcal{V}$ and H. Moreover we can suppose that $\mathcal{V}$ has real rank 2. Let us suppose now that $dim_{\mathbb{R}} A_\sigma = 1$ and let S in A_σ be non-zero. Then we have $TM = \mathcal{V} \oplus H_1 \oplus H_2$ with $H_1 = S(\mathcal{V})$ and H_2 the orthogonal complement of H_1 in H. Let now S_1 be in C_σ. Then $SS_1 + S_1 S$ is in A_σ and then $SS_1 + S_1 S = \lambda S$ where λ is a real constant. It follows by some simple algebraic considerations that S_1 preserves $H_i, i = 1, 2$ and since the latter are 2-dimensional we obtain that C_σ has dimension 3 hence $b_2(M) = 4$ by Lemma 5.2, (i). By Lemma 5.4, (i) and the fact that (M^6, g, J) is Kähler, the only possibilities for $b_3(M)$ are $0, 2, 4, 6$. Therefore, the possible values of $\chi(M) = 2 + 2b_2(M) - b_3(M) = 10 - b_3(M)$ are 10, 8, 6, 4 a fact which in contradiction with the fact that $\chi(M)$ is divisible by 12 (cf. Lemma 5.4, (ii)).

We showed that for any involution in $\mathcal{S}$ the space A_σ vanishes and this implies that any two elements of $\mathcal{S}$ must commute. At a given point of M, the elements of $\mathcal{H}^{1,1}$ form a commutative subalgebra of $\mathfrak{u}(3)$ and this implies finally $b_2(M) \leq 3$. □

Remark 5.2 (i) In view of the Theorem 1.3 it seems quite likely that a case by case discussion could give the real cohomology type of a geometrically formal, 6-dimensional Kähler manifold provided that one founds a method to analyze obstructions to geometric formality at the level of the third cohomology group. For the time being all the informations about the third Betti number we have are the estimates $b_3(M) \leq 10$ if $b_2(M) = 2$ and $b_3(M) \leq 8$ if $b_3(M) = 3$; these follow easily from the first part of the proof of lemma 5.4.

(ii) If M is Kähler and geometrically formal of dimension divisible by 4, the commutativity result of Theorem 1.3 may not hold since, a priori, (g, J) could admit a compatible, complex symplectic structure, which is also J-invariant.

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