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On the cohomology algebra of some classes of geometrically formal manifolds

J.-F. Grosjean and P.-A. Nagy

Abstract

We investigate harmonic forms of geometrically formal metrics, which are defined as those having the exterior product of any two harmonic forms still harmonic. We prove that a formal Sasakian metric can exist only on a real cohomology sphere and that holomorphic forms of a formal Kähler metric are parallel with respect to the Levi–Civita connection. In the general Riemannian case a formal metric with maximal second Betti number is shown to be flat. Finally we prove that a 6-dimensional manifold with $b_1 \neq 1, b_2 \geqslant 2$ and not having the real cohomology algebra of $\mathbb{T}^3 \times S^3$ carries a symplectic structure as soon as it admits a formal metric.

1. *Introduction*

Let (M^n, g) be a compact oriented Riemannian manifold. We denote by $\Lambda^p M, 0 \leqslant p \leqslant n$ the space of smooth, real-valued, differential p-forms of M. We have then a differential complex

$$\cdots \to \Lambda^p M \xrightarrow{d} \Lambda^{p+1} M \to \ldots$$

where d is the exterior derivative. The pth cohomology group of this complex, known as the pth deRham cohomology group will be denoted by $H^p_{\mathrm{DR}}(M)$. The Riemannian metric g induces a scalar product at the level of differential forms, and hence one can consider also the operator $d^\star$, the formal adjoint of d. For $0 \leqslant p \leqslant n$ we define the space of harmonic p-forms by setting

$$\mathcal{H}^p(M, g) = \{\alpha \in \Lambda^p M : \Delta\alpha = 0\}.$$

Here the Laplacian Δ is defined by

$$\Delta = dd^\star + d^\star d.$$

Classical Hodge theory produces an isomorphism

$$H^p_{\mathrm{DR}}(M) \cong \mathcal{H}^p(M, g) \tag{1.1}$$

for all $0 \leqslant p \leqslant n$. While $H^\star(M) = \bigoplus_{p \geqslant 0} H^p_{\mathrm{DR}}(M)$ is a graded algebra, generally $\mathcal{H}^\star(M, g) = \bigoplus_{p \geqslant 0} \mathcal{H}^p(M, g)$ is not an algebra with respect to the wedge product operation for there is no reason that the isomorphism (1.1) descends to the level of harmonic forms. Our next definition is related to this fact.

Definition 1.1. ([**7**]). Let M^n be a compact and oriented manifold.
(i) A Riemannian metric g on M is formal if the exterior product of any two harmonic (with respect to g) forms remains harmonic,
(ii) The manifold M is geometrically formal if it admits a formal metric.

A closely related notion is that of topological formality (see [**2**] for instance), which implies that the rational homotopy type of the manifold is a formal consequence of its cohomology

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ring [**11**]. From the existence of a formal metric it follows that the underlying manifold is topologically formal, and this provides obstructions to the existence to such metrics; for instance they cannot exist on nilmanifolds since those have non-trivial Massey products, a fact which is in itself an obstruction to formality [**2**, **13**]. On the other hand, simply connected, compact manifolds of dimension not exceeding 6 are topologically formal [**3**, **9**].
Now the existence of formal metrics is more directly related to the geometry of the ambient manifold and known obstructions are related to the length of harmonic forms.

THEOREM 1.1. ([**7**]). *Let (M^n, g) be compact and oriented such that g is a formal metric. Then*

(i) *the inner product of any two harmonic forms is a constant function;*
(ii) $b_p(M) \leqslant \binom{n}{p}$ *for all* $1 \leqslant p \leqslant n$;
(iii) *if in (ii) equality occurs for $p = 1$ then g is a flat metric.*

Standard examples of formal metrics are provided by compact symmetric spaces for in this case all harmonic forms must be parallel with respect to the Levi–Civita connection. Kotschik proved that in dimension 4 every geometrically formal manifold has the real cohomology algebra of a compact symmetric space. One of the current questions related to the notion of geometric formality is then to examine up to what extent this is true in general.
In the context of Sasakian geometry, the odd-dimensional analogue of Kähler geometry, we prove the following.

THEOREM 1.2. *Let (M^{2n+1}, g) be a compact Sasakian manifold. If g is a formal metric then M is a real cohomology sphere.*

Next we obtain obstructions to the existence of formal Kähler metrics, through the study of their holomorphic forms. In this context topological formality is no longer restrictive since any Kähler manifold is known to have this property [**2**].

THEOREM 1.3. *Let (M^{2n}, g, J) be a compact Kähler manifold such that the metric g is formal. Then every harmonic form Ω of real type $(p, 0) + (0, p)$ (hence every holomorphic p-form) is parallel with respect to the Levi–Civita connection. Moreover, Ω induces in a canonical way a local splitting of M as the Riemannian product of two compact Kähler manifolds M_1 and M_2 so that Ω is zero on M_1, non-degenerate on M_2 which is Ricci flat.*

REMARK 1.1. (i) Theorem 1.3 was already proved in [**8**] for $p = 2$, using arguments relying heavily on the algebraic structure of the space of harmonic 2-forms. For higher degree forms, such results are no longer available.
(ii) If in Theorem 1.3 we furthermore assume the metric being locally irreducible and not symmetric, it follows from Berger's holonomy classification theorems (see [**10**]) that the only cases when we can have a non-vanishing holomorphic form are when $\mathrm{Hol}(g) = Sp(m)(n = 2m)$ or $\mathrm{Hol}(g) = SU(n)$.
(iii) From the above it also follows that if M admits a locally irreducible Kähler and formal metric which is not Ricci flat then the Todd genus satisfies $Td(M) = 1$.

In the second part of the paper we study general properties of 2-forms which are harmonic with respect to a formal metric. We observe that any such 2-form diagonalises with constant eigenvalues and constant rank eigendistributions. This is extending results from [**8**] to the general Riemannian case and can also be used as a starting point to give sufficient conditions, essentially phrased in terms of Betti numbers lower bounds, for a formal metric to admit a compatible symplectic form in dimension 6. We prove the following.

THEOREM 1.4. *Let M^6 be geometrically formal. If $b_1(M) \neq 1$ and $b_2(M) \geqslant 2$ and, moreover, M has not the real cohomology algebra of $\mathbb{T}^3 \times S^3$ then any formal metric on M admits a compatible symplectic form.*

The above result essentially says that in dimension 6 a geometrically formal manifold M always carries a symplectic structure compatible with the formal metric with the exception of the cases when $b_1(M) = 1$ or $b_1(M) \neq 1, b_2(M) = 0, 1$ or when the real cohomology algebra is that of $\mathbb{T}^3 \times S^3$. This suggests that symplectic techniques could be used to investigate, under these conditions, the topology and geometry of these manifolds. In dimension 4, the existence of symplectic forms on geometrically formal manifolds has been extensively treated in [**7**].

When $b_2(M) \geqslant 3$ Theorem 1.4 follows essentially by algebraic arguments mainly using the above-mentioned fact on the diagonalisation of harmonic 2-forms of a formal metric. To prove it when $b_2(M) = 2$ we first show that the absence of a compatible symplectic form forces the presence of enough harmonic 3-forms (actually $b_3(M) = 6$ in this case). Then we need to perform a rather delicate local analysis, involving the internal symmetries of the set harmonic 3-forms in order to arrive at $b_1(M) \geqslant 2$, a case which can be ruled out algebraically.

In the final part of the paper we are concerned with giving a characterisation of geometrically formal Riemannian manifold with maximal second Betti number. We prove the following.

THEOREM 1.5. *Let M^n be geometrically formal with $n \geqslant 3$. If $b_2(M)$ is maximal, that is, $b_2(M) = \binom{n}{2}$, then any formal metric on M is flat.*

This clarifies the equality case in Theorem 1.1, (iii) for degree 2-forms. Note that the assertion in Theorem 1.5 is straightforward when n is odd for if $n = 2k+1$ the formality and the maximality of b_2 imply that $b_{2k}(M)$ is maximal. Hodge duality implies then the maximality of $b_1(M)$ and hence the flatness of the metric (see Section 5 for more details). When n is even, our point of departure consists in observing that the metric must admit a compatible almost-Kähler structure and then work out this situation within the same circle of arguments which have led to the proof of Theorem 1.3.

To conclude, it would be interesting to have results similar to Theorem 1.4 in arbitrary even dimensions and of course to give necessary but also sufficient conditions for a geometrically formal metric to admit a compatible symplectic structure. In doing so, the difficulties one faces are related to understanding, at the algebraic level, the constraints imposed by geometric formality on forms of degree at least 3.

2. *Some algebraic facts*

Let (V^{2n}, g, J) be a Hermitian vector space and let $\Lambda^\star V$ be its exterior algebra over the reals. Consider the operator $\mathcal{J} : \Lambda^p V \to \Lambda^p V$ acting on a p-form α by

$$(\mathcal{J}\alpha)(v_1, \ldots, v_p) = \sum_{k=1}^{p} \alpha(v_1, \ldots, Jv_k, \ldots, v_p)$$

for all $v_1, \ldots, v_p$ in V. The operator $\mathcal{J}$ acts as a derivation on $\Lambda^\star$ and gives the complex bi-grading of the exterior algebra in the following sense. Let $\lambda^{p,q}V$ be given as the $-(p-q)^2$-eigenspace of $\mathcal{J}^2$. Then

$$\Lambda^s V = \sum_{p+q=s} \lambda^{p,q} V$$

is an orthogonal, direct sum. Note that $\lambda^{p,q}V = \lambda^{q,p}V$. Of special importance in our discussion are the spaces $\lambda^p V = \lambda^{p,0}V$; forms α in λ^p are such that $(X_1, \ldots, X_p) \to \alpha(JX_1, X_2, \ldots, X_p)$ is still an alternating form which equals $p^{-1}\mathcal{J}\alpha$. We shall also use the extension of J to $\Lambda^\star V$ given by

$$(J\alpha)(v_1, \ldots, v_p) = \alpha(Jv_1, \ldots, Jv_p)$$

for all α in $\Lambda^p V$ and $v_1, \ldots, v_p$ in V. Let $\lambda^p V \otimes_1 \lambda^q V$ be the space of tensors $Q : \lambda^p V \to \lambda^q V$ which satisfy

$$[(\mathbb{J}Q)(X_1, \ldots, X_p)](Y_1, \ldots, Y_q) = -[\mathbb{J}(Q(X_1, \ldots, X_p))](Y_1, \ldots, Y_q)$$

(here $\mathbb{J}$ as a map of $\lambda^p V$ stands in fact for $p^{-1}\mathcal{J}$). We also define $\lambda^p V \otimes_2 \lambda^q V$ to be the space of tensors $Q : \lambda^p V \to \lambda^q V$ such that $Q\mathbb{J} = \mathbb{J}Q$.

Lemma 2.1. *Let $a : \lambda^p V \otimes \lambda^q V \to \Lambda^{p+q} V$ be the total anti-symmetrisation map. Then*

(i) *the image of the restriction of a to $\lambda^p V \otimes_1 \lambda^q V \to \Lambda^{p+q} V$ is contained in $\lambda^{p,q} V$;*
(ii) *the image of the restriction of a to $\lambda^p V \otimes_2 \lambda^q V \to \Lambda^{p+q} V$ is contained in $\lambda^{p+q} V$.*

Proof. We shall provide a direct proof, but only for (i), that of (ii) being similar. Choose Q in $\lambda^p V \otimes_1 \lambda^q V$. Then

$$a(Q) = \sum_{I=(i_1,\ldots,i_p)} e^\flat_{i_1} \wedge \ldots \wedge e^\flat_{i_p} \wedge Q(e_{i_1}, \ldots, e_{i_p}),$$

where for v in V we denote by $v^\flat$ the dual, with respect to the metric, 1-form. Then

$$\begin{aligned} \mathcal{J}(a(Q)) = &\sum_{I=(i_1,\ldots,i_p)} \mathcal{J}(e^\flat_{i_1} \wedge \ldots \wedge e^\flat_{i_p}) \wedge Q(e_{i_1}, \ldots, e_{i_p}) \\ &+ \sum_{I=(i_1,\ldots,i_p)} e^\flat_{i_1} \wedge \ldots \wedge e^\flat_{i_p} \wedge \mathcal{J}Q(e_{i_1}, \ldots, e_{i_p}). \end{aligned}$$

For any $1 \leqslant r \leqslant p$ we compute

$$\begin{aligned} &\sum_{I=(i_1,\ldots,i_p)} e^\flat_{i_1} \wedge \ldots \wedge Je^\flat_{i_r} \wedge \ldots \wedge e^\flat_{i_p} \wedge Q(e_{i_1}, \ldots, e_{i_p}) \\ &= - \sum_{I=(i_1,\ldots,i_p)} e^\flat_{i_1} \wedge \ldots \wedge (Je_{i_r})^\flat \wedge \ldots \wedge e^\flat_{i_p} \wedge Q(e_{i_1}, \ldots, e_{i_p}) \\ &= \sum_{I=(i_1,\ldots,i_p)} e^\flat_{i_1} \wedge \ldots \wedge e^\flat_{i_r} \wedge \ldots \wedge e^\flat_{i_p} \wedge Q(e_{i_1}, \ldots, Je_{i_r}, \ldots, e_{i_p}) \\ &= \sum_{I=(i_1,\ldots,i_p)} e^\flat_{i_1} \wedge \ldots \wedge e^\flat_{i_p} \wedge (\mathbb{J}Q)(e_{i_1}, \ldots, e_{i_p}). \end{aligned}$$

On the other side we have $\mathcal{J}Q(e_{i_1},\ldots,e_{i_p}) = q\mathbb{J}[Q(e_{i_1},\ldots,e_{i_p})] = -q(\mathbb{J}Q)(e_{i_1},\ldots,e_{i_p})$ and putting all these together we arrive easily at

$$\mathcal{J}(a(Q)) = (p-q)\sum_{I=(i_1,\ldots,i_p)} e^{\flat}_{i_1}\wedge\ldots\wedge e^{\flat}_{i_p}\wedge(\mathbb{J}Q)(e_{i_1},\ldots,e_{i_p}).$$

Applying $\mathcal{J}$ once more time while going through the same steps yields $\mathcal{J}^2 a(Q) = -(p-q)^2 a(Q)$ and the proof is completed. □

The main technical observation in this section is as given below.

PROPOSITION 2.1. *The following hold:*
(i) *the total alternation map $a : \lambda^p V\otimes_1 \lambda^q V \to \Lambda^{p+q}V$ is injective for any $p\neq q$;*
(ii) *the kernel of $a : \lambda^p V\otimes \lambda^q V \to \Lambda^{p+q}V$ is contained in $\lambda^p V\otimes_2 \lambda^q V$.*

Proof. (i) If Q belongs to $\lambda^p V\otimes_1\lambda^q V$ and X is in V we define Q_X and Q^X in $\lambda^{p-1}V\otimes_1\lambda^q V$ and $\lambda^p V\otimes_1\lambda^{q-1}V$, respectively, by

$$Q_X = Q(X,\cdot) \quad \text{and} \quad Q^X = X\lrcorner Q.$$

It is easy to see that those are well defined. Assume now that $a(Q)=0$. Then

$$\begin{aligned}
0 = X\lrcorner a(Q) &= \sum_{i_1,\ldots,i_p} X\lrcorner(e^{\flat}_{i_1}\wedge\ldots\wedge e^{\flat}_{i_p})\wedge Q(e_{i_1},\ldots,e_{i_p})\\
&\quad + (-1)^p\sum_{i_1,\ldots,i_p} e^{\flat}_{i_1}\wedge\ldots\wedge e^{\flat}_{i_p}\wedge(X\lrcorner Q(e_{i_1},\ldots,e_{i_p}))\\
&= p\sum_{i_1,\ldots,i_{p-1}} e^{\flat}_{i_1}\wedge\ldots\wedge e^{\flat}_{i_{p-1}}\wedge Q(X,e_{i_1},\ldots,e_{i_{p-1}})\\
&\quad + (-1)^p\sum_{i_1,\ldots,i_p} e^{\flat}_{i_1}\wedge\ldots\wedge e^{\flat}_{i_p}\wedge Q^X(e_{i_1},\ldots,e_{i_p})\\
&= pa(Q_X) + (-1)^p a(Q^X).
\end{aligned}$$

By the previous Lemma $a(Q_X)$ is in $\lambda^{p-1,q}V$ while $a(Q^X)$ belongs to $\lambda^{p,q-1}V$ and hence both must vanish since elements of distinct spaces as $p\neq q$. Now an induction argument leads directly to the proof of the claim in (i).

To prove (ii) we first note that $\lambda^p V\otimes\lambda^q V = (\lambda^p V\otimes_1\lambda^q V)\oplus(\lambda^p V\otimes_2\lambda^q V)$ and the claim follows from Lemma 2.1. □

Let $L:\Lambda^\star V\to\Lambda^\star V$ be the exterior multiplication with the Kähler form $\omega = g(J\cdot,\cdot)$. Recall that the space $\Lambda^\star_0 V$ of primitive forms is defined to be the kernel of $L^\star$, the adjoint of L with respect to the inner product g. We consider the operators $P_k:\Lambda^r V\times\Lambda^s V\to\Lambda^{r+s-2k}V$ defined by

$$P_k(\alpha,\beta) := \sum_{1\leqslant i_1,\ldots,i_k\leqslant 2n} (e_{i_1}\lrcorner\ldots\lrcorner e_{i_k}\lrcorner\alpha)\wedge(Je_{i_1}\lrcorner\ldots\lrcorner Je_{i_k}\lrcorner\beta)$$

for all (α,β) in $\Lambda^r V\times\Lambda^s V$, and where $\{e_i, 1\leqslant i\leqslant 2n\}$ is some orthonormal basis in V. Clearly, $P_0(\alpha,\beta)=\alpha\wedge\beta$ for all (α,β) in $\Lambda^r V\times\Lambda^s V$.

PROPOSITION 2.2. *For any $\alpha\in\Lambda^r V$ and $\beta\in\Lambda^s V$, we have*
(i) *$L^\star P_k(\alpha,\beta) = P_k(L^\star\alpha,\beta)+P_k(\alpha,L^\star\beta)+(-1)^{r-k-1}P_{k+1}(\alpha,\beta)$ for all $k\geqslant 0$;*
(ii) *$(L^\star)^p(\alpha\wedge\beta) = (-1)^{p(p-1)/2}p!\langle\alpha,J\beta\rangle$ for any primitive p-forms α and β.*

Proof. (i) Let $\alpha \in \Lambda^r V$ and $\beta \in \Lambda^s V$. Then

$$\begin{aligned}
L^\star P_k(\alpha,\beta) &= \frac{1}{2}\sum_{i,i_1\ldots i_k} Je_i\lrcorner e_i\lrcorner((e_{i_1}\lrcorner\ldots e_{i_k}\lrcorner\alpha)\wedge(Je_{i_1}\lrcorner\ldots Je_{i_k}\lrcorner\beta))\\
&= \frac{1}{2}\sum_{i,i_1\ldots i_k} Je_i\lrcorner((e_i\lrcorner e_{i_1}\lrcorner\ldots e_{i_k}\lrcorner\alpha)\wedge(Je_{i_1}\lrcorner\ldots Je_{i_k}\lrcorner\beta))\\
&\quad+\frac{1}{2}(-1)^{r-k}\sum_{i,i_1\ldots i_k} Je_i\lrcorner((e_{i_1}\lrcorner\ldots e_{i_k}\lrcorner\alpha)\wedge(e_i\lrcorner Je_{i_1}\lrcorner\ldots Je_{i_k}\lrcorner\beta))\\
&= P_k(L^\star\alpha,\beta)+\frac{1}{2}(-1)^{r-k-1}\sum_{i_1\ldots i_{k+1}}(e_{i_1}\lrcorner\ldots e_{i_{k+1}}\lrcorner\alpha)\wedge(Je_{i_1}\lrcorner\ldots Je_{i_{k+1}}\lrcorner\beta)\\
&\quad+\frac{1}{2}(-1)^{r-k}\sum_{i_1\ldots i_{k+1}}(Je_{i_1}\lrcorner e_{i_2}\lrcorner\ldots e_{i_{k+1}}\lrcorner\alpha)\wedge(e_{i_1}\lrcorner Je_{i_2}\lrcorner\ldots Je_{i_{k+1}}\lrcorner\beta)\\
&\quad+P_k(\alpha,L^\star\beta)
\end{aligned}$$

and the claim in (i) follows.

To prove (ii) we first obtain by induction from (i) that $(L^\star)^p(\alpha\wedge\beta)=(-1)^{p(p-1)/2}P_p(\alpha,\beta)$ whenever α,β belong to Λ^p_0V. To conclude it is enough to directly use the definition of P_p to get $P_p(\alpha,\beta)=p!\langle\alpha,J\beta\rangle$. □

2.1. *Formal Sasakian metrics*

Part of the algebraic facts developed above can also be used to describe completely the cohomology algebra of a geometrically formal, Sasakian metric. For an introduction to Sasakian geometry, the odd-dimensional analogue of Kähler geometry, we refer the reader to [**1**].

THEOREM 2.1. *Let (M^{2n+1},g) be a compact Sasakian manifold. If the metric g is formal then $b_p(M)=0$ for all $1\leqslant p\leqslant 2n$, in other words M is a real cohomology sphere.*

Proof. Recall that the tangent bundle of M splits as $TM=\mathcal{V}\oplus H$ an orthogonal direct sum, where $\mathcal{V}$ is spanned by the so-called Reeb vector field, to be denoted by ζ. The contact distribution H admits a g-compatible complex structure $J:H\to H$ which, moreover, satisfies $d\theta=\omega$, where θ is the 1-form dual to ζ and $\omega=g(J\cdot,\cdot)$. We call a differential p-form horizontal, and denote the corresponding space by Λ^pH if the interior product with ζ vanishes. Now, let $d_H:\Lambda^\star H\to\Lambda^\star H$ be the projection of the usual exterior derivative d onto H. If $d_H^\star$ is its formal adjoint with respect to the restriction of g on H, we have (see [**12**]) on $\Lambda^pM=\Lambda^pH\oplus[\theta\wedge\Lambda^{p-1}H]$

$$d^\star=\begin{pmatrix} d_H^\star & -\mathcal{L}_\zeta\\ L^\star & -d_H^\star\end{pmatrix}, \tag{2.1}$$

where $\mathcal{L}_\zeta$ denotes the Lie derivative. As a last reminder, we mention that the extension of J to $\Lambda^\star H$ defined as in the previous section preserves the space of harmonic forms.

Let now α be a harmonic form on M. It is a known fact that if $0\leqslant p\leqslant n$, every harmonic form α on M is horizontal and invariant by the Reeb vector field. Moreover, α must be primitive, that is, $L^\star\alpha=0$. Using the formality assumption on g we obtain that $\alpha\wedge J\alpha$ is still harmonic. Since this is a horizontal form, invariant under the Reeb vector field it follows from (2.1) that $L^\star(\alpha\wedge J\alpha)=0$. We conclude that α vanishes by means of Proposition 2.2, (ii). □

The proof of Theorem 1.2 in Section 1 is now complete.

3. *Holomorphic forms with harmonic squares*

Let (M^{2n}, g, J) be a compact Kähler manifold and consider a harmonic p-form Ω in $\lambda^p M$, that is of type $(0,p)+(p,0)$. It is a well-known fact, see [**5**] for instance, that Ω must be holomorphic, that is

$$\nabla_{JX}\Omega = \nabla_X(\mathbb{J}\Omega) \tag{3.1}$$

for all X in TM. Together with Ω comes $S : \Lambda^{p-1}M \to \Lambda^1 M$ defined by $S(X_1,\ldots,X_{p-1}) = \Omega(X_1,\ldots,X_{p-1},\cdot)$. That Ω has real type $(0,p)+(p,0)$ translates into

$$(S(JX_1,\ldots,X_{p-1}))^\sharp = -J(S(X_1,\ldots,X_{p-1}))^\sharp \tag{3.2}$$

whenever $X_1,\ldots,X_{p-1}$ belong to TM, and where for any 1-form θ, $\theta^\sharp$ denotes the associated vector field with respect to the metric g. Let now $Q : \Lambda^{p-1}M \to \lambda^p M$ be given by

$$Q(X_1,\ldots,X_{p-1}) = \nabla_{(S(X_1,\ldots,X_{p-1}))^\sharp}\Omega$$

for all $X_1,\ldots,X_{p-1}$ in TM. The next lemma provides information about the complex type of Q.

Lemma 3.1. *The tensor Q belongs to $\lambda^{p-1}M \otimes_1 \lambda^p M$.*

Proof. Follows immediately from (3.1) and (3.2). □

Proposition 3.1. *Let Ω in $\lambda^p M$ be a harmonic form. If the metric g is formal, then*

$$\nabla_{(S(X_1,\ldots,X_{p-1}))^\sharp}\Omega = 0 \tag{3.3}$$

holds, for all $X_1,\ldots,X_{p-1}$ in TM.

Proof. Let $\{e_i, 1 \leqslant i \leqslant 2n\}$ be a geodesic frame at a point m in M. If p is even $\Omega\wedge\Omega$ is harmonic and we have at m

$$\begin{aligned} 0 = -d^\star(\Omega\wedge\Omega) &= \sum_{i=1}^{2n} e_i \lrcorner \nabla_{e_i}(\Omega\wedge\Omega) \\ &= 2\sum_{i=1}^{2n} e_i\lrcorner(\nabla_{e_i}\Omega\wedge\Omega) = 2\sum_{i=1}^{2n}\nabla_{e_i}\Omega\wedge(e_i\lrcorner\Omega) \end{aligned}$$

since Ω is itself co-closed. In other words $a(Q)=0$ and we conclude by means of Lemma 3.1 and Proposition 2.1 that $Q=0$. If p is odd, the harmonicity of $\Omega\wedge\mathbb{J}\Omega$ gives

$$\begin{aligned} 0 = -d^\star(\Omega\wedge\mathbb{J}\Omega) &= \sum_{i=1}^{2n} e_i\lrcorner(\nabla_{e_i}\Omega\wedge\mathbb{J}\Omega + \Omega\wedge\nabla_{e_i}\mathbb{J}\Omega) \\ &= \sum_{i=1}^{2n} -\nabla_{e_i}\Omega\wedge(e_i\lrcorner\mathbb{J}\Omega) + (e_i\lrcorner\Omega)\wedge\nabla_{e_i}(\mathbb{J}\Omega), \end{aligned}$$

where we take into account the co-closedness of Ω and $\mathbb{J}\Omega$. Now $\nabla_{e_i}\mathbb{J}\Omega = \nabla_{Je_i}\Omega$ and hence

$$\begin{aligned} 0 &= \sum_{i=1}^{2n} -\nabla_{e_i}\Omega\wedge(Je_i\lrcorner\Omega) + (e_i\lrcorner\Omega)\wedge\nabla_{Je_i}\Omega \\ &= -2\sum_{i=1}^{2n}\nabla_{e_i}\Omega\wedge(Je_i\lrcorner\Omega). \end{aligned}$$

This is easily reinterpreted to say that $a(\mathbb{J}Q) = 0$, and then Lemma 3.1 together with Proposition 2.1 leads to the vanishing of Q and hence to the claimed result. □

REMARK 3.1. From the proof of the result above we see that it actually holds for harmonic forms Ω in $\lambda^p M$ such that $\Omega \wedge \Omega$ for p even and $\Omega \wedge \mathbb{J}\Omega$ for p odd are co-closed.

We need now to recall some facts about the algebraic structure of harmonic forms of type $(1,1)$.

PROPOSITION 3.2. ([**8**]). *Let (M^{2n}, g, J) be a compact Kähler manifold such that the metric g is formal. If $\alpha = g(F\cdot, \cdot)$ is harmonic in $\lambda^{1,1}M$ then we have an orthogonal and J-invariant splitting*

$$TM = \bigoplus_{i=0}^{p} E_i$$

which is preserved by F and such that $F = \lambda_i J_i$ on E_i, for all $0 \leqslant i \leqslant p$. Here J_i are almost-complex structure on E_i and λ_i are real constants, for $0 \leqslant i \leqslant p$.

Now we would like to conclude from Proposition 3.1 that Ω is actually parallel. This is eventually seen to be the case if Ω is non-degenerate at every point of the manifold. To rule out the general case we must study the null distribution of Ω. For each m in M define $\mathcal{V}_m = \{X \in T_m M : X \lrcorner \Omega = 0\}$. Our first concern is to show that $m \to \mathcal{V}_m$ gives a smooth, *constant* rank distribution on M.

LEMMA 3.2. *Let (M^{2n}, g, J) be a compact Kähler manifold such that the metric g is formal. If Ω in $\lambda^p M$ is harmonic the following hold*:

(i) *the distribution $\mathcal{V}$ is of constant rank;*
(ii) *both distributions $\mathcal{V}$ and $H = \mathcal{V}^{\perp}$ are integrable and H is totally geodesic.*

Proof. (i) Let α_Ω in $\lambda^{1,1}M$ be defined by $\alpha_\Omega(X,Y) = \langle JX \lrcorner \Omega, Y \lrcorner \Omega \rangle$ for all X, Y in TM. Because g is formal we have that $(L^\star)^{p-1}(\Omega \wedge J\Omega)$ is a harmonic two form. On the other hand, from Proposition 2.2, (i) it follows by induction that $(L^\star)^{p-1}(\Omega \wedge J\Omega) = (-1)^{(p-2)(p-3)/2} P_{p-1}(\Omega, J\Omega)$ by also using that Ω is primitive. Now a direct computation using the definition of P_{p-1} shows that

$$\begin{aligned} P_{p-1}(\Omega, J\Omega)(X,Y) &= (-1)^{p-1}(p-1)!(\langle X \lrcorner \Omega, JY \lrcorner \Omega \rangle - \langle Y \lrcorner \Omega, JX \lrcorner \Omega \rangle) \\ &= 2(-1)^p (p-1)! \alpha_\Omega(X,Y) \end{aligned}$$

for all X, Y in TM. We conclude that α_Ω is a harmonic form of type $(1,1)$ and hence the formality of g and Proposition 3.2 ensure that α_Ω has constant rank. By a positivity argument, the nullity of α_Ω coincides with that of Ω and the claim is proved.

(ii) The distribution $\mathcal{V}$ (hence H) is J-invariant since α_Ω lives in $\lambda^{1,1}M$. By (i) we obtain a globally defined splitting $TM = \mathcal{V} \oplus H$ which is therefore orthogonal and J-invariant. From the definition of $\mathcal{V}$ it follows by an orthogonality argument that the distribution H is spanned by $S(X_1, \ldots, X_{p-1})$ with $X_1, \ldots, X_{p-1}$ in TM and hence

$$\nabla_X \Omega = 0 \quad \text{for all } X \in H \tag{3.4}$$

by Proposition 3.1. Taking now a direction, say, V in $\mathcal{V}$ gives that $\nabla_X V$ belongs to $\mathcal{V}$ and this shows the total geodesicity, and hence the integrability of H. The integrability of $\mathcal{V}$ is an easy consequence of the closedness of Ω. Indeed, taking $X_1, \ldots, X_{p-1}$ in H and V, W in $\mathcal{V}$, we have

$$\begin{aligned} 0 = d\Omega(X_1, \ldots, X_{p-1}, V, W) &= \sum_{i=1}^{p-1} (-1)^{i+1} (\nabla_{X_i}\Omega)(X_1, \ldots, \widehat{X_i}, \ldots, X_{p-1}, V, W) \\ &\quad - (\nabla_V \Omega)(X_1, \ldots, X_{p-1}, W) + (\nabla_W \Omega)(X_1, \ldots, X_{p-1}, V) \\ &= \Omega(X_1, \ldots, X_{p-1}, [V, W]). \end{aligned}$$

Since Ω vanishes on $\mathcal{V}$ by the definition of the latter it follows that $[V, W] \lrcorner \Omega = 0$ and our integrability claim follows by using again the definition of $\mathcal{V}$. □

To prove the parallelism of Ω, which amounts to having $\mathcal{V}$ totally geodesic we need to establish one more fact. Recall [**14**] that the transversal Ricci tensor $\mathrm{Ric}^H : H \to H$ of the totally geodesic distribution H is defined by

$$g(\mathrm{Ric}^H X, Y) = \sum_i R(X, e_i, Y, e_i)$$

for all X, Y in H and local orthonormal frames $\{e_i\}$ in H. When $\mathcal{V}$ integrates to give a Riemannian submersion, which is always true locally, Ric^H corresponds to the usual Ricci tensor of the base manifold.

Lemma 3.3. *The transversal Ricci tensor* Ric^H *of the distribution* H *vanishes.*

Proof. For any α in $\Lambda^2 M$ and for all φ in $\Lambda^\star M$ let us define

$$[\alpha, \varphi] = \sum_{i=1}^{2n} e_i \lrcorner \alpha \wedge e_i \lrcorner \varphi,$$

where $\{e_i, 1 \leqslant i \leqslant 2n\}$ is some local orthonormal frame in TM. Since H is totally geodesic, after differentiation of (3.4) in directions coming from H we get $[R(X, Y), \Omega] = 0$ for all X, Y in H. Since $V \lrcorner \Omega = 0$ for V in $\mathcal{V}$ it follows that $\sum_i R(X, Y) e_i \wedge e_i \lrcorner \Omega = 0$ for all X in H, and where $\{e_i\}$ is a local orthonormal frame in H, to be fixed in what follows. Therefore, we get

$$\begin{aligned} 0 = \sum_{j,i} e_j \lrcorner (R(X, e_j) e_i \wedge e_i \lrcorner \Omega) &= \mathrm{Ric}^H X \lrcorner \Omega - \sum_{j,i} R(X, e_j) e_i \wedge e_j \lrcorner e_i \lrcorner \Omega \\ &= \mathrm{Ric}^H X \lrcorner \Omega + \frac{1}{2} \sum_{j,i} R(e_j, e_i) X \wedge e_j \lrcorner e_i \lrcorner \Omega \end{aligned}$$

for all X in H, where for obtaining the second line we used the algebraic Bianchi identity for R. As consequences of the Kähler condition and of the fact that Ω is in $\lambda^p M$ we have that $R(JX, JY) = R(X, Y)$, while $JX \lrcorner JY \lrcorner \Omega = -X \lrcorner Y \lrcorner \Omega$ for all X, Y in TM. Hence the last sum above vanishes and we end up with $\mathrm{Ric}^H X \lrcorner \Omega = 0$ for all X in TM whence the claim, since Ω is non-degenerate on H. □

At the same time, the situation when Ric^H vanishes is well described by the following.

Theorem 3.1 ([**8**]). *Let* (M^{2n}, g, J) *be a compact Kähler manifold equipped with a Riemannian foliation with complex leaves. If the foliation is transversally totally geodesic with non-negative transversal Ricci tensor then it has to be totally geodesic, therefore locally a Riemannian product.*

Proof of Theorem 1.3. Since Ric^H vanishes, it follows by Theorem 3.1 that $\mathcal{V}$ is totally geodesic, and hence parallel with respect to the Levi–Civita connection ∇. This implies immediately the parallelism of Ω, by means of (3.4). The local product decomposition of (M^{2n}, g, J) follows by using the deRham splitting theorem for the ∇-parallel decomposition $TM = \mathcal{V} \oplus H$, combined with Lemma 3.3. $\square$

4. *Harmonic* 2-*forms*

We shall develop in this section the general Riemannian counterpart of Proposition 3.2. From now on, we shall use the metric to identify a 2-form α with a skew-symmetric endomorphism A of TM; explicitly $\alpha = g(A\cdot, \cdot)$. Moreover, the space $\mathcal{A}$ is the space of skew-symmetric endomorphisms of TM which are associated to an element of $\mathcal{H}^2(M, g)$. If φ belongs to $\Lambda^\star M$ let $L_\varphi : \Lambda^\star M \to \Lambda^\star M$ be given as exterior multiplication by φ and let $L_\varphi^\star$ be the adjoint of L_φ.

Proposition 4.1. *Let M^n be geometrically formal and let g be a formal metric on M. We have :*

$$A_2A_1A_3 + A_3A_1A_2 \in \mathcal{A}$$

whenever $A_i, 1 \leqslant i \leqslant 3$ belong to $\mathcal{A}$.

Proof. Let α belong to $\mathcal{H}^2(M, g)$. Since g is formal and $L_\alpha^\star$ is up to sign equal to $\star L_\alpha \star$ it follows that both L_α and $L_\alpha^\star$ preserve the space of harmonic forms of (M, g). Therefore, if $\alpha_i, 1 \leqslant i \leqslant 3$ belong to $\mathcal{H}^2(M, g)$ then $L_{\alpha_1}^\star L_{\alpha_2}\alpha_3$ is an element of $\mathcal{H}^2(M, g)$. Let $A_i, 1 \leqslant i \leqslant 3$ be the skew-symmetric endomorphisms associated to the forms $\alpha_i, 1 \leqslant i \leqslant 3$ and let $\{e_i, 1 \leqslant i \leqslant n\}$ be a local orthonormal basis in TM. We shall now compute

$$L_{\alpha_1}^\star L_{\alpha_2}\alpha_3 = \frac{1}{2}\sum_{i,j=1}^{n} \alpha_1(e_i, e_j) e_j \lrcorner [e_i \lrcorner (\alpha_2 \wedge \alpha_3)].$$

However,

$$e_j \lrcorner \Big[e_i \lrcorner (\alpha_2 \wedge \alpha_3) \Big] = \alpha_2(e_i, e_j)\alpha_3 - (e_i \lrcorner \alpha_2) \wedge (e_j \lrcorner \alpha_3) + (e_j \lrcorner \alpha_2) \wedge (e_i \lrcorner \alpha_3) + \alpha_3(e_i, e_j)\alpha_2.$$

Further computation yields, after some elementary manipulations

$$L_{\alpha_1}^\star L_{\alpha_2}\alpha_3 = \langle \alpha_1, \alpha_2 \rangle \alpha_3 + \langle \alpha_1, \alpha_3 \rangle \alpha_2 + \langle A_3A_1A_2 + A_2A_1A_3 \cdot, \cdot \rangle. \qquad \square$$

In what follows we shall say that a symplectic form on M is compatible with the metric g if its associated skew-symmetric endomorphism defines an almost-complex structure on M.

Proposition 4.2. *Let M^n be geometrically formal and let g denote a formal metric on M. Moreover, let α belong to $\mathcal{H}^2(M, g)$ with associated endomorphism A in $\mathcal{A}$. Then*

(i) *the eigenvalues of A^2 are constant with eigenbundles of constant rank;*

(ii) *let μ_i be (the pairwise distinct) eigenvalues of A^2, with $\mu_0 = 0$ and let E_i be the eigenbundles of A^2 corresponding to μ_i. Then for $1 \leqslant i \leqslant p$, E_i is of even dimension and we have an orthogonal decomposition*

$$\alpha = \sum_{i=1}^{p} \sqrt{-\mu_i}\,\omega_i,$$

where $\omega_i, 1 \leqslant i \leqslant p$ *belong to* $\mathcal{H}^2(M,g)$. *Moreover,* $\omega_i = g(J_i\cdot,\cdot)$ *on* E_i, *for some g-compatible almost-complex structure* J_i *on* $E_i, 1 \leqslant i \leqslant p$;

(iii) *if* α *is non-degenerate then* g *admits a compatible symplectic form.*

Proof. (i) From Proposition 4.1 we get by induction that A^{2k+1} belongs to $\mathcal{A}$ whenever A is in $\mathcal{A}$. Since $\mathcal{A}$ is finite-dimensional, there exists $P \in \mathbb{R}[X]$ so that $P(A^2) = 0$ and, moreover, by using the symmetry of A^2 the polynomial P can be supposed to have only real and simple roots $\mu_i, 1 \leqslant i \leqslant p$. Let m_i be the dimension of the μ_i-eigenbundle, $1 \leqslant i \leqslant p$. To see that m_i, μ_i, $0 \leqslant i \leqslant p$ are constant over M, we use the fact that A^{2k+1} belongs to $\mathcal{A}$ for any $k \in \mathbb{N}$ by Proposition 4.1 and from the fact that elements in $\mathcal{A}$ have pointwisely constant scalar products we deduce that $\mathrm{Tr}\ (A^{2k}) = -\langle A^{2k-1}, A\rangle = c_k$ for some constant c_k and for any integer k. It follows that $\sum_{i=1}^p m_i \mu_i^k = c_k$ for all k in $\mathbb{N}$ and hence this Vandermonde system leads to the constancy of the functions m_i, μ_i, $1 \leqslant i \leqslant p$.

(ii) With the notation $\lambda_i = \sqrt{-\mu_i}$, the orthogonal projection of α on E_i is given by $\lambda_i \omega_i$, where $\omega_i = g(J_i\cdot,\cdot)$ for some almost-complex structure J_i on E_i, $1 \leqslant i \leqslant p$. Now,

$$g(A^{2k+1}\cdot,\cdot) = \sum_{i=1}^{p} \lambda_i^{2k+1}\omega_i$$

is harmonic for all natural k and by an argument similar to the one used in the proof of the Proposition 3.1 of [**8**] we deduce that ω_i belong to $\mathcal{H}^2(M,g)$.

(iii) By (ii) the form $\sum_{i=1}^p \omega_i$ belongs to $\mathcal{H}^2(M,g)$ and it is g-compatible if α is non-degenerate. □

The technical advantage of Proposition 4.2 is essentially to say that all distributions appearing as ranges or kernels of harmonic 2-forms are of constant rank over the manifold, and in this respect they can, as we shall see in the next section, be treated as algebraic objects.

4.1. 6-*dimensions*

We shall present here a geometric application of the algebraic facts from the previous section. More precisely, we are going to obtain sufficient conditions for a geometrically formal 6-manifold to admit a compatible symplectic structure. We need first a number of preliminary results.

LEMMA 4.1. *Let* M^n *be geometrically formal and let* g *be a formal metric on* M. *Let* α *belong to* $\mathcal{H}^2(M,g)$ *with kernel* $\mathcal{V}$ *and such that on* $H = \mathcal{V}^{\perp}, \alpha = g(J\cdot,\cdot)$ *for some almost-complex structure* J *of* H. *Then for any* ϕ *in* $\mathcal{H}^p(M,g)$ *we have that* ϕ^{ij} *belongs to* $\mathcal{H}^p(M,g)$, *where for any* i, j *with* $i + j = p$ *we have denoted by* ϕ^{ij} *the orthogonal projection of* ϕ *onto* $\Lambda^i\mathcal{V}\widehat{\otimes}\Lambda^j H \subseteq \Lambda^p M$. *Here* $\Lambda^i\mathcal{V}\widehat{\otimes}\Lambda^j H$ *is the image of* $\Lambda^i\mathcal{V}\otimes\Lambda^j H$ *in* $\Lambda^{i+j}M$ *under the anti-symmetrisation map.*

Proof. We first note that

$$L_\alpha^\star(\psi \wedge \alpha) = \tfrac{1}{2}(-1)^p(\dim\, H)\psi + (L_\alpha^\star\psi)\wedge\alpha + (-1)^p Q\psi$$

whenever ψ is a p-form on M, where the operator Q is given by $Q\psi = \sum_{e_i \in H}(e_i \lrcorner \psi)\wedge e^i$ for an arbitrary local frame $\{e_i\}$ in H. Hence Q preserves the space of harmonic forms and on the other hand a standard computation shows that the non-zero eigenvalues of Q on $\Lambda^p M$ are $(-1)^{p-1}j$ for $1 \leqslant j \leqslant \dim\, H$ and $i = p - j \leqslant \dim\,\mathcal{V}$ with corresponding eigenbundles $\Lambda^i\mathcal{V}\widehat{\otimes}\Lambda^j H$. However, formality actually implies that all powers of Q preserve $\mathcal{H}^p(M,g)$, and the claim follows. □

Lemma 4.2. *Let M^6 be geometrically formal and let g be a formal metric on M. If g does not admit a compatible symplectic form then every non-zero harmonic 2-form on M has 4-dimensional kernel.*

Proof. Let $\alpha \neq 0$ belong to $\mathcal{H}^2(M,g)$. It cannot be non-degenerate for Proposition 4.2, (iii) would imply the existence of a g-compatible symplectic form. It remains to see that α cannot have 2-dimensional kernel. Arguing by contradiction, let us suppose that $\mathcal{V} = \mathrm{Ker}(\alpha)$ is 2-dimensional, so that $H = \mathcal{V}^{\perp}$ is of dimension 4. Moreover, from α we get again by using Proposition 4.2 a harmonic 2-form $\alpha' = g(J\cdot,\cdot)$ on H for some almost-complex structure J on H. Then $\alpha' + \star(\alpha' \wedge \alpha')$ gives a globally defined symplectic form on M, compatible with g, and hence the desired contradiction. □

In what follows the distribution spanned by an orthonormal system of vector fields $\{X_1, \ldots, X_q\}$ on M shall be denoted by $(X_1, \ldots, X_q)$.

Proposition 4.3. *Let M^6 be geometrically formal with $b_1(M) = 0$ and $b_2(M) \geqslant 2$. If g is a formal metric on M which does not admit a compatible symplectic form we must have $b_2(M) = 2, b_3(M) = 6$.*

Proof. Let $\alpha \neq 0$ belong to $\mathcal{H}^2(M,g)$. By Lemma 4.2 the distribution $\mathcal{V} = \mathrm{Ker}(\alpha)$ must be 4-dimensional, so after constant rescaling α can be written as $\alpha = g(J\cdot,\cdot)$, where J is an almost-complex structure on the plane distribution $H = \mathcal{V}^{\perp}$. We now note there are no non-zero harmonic 2-forms contained in $\Lambda^2\mathcal{V}$, for by Lemma 4.2 any such form must have 4-dimensional kernel and hence must vanish. It follows then from Lemma 4.1 that $\mathcal{H}^2(M,g)$ is contained in $(\Lambda^1\mathcal{V}\widehat{\otimes}\Lambda^1 H) \oplus \mathbb{R}\alpha$. Further on, because $b_2(M) \geqslant 2$, there must be a non-zero β in $\Lambda^1\mathcal{V}\widehat{\otimes}\Lambda^1 H$, and again by Lemma 4.2 this has 4-dimensional kernel to be denoted by $\mathcal{V}'$. By rescaling if necessary we may also assume that β is of unit length.

Let now F_1 and F_2 be the orthogonal projections of $H' = (\mathcal{V}')^{\perp}$ onto $\mathcal{V}$ and H, respectively. The projection F_1 is not the zero space because otherwise we would have $H' \subseteq H$, hence β in $\Lambda^2 H$, an absurdity. We cannot have $F_2 = \{0\}$ either: it would imply that $H' \subseteq \mathcal{V}$ and hence $\beta \in \Lambda^2\mathcal{V}$, which again is impossible. Therefore, both of F_1 and F_2 have rank at least 1 and given that $H' = F_1 \oplus F_2$ and H' has rank 2, their respective ranks must actually equal 1. Since the manifold is oriented, every real line bundle over M is trivial and this leads to the existence of a globally defined orthonormal frame $\{\zeta, e_2\}$ on H', spanning F_1 and F_2. Since β belongs to $\Lambda^2 H'$, it follows that

$$\beta = e^2 \wedge \zeta.$$

Now the orthogonal complement of (e_2) in H is 1-dimensional, and hence trivial as a real line bundle. Therefore it is spanned by some a unit vector field, say e_1, and since α belongs to $\Lambda^2 H$ we get

$$\alpha = e^1 \wedge e^2.$$

choose now a non-zero harmonic 3-form T on M. By Lemma 4.1 the components T^{11} in $\Lambda^3\mathcal{V}$ and $T^{12} = \theta \wedge \alpha$ in $\Lambda^1\mathcal{V}\widehat{\otimes}\Lambda^2 H$ of T are harmonic. However, $\star L_\alpha T^{11}$ and $L_\alpha^{\star} T^{12} = \theta$ are harmonic 1-forms and since $b_1(M) = 0$ these 1-forms are vanishing fact which implies the nullity of T^{11} and T^{12}. Hence T can be written as

$$T = \omega_1 \wedge e^1 + \omega_2 \wedge e^2$$

with $\omega_k, k = 1, 2$ in $\Lambda^2\mathcal{V}$. Again, $L_\phi T$ and $L^\star_\phi T$ vanish for any harmonic 2-form ϕ because $b_1(M) = 0$, and hence from $L_\beta T = 0$ and $L^\star_\beta T = 0$ we get that

$$\zeta \wedge \omega_1 = 0, \quad \zeta \lrcorner \omega_2 = 0.$$

It follows easily that harmonic 3-forms on M are contained in a rank 6 sub-bundle of $\Lambda^3 M$, thus using that scalar products of harmonic 3-forms are (pointwisely) constant we obtain that $b_3(M) \leqslant 6$. Since M has nowhere vanishing vector fields, it has vanishing Euler characteristic, and from $b_1(M) = 0, b_2(M) \geqslant 2$ we get

$$b_3(M) = 2(1 + b_2(M)) \geqslant 6$$

showing that actually $b_2(M) = 2$ and $b_3(M) = 6$. □

THEOREM 4.1. *Let M^6 be geometrically formal with $b_1(M) \neq 1$ and $b_2(M) \geqslant 2$. If g is a formal metric on M which does not admit a compatible symplectic form then either:*

(i) *M has the real cohomology algebra of $\mathbb{T}^3 \times S^3$, or*

(ii) *$b_1(M) = 0, b_2(M) = 2, b_3(M) = 6$.*

Proof. In view of the Proposition above it suffices to treat the cases when $b_1(M) \neq 0$. Again, we do a case by case discussion. Let $\mathcal{V}$ be the distribution spanned by the harmonic 1-forms and let $\zeta_k, 1 \leqslant k \leqslant b_1(M)$ be a frame of harmonic 1-forms in $\mathcal{V}$. As an immediate consequence of Lemma 4.1 and of the fact that $H = \mathcal{V}^\perp$ does not contain, by definition, harmonic 1-forms it follows that harmonic 2-forms are contained in $\Lambda^2\mathcal{V} \oplus \Lambda^2 H$.

If $b_1(M) = 2$, H is of rank 4 and since $b_2(M) \geqslant 2$ there must be a non-zero harmonic 2-form contained in $\Lambda^2 H$. In view of Lemma 4.2 it has rank 4 kernel and therefore vanishes, a contradiction.

Suppose now that $b_1(M) = 3$ so that H is of rank 3. If α is a non-zero harmonic 2-form contained in $\Lambda^2 H$, then $\zeta_1 \lrcorner \zeta_2 \lrcorner \zeta_3 \lrcorner (\star\alpha)$ is a non-zero harmonic form in $\Lambda^1 H$ which is a contradiction. Therefore $\mathcal{H}^2(M, g) \subseteq \Lambda^2\mathcal{V}$ and similarly, by using Lemma 4.1 we get $\mathcal{H}^3(M, g) \subseteq \Lambda^3\mathcal{V} \oplus \Lambda^3 H$. It is now straightforward that M has the cohomology algebra of $\mathbb{T}^3 \times S^3$.

If $b_1(M) = 4$, then $\zeta_1 \wedge \zeta_2 + \zeta_3 \wedge \zeta_4 + \star(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \zeta_4)$ is a compatible symplectic form, a contradiction.

Now we cannot have $b_1(M) = 5$ (see [**7**]) and when $b_1(M) = 6$ there exists an orthonormal frame of harmonic 1-forms and hence a compatible symplectic structure, a contradiction. This completes the proof of the Theorem. □

The proof of Theorem 1.4, when $b_2(M) \geqslant 3$ follows now immediately from the above.

REMARK 4.1. The proof of Proposition 4.3 can also be adapted to show that if g is a formal metric on M^6 which does not admit a compatible symplectic structure then $b_3(M) \leqslant 6$ when $b_1(M) = 0, b_2(M) = 1$.

4.2. *The case when $b_1 = 0, b_2 = 2, b_3 = 6$*

We shall examine now the case when the geometrically formal manifold M^6 has a formal metric g which does not admit a compatible symplectic form and, moreover, $b_1(M) = 0, b_2(M) = 2, b_3(M) = 6$. We have seen that harmonic 2-forms must be of the form $e^{12} = e^1 \wedge e^2, e^2 \wedge \zeta$ for some orthonormal system e_1, e_2, ζ in TM. Let us denote by E the rank 3 distribution orthogonal to e_1, e_2, ζ. It inherits a transversal volume form, that is, a nowhere vanishing 3-form ν_E in $\Lambda^3 E$ given by $\nu_E = \star(e^{12} \wedge \zeta)$. We shall write $\star_E : \Lambda^\star E \to \Lambda^\star E$ for the Hodge star operator obtained when E is equipped with the restriction of the metric g and orientation given by ν_E.

LEMMA 4.3. *The following hold:*

$$\begin{aligned} de^1 &= A \wedge e^1 + B \wedge e^2 + \lambda e^{12}, \\ de^2 &= q\zeta \wedge e^1 - A \wedge e^2 + \mu e^{12}, \\ d\zeta &= A \wedge \zeta - \mu e^1 \wedge \zeta + e^2 \wedge D, \end{aligned}$$

where A, B, D are 1-forms on $E \oplus (\zeta)$ and λ, q, μ are functions on M.

Proof. Because e^{12} is closed we get $de^1 \wedge e^2 = de^2 \wedge e^1$ and it follows that none of de^1, de^2 can have components in $\Lambda^2(e_1, e_2)^\perp$. Therefore one can write

$$\begin{aligned} de^1 =& A \wedge e^1 + B \wedge e^2 + \lambda e^{12}, \\ de^2 =& C \wedge e^1 + D' \wedge e^2 + \mu e^{12} \end{aligned}$$

for some one-forms A, B, C, D' in $\Lambda^1(e_1, e_2)^\perp$ and some smooth functions λ, μ on M. Now the remaining information contained in $de^1 \wedge e^2 = de^2 \wedge e^1$ is that $D' = -A$. Since $e^2 \wedge \zeta$ is equally closed we have $de^2 \wedge \zeta = d\zeta \wedge e^2$ and hence $de^2 \wedge \zeta \wedge e^2 = 0$ leading to $C \wedge \zeta = 0$. Thus we may write $C = q\zeta$ for some smooth function q on M. Moreover, by an argument already used for e^{12}, $d\zeta$ has no component in $\Lambda^2(e_2, \zeta)^\perp$ and hence after a small computation we can fully rewrite the closedness of $e^2 \wedge \zeta$ as

$$d\zeta = A \wedge \zeta - \mu e^1 \wedge \zeta + e^2 \wedge D + \nu e^{12}$$

for some one form D on $E \oplus (\zeta)$ and a smooth function ν on M. Now the harmonicity of e^{12} tells us that

$$0 = d^\star e^{12} = d^\star e^1 \cdot e^2 - [e_1, e_2] - d^\star e^2 \cdot e_2$$

in other words the distribution (e_1, e_2) is integrable. Henceforth, $\nu = d\zeta(e_1, e_2) = -\langle \zeta, [e_1, e_2] \rangle$ vanishes and our Lemma is proved. □

COROLLARY 4.1. (i) *The distribution E is integrable.*
(ii) *The distributions (e_1, e_2) and (e_2, ζ) are integrable as well.*

Proof. (i) By inspecting the structure equations in the lemma above, we see that either of $d\zeta, de^1, de^2$ vanish on $\Lambda^2 E$ and the claim follows.
(ii) Follows by arguments similar to the last part of the proof of the Lemma 4.3. □

We shall now bring into consideration the fact that $b_3(M) = 6$. Let

$$T_1, T_2, T_3, \star T_1, \star T_2, \star T_3 \tag{4.1}$$

be an (pointwisely) orthonormal basis in $\mathcal{H}^3(M, g)$. From the proof of Proposition 4.3 we must have

$$T_k = (e^1 \wedge \zeta) \wedge \alpha_k + e^2 \wedge \star_E \beta_k,$$

where α_k, β_k belong to $\Lambda^1 E$ for all $1 \leqslant k \leqslant 3$. The next Lemma recasts the orthogonality of the system (4.1) into a simpler algebraic form.

LEMMA 4.4. *For $1 \leqslant k \leqslant 3$ we define $\gamma_k = \alpha_k + i\beta_k$ in $\Lambda^1(E, \mathbb{C})$. We have*

$$\begin{aligned} \star_E \gamma_1 &= k\overline{\gamma}_2 \wedge \overline{\gamma}_3, \\ \star_E \gamma_2 &= -k\overline{\gamma}_1 \wedge \overline{\gamma}_3, \\ \star_E \gamma_3 &= k\overline{\gamma}_1 \wedge \overline{\gamma}_2 \end{aligned}$$

for some smooth function $k : M \to \mathbb{C}$ such that $|k| = 1$ and $k\overline{\gamma}_1 \wedge \overline{\gamma}_2 \wedge \overline{\gamma}_3 = \nu_E$.

Proof. The Hodge star operator of the forms $T_k, 1 \leqslant k \leqslant 3$ reads

$$\star T_k = -(e^1 \wedge \zeta) \wedge \beta_k + e^2 \wedge \star_E \alpha_k$$

and the orthonormality of (4.1) is equivalent with the following

$$\begin{aligned} &|\alpha_k|^2 + |\beta_k|^2 = 1, \\ &\langle \alpha_i, \alpha_j \rangle + \langle \beta_i, \beta_j \rangle = 0, \quad i \neq j, \\ &\langle \alpha_i, \beta_j \rangle = \langle \alpha_j, \beta_i \rangle. \end{aligned}$$

It is easy to see that $\{\gamma_i, 1 \leqslant i \leqslant 3\}$ gives a basis of $\Lambda^1(E, \mathbb{C})$ (not orthonormal though) and then $\{\gamma_i \wedge \gamma_j : 1 \leqslant i \neq j \leqslant 3\}$ is a basis in $\Lambda^2(E, \mathbb{C})$. Of course, by using complex conjugation we obtain another set of basis in the above-mentioned spaces. We now compute

$$\begin{aligned} \star_E \gamma_j \wedge \overline{\gamma}_j &= (\star_E \alpha_j + i \star_E \beta_j) \wedge (\alpha_j - i \star_E \beta_j) \\ &= (\star_E \alpha_j \wedge \alpha_j + \star_E \beta_j) + i(\star_E \beta_j \wedge \alpha_j - \star_E \alpha_j \wedge \beta_j) \\ &= \nu_E. \end{aligned}$$

Similarly, we also find that $\star_E \gamma_j \wedge \overline{\gamma}_p = 0$ for $p \neq j$ and the result follows. That $|k| = 1$ follows routineously by taking norms. □

The triple of 1-forms $(\gamma_1, \gamma_2, \gamma_3)$ has also an internal symmetry, of particular relevance for what follows. Write

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

and then notice the transition formula $\gamma = P\overline{\gamma}$ for some $P = (P_{ij}, 1 \leqslant i, j \leqslant 3) : M \to M_3(\mathbb{C})$. This is possible because both γ and $\overline{\gamma}$ leave basis in $\Lambda^1(E, \mathbb{C})$. It follows immediately that $P\overline{P} = I_3$ holds and, moreover, from the definition of P we see that it is symmetric, that is, $P = P^{\mathrm{T}}$. To exploit the closedness the frame (4.1) we need the following.

LEMMA 4.5. *If α belongs to $\Lambda^\star E$ we have*

$$d\alpha = d_E \alpha + \zeta \wedge L^E_\zeta \alpha + e^1 \wedge (L^E_{e_1} \alpha + \zeta \wedge R \lrcorner \alpha) + e^2 \wedge L^E_{e_2} \alpha,$$

where d_E denotes the orthogonal projection of d onto $\Lambda^\star E$ and for any vector field X in E, L^E_X is the orthogonal projection of the Lie derivative $L_X \alpha$ onto $\Lambda^\star E$. Moreover, the vector field R in E is given by the projection on E of $[e_1, \zeta]$.

Proof. Follows eventually by expanding d along the decomposition

$$\Lambda^\star M = \Lambda^\star E \otimes \Lambda^\star(e_1, e_2, \zeta),$$

while making use of the integrability of the distributions listed in Corollary 4.1. □

Let us denote by $\hat{A}, \hat{B}, \hat{D}$ the components on E of the 1-forms A, B, D, so that $A = \hat{A} + x\zeta, B = \hat{B} + y\zeta, D = \hat{D} + z\zeta$ for some smooth functions x, y, z on M.

Lemma 4.6. *The harmonicity of the forms $T_k, 1 \leqslant k \leqslant 3$ is equivalent with the following system of equations:*

(i) $d_E\gamma_k = -2\hat{A} \wedge \gamma_k - iq \star_E \gamma_k$;
(ii) $d_E(\star_E\gamma_k) = \hat{A} \wedge \star_E\gamma_k$;
(iii) $L^E_\zeta(\star_E\gamma_k) - x \star_E \gamma_k - i\hat{B} \wedge \gamma_k = 0$;
(iv) $L^E_{e_1}(\star_E\gamma_k) + \mu \star_E \gamma_k - i\hat{D} \wedge \gamma_k = 0$;
(v) $L^E_{e_2}\gamma_k + (z - \lambda)\gamma_k - iR\lrcorner \star_E \gamma_k = 0$

for $1 \leqslant k \leqslant 3$.

Proof. For any $1 \leqslant k \leqslant 3$ the closedness of the forms T_k is equivalent with

$$\begin{aligned} 0 = dT_k = d(e^1 \wedge \zeta) \wedge \alpha_k + e^1 \wedge \zeta \wedge d\alpha_k \\ + de^2 \wedge [\star_E\beta_k] - e^2 \wedge d[\star_E\beta_k]. \end{aligned}$$

Using now Lemma 4.5 we obtain further

$$\begin{aligned} 0 = &\, d(e^1 \wedge \zeta) \wedge \alpha_k + de^2 \wedge \star_E\beta_k \\ &+ e^1 \wedge \zeta \wedge \left[d_E\alpha_k + e^2 \wedge L^E_{e_2}\alpha_k\right] \\ &- e^2 \wedge d_E(\star_E\beta_k) - e^2 \wedge \zeta \wedge L^E_\zeta(\star_E\beta_k) + e^{12} \wedge L^E_{e_1}(\star_E\beta_k) \\ &- e^{12} \wedge \zeta \wedge (R\lrcorner \star_E \beta_k). \end{aligned}$$

However, accordingly to Lemma 4.3 we eventually get

$$d(e^1 \wedge \zeta) = 2\hat{A} \wedge e^1 \wedge \zeta + \hat{B} \wedge e^2 \wedge \zeta - \hat{D} \wedge e^{12} + (\lambda - z)e^{12} \wedge \zeta.$$

Hence after identifying the components of $e^1 \wedge \zeta, e^2 \wedge \zeta, e^{12}, e^{12} \wedge \zeta, e^2$ we find the system of equations

$$\begin{aligned} &2\hat{A} \wedge \alpha_k - q \star_E \beta_k + d_E\alpha_k = 0, \\ &\hat{B} \wedge \alpha_k + x \star_E \beta_k - L^E_\zeta(\star_E\beta_k) = 0, \\ &-\hat{D} \wedge \alpha_k + \mu \star_E \beta_k + L^E_{e_1}(\star_E\beta_k) = 0, \\ &(\lambda - z)\alpha_k - L^E_{e_2}\alpha_k - R\lrcorner \star_E \beta_k = 0, \\ &\hat{A} \wedge \star_E\beta_k = d_E(\star_E\beta_k). \end{aligned}$$

However, the forms $\star T_k, 1 \leqslant k \leqslant 3$ are closed as well, in other words the system above has the symmetry $(\alpha_k, \beta_k) \to (\beta_k, -\alpha_k)$. It is now straightforward to rephrase these by means of the complex-valued forms $\gamma_k, 1 \leqslant k \leqslant 3$. □

We are now in position to examine the geometric consequences imposed by our initial situation.

Lemma 4.7. *The following hold:*

(i) $\hat{A} = 0$;
(ii) $d_Ek = 0$.

Proof. We will prove both claims at the same time. Using Lemma 4.6, (i) we compute

$$\begin{aligned} d_E(\gamma_2 \wedge \gamma_3) &= -4\hat{A} \wedge \gamma_2 \wedge \gamma_3 - iq(\star_E \gamma_2 \wedge \gamma_3 - \star_E \gamma_3 \wedge \gamma_2) \\ &= -4\hat{A} \wedge \gamma_2 \wedge \gamma_3 \end{aligned}$$

by using standard properties of the Hodge star operator. However from (ii) of the same Lemma, actualised by Lemma 4.4 one infers that

$$d_E(k\overline{\gamma}_2 \wedge \overline{\gamma}_3) = k\hat{A} \wedge \overline{\gamma}_2 \wedge \overline{\gamma}_3.$$

It follows that $(5\hat{A} - k^{-1}d_E k) \wedge \overline{\gamma}_2 \wedge \overline{\gamma}_3 = 0$ and repeating the procedure for the other two equations in Lemma 4.6, (i) we arrive easily to $5\hat{A} - k^{-1}d_E k = 0$. However, $\hat{A}$ is real-valued while $k^{-1}d_E k$ belongs to $\Lambda^1(E, i\mathbb{R})$ since $|k| = 1$ and the proof of the Lemma follows. □

We examine the rest of the equations in Lemma 4.6. For a triple

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

of one forms in $\Lambda^1(E, \mathbb{C})$ we consider the triple of 2-forms in $\Lambda^2(E, \mathbb{C})$ given by

$$\alpha \times \alpha = \begin{pmatrix} \alpha_2 \wedge \alpha_3 \\ \alpha_3 \wedge \alpha_1 \\ \alpha_1 \wedge \alpha_2 \end{pmatrix}.$$

Note that in the new notation Lemma 4.4 now reads

$$\star_E \gamma = k\overline{\gamma} \times \overline{\gamma} \tag{4.2}$$

and after taking the conjugate we also get

$$\star_E \overline{\gamma} = k^{-1} \gamma \times \gamma \tag{4.3}$$

since $\overline{k} = k^{-1}$. For any $\alpha = \sum_{k=1}^{3} \alpha_k \gamma_k$ in $\Lambda^1(E, \mathbb{C})$ we consider the matrix

$$r_\alpha = \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix}$$

Note that $r_\alpha^{\mathrm{T}} = -r_\alpha$ and we shall let r_α operate on triple of forms in $\Lambda^k(E, \mathbb{C}), k = 1, 2$ by matrix multiplication. Moreover, a straightforward computation shows that

$$\alpha \wedge \gamma = \begin{pmatrix} \alpha \wedge \gamma_1 \\ \alpha \wedge \gamma_2 \\ \alpha \wedge \gamma_3 \end{pmatrix} = r_\alpha(\gamma \times \gamma).$$

These observations now allow us to bring the remaining equations into final form.

LEMMA 4.8. *The following hold:*
(i) $L^E_\zeta(\star_E \gamma) - x \star_E \gamma - ikr_{\hat{B}}(\star_E \overline{\gamma}) = 0$;
(ii) $L^E_{e_1}(\star_E \gamma) + \mu \star_E \gamma - ikr_{\hat{D}}(\star_E \overline{\gamma}) = 0$;
(iii) $L^E_{e_2}\gamma + (z - \lambda)\gamma + ikr_\eta \overline{\gamma} = 0$
where the 1*-form* η *in* $\Lambda^1 E$ *is given as* $\eta = g(R, \cdot)$.

Proof. We shall prove only (i) the other two claims being entirely analogous. Indeed, writing (iii) of Lemma 4.6 in matrix form we have

$$L^E_\zeta(\star_E \gamma) - x \star_E \gamma - i\hat{B} \wedge \gamma = 0.$$

However, $\hat{B} \wedge \gamma = r_{\hat{B}}(\gamma \times \gamma) = kr_{\hat{B}}(\star_E \overline{\gamma})$ by (4.3) and we are done. $\square$

Proposition 4.4. *The following hold:*
(i) $L_{e_1}P = L_{e_2}P = L_{\zeta}P = 0$;
(ii) $P\overline{r_{\hat{B}}}P + k^2 r_{\hat{B}} = 0$;
(iii) $P\overline{r_{\hat{D}}}P + k^2 r_{\hat{D}} = 0$;
(iv) $P\overline{r_{\eta}}P + k^2 r_{\eta} = 0$.

Proof. Taking the conjugate in (i) of Lemma 4.8 we get

$$L_{\zeta}^{E}(\star_E \overline{\gamma}) - x \star_E \overline{\gamma} + ik^{-1}\overline{r_{\hat{B}}}(\star_E \gamma) = 0. \tag{4.4}$$

Now $\star_E \gamma = \star_E(P\overline{\gamma}) = P(\star_E \overline{\gamma})$ and hence (i) of Lemma 4.8 gives

$$(L_{\zeta}^{E}P) \star_E \overline{\gamma} + PL_{\zeta}^{E}(\star_E \overline{\gamma}) + xP(\star_E \overline{\gamma}) + ikr_{\hat{B}}(\star_E \overline{\gamma}) = 0.$$

Substituting here the expression of $L_{\zeta}^{E}(\star_E \overline{\gamma})$ as given by (4.4) we obtain further

$$(L_{\zeta}^{E}P) \star_E \overline{\gamma} + P\Big[x \star_E \overline{\gamma} - ik^{-1}\overline{r}_{\hat{B}}(\star_E \gamma)\Big] - (xP + ikr_{\hat{B}}) \star_E \overline{\gamma} = 0$$

whence

$$(L_{\zeta}^{E}P - ik^{-1}P\overline{r_{\hat{B}}}P - ikr_{\hat{B}}) \star_E \overline{\gamma} = 0,$$

where we have used once more that $\gamma = P\overline{\gamma}$. Given that $\star_E \overline{\gamma}$ gives a basis in $\Lambda^2(E, \mathbb{C})$ we infer that

$$L_{\zeta}^{E}P - ik^{-1}P\overline{r_{\hat{B}}}P - ikr_{\hat{B}} = 0.$$

However, P is symmetric and $r_{\hat{B}}$ is skew-symmetric therefore $P\overline{r_{\hat{B}}}P$ is skew-symmetric as well, and hence identifying the symmetric and skew-symmetric part in the equation above we arrive at $L_{\zeta}^{E}P = 0$ and $P\overline{r_{\hat{B}}}P + k^2 r_{\hat{B}} = 0$. The other two claims in (i) and assertions in (iii) and (iv) are proved by applying a completely similar procedure to the equations in (ii) and (iii) of Lemma 4.8. $\square$

Corollary 4.2. *We must have $\hat{B} = \hat{D} = \eta = 0$.*

Proof. We first work out the equation in (ii) of Lemma 4.4. It implies that

$$(P\overline{r_{\hat{B}}}P) \star_E \overline{\gamma} + k^2 r_{\hat{B}}(\star_E \overline{\gamma}) = 0.$$

Now $r_{\hat{B}}(\star_E \overline{\gamma}) = k^{-1} r_{\hat{B}}(\gamma \times \gamma) = k^{-1}\hat{B} \wedge \gamma$. On the other hand we have

$$\begin{aligned}(P\overline{r_{\hat{B}}}P) \star_E \overline{\gamma} &= (P\overline{r_{\hat{B}}}) \star_E (P\overline{\gamma}) \\ &= P\overline{r_{\hat{B}}} \star_E \gamma = kP\overline{r_{\hat{B}}}(\overline{\gamma} \times \overline{\gamma}) \\ &= kP(\overline{\hat{B}} \wedge \overline{\gamma}) \\ &= k\hat{B} \wedge P\overline{\gamma} = k\hat{B} \wedge \gamma\end{aligned}$$

since $\hat{B}$ is real-valued. Altogether $(k + k^{-1}k^2)\hat{B} \wedge \gamma = 0$ whence the vanishing of $\hat{B}$ since $|k| = 1$. The vanishing of $\hat{D}$ follows from (iii) of Lemma 4.4 and the varnishing of η follows from (iv) of Lemma 4.4, respectively, by using the same argument. $\square$

We now continue the study of the distribution (e_1, e_2, ζ).

LEMMA 4.9. *The following hold:*
(i) $d^\star e^1 = -\mu$;
(ii) $d^\star e^2 = \lambda = -z$;
(iii) $d^\star \zeta = x$.

Proof. First of all we update Lemma 4.3 to

$$\begin{aligned} de^1 =& x\zeta \wedge e^1 + y\zeta \wedge e^2 + \lambda e^{12}, \\ de^2 =& q\zeta \wedge e^1 - x\zeta \wedge e^2 + \mu e^{12}, \\ d\zeta =& -\mu e^1 \wedge \zeta + ze^2 \wedge \zeta. \end{aligned} \tag{4.5}$$

by using that $\hat{A} = \hat{B} = \hat{D} = 0$.
(i) Since e^{12} is harmonic we have

$$0 = d^\star(e^{12}) = d^\star e^1 \cdot e_2 - [e_1, e_2] - d^\star e^2 \cdot e_1.$$

Hence $d^\star e^1 = \langle [e_1, e_2], e_2 \rangle = -de^2(e_1, e_2) = -\mu$ and $d^\star e^2 = -\langle [e_1, e_2], e_1 \rangle = de^1(e_1, e_2) = \lambda$. This proves (i) and the first-half of (ii) to prove the rest it is enough to repeat the argument above starting from $d^\star(e^2 \wedge \zeta) = 0$. $\square$

THEOREM 4.2. *A geometrically formal manifold M^6 with $b_1(M) = 0, b_2(M) = 2, b_3(M) = 6$ and formal metric g must admit a g-compatible symplectic structure.*

Proof. Suppose that there is no g-compatible symplectic structure on M. Then our whole previous discussion applies and based upon it we will obtain a contradiction. We proceed first towards updating the expressions of the Lie derivatives of $\gamma, \star_E \gamma$ as given by Lemma 4.8. Since $k^2 = \det(P)$ and P has no Lie derivatives in the direction of (e_1, e_2, ζ) it follows that $L_{e_1} k = L_{e_2} k = L_\zeta k = 0$. Therefore, (i) of Lemma 4.8 gives

$$L^E_\zeta(\overline{\gamma} \times \overline{\gamma}) - x\overline{\gamma} \times \overline{\gamma} = 0.$$

Note that actually $L^E_\zeta \gamma = L_\zeta \gamma$ since η (hence R) vanishes. A short computation using only that γ gives a basis in $\Lambda^1(E, \mathbb{C})$ leads to

$$L_\zeta \gamma - \frac{x}{2}\gamma = 0.$$

It follows that $L_\zeta(\gamma_1 \wedge \gamma_2 \wedge \gamma_3) = (3x/2)\gamma_1 \wedge \gamma_2 \wedge \gamma_3$ whence $L_\zeta \nu_E = (3x/2)\nu_E$. However, $L_\zeta(e^{12} \wedge \zeta) = 0$ as well, because $e^{12}, e^{12} \wedge \zeta$ are closed (the latter after a computation based on (4.5)) and we get that the volume form $\nu_M = e^{12} \wedge \zeta \wedge \nu_E$ satisfies $L_\zeta \nu_M = (3x/2)\nu_M$. However,

$$L_\zeta \nu_M = d(\zeta \lrcorner \nu_M) = -d^\star \zeta \cdot \nu_M = -x\nu_M$$

by Lemma 4.9, (iii) and it follows that we must have $x = 0$. When working out, in the same spirit, the equation contained in (ii) of Lemma 4.8 we obtain that $\mu = 0$. Now (iii) of Lemma 4.8 ensures, as before, that $L_{e_2}\nu_E + 3(z - \lambda)\nu_E = 0$. At the same time

$$L_{e_2}(e^{12} \wedge \zeta) = -d(e^1 \wedge \zeta) = (-\lambda + z)e^{12} \wedge \zeta$$

after making use of (4.5). It follows that $L_{e_2}\nu_M = -2(z - \lambda)\nu_M = 4\lambda\nu_M$ as $z = -\lambda$ by Lemma 4.9, (ii). However, once again from $L_{e_2}\nu_M = -d^\star e^2 \cdot \nu_M = -\lambda \cdot \nu_M$ we obtain that $\lambda = 0$.

Inspecting now the structure equations in (4.5) we see that $d\zeta = 0$ and again from Lemma 4.9 $d^\star \zeta = 0$, in other words ζ is a harmonic, nowhere vanishing 1-form on M which contradicts that $b_1(M) = 0$. $\square$

The proof of the Theorem 4.1 in Section 1 is now complete.

5. *Formal metrics with maximal b_2*

We study in this section geometrically formal manifolds M^n having maximal second Betti number, that is,

$$b_2(M) = \binom{n}{2}.$$

To prove Theorem 1.5, we split our discussion into two cases according to the parity of n.

Proposition 5.1. *Let M^n be geometrically formal and let g be a formal metric on M. The following hold:*

(i) *if $b_p(M)$ and $b_q(M)$ are maximal for $p + q \leqslant n$ then $b_{p+q}(M)$ is also maximal;*
(ii) *if $b_p(M)$ and $b_q(M)$ are maximal for $0 \leqslant p \leqslant q \leqslant n$ and then so is $b_{q-p}(M)$;*
(iii) *if $b_p(M)$ is maximal for some $1 \leqslant p \leqslant n-1$ and $(p, n) = 1$ then g is a flat metric.*

Proof. (i) If $\{\alpha_i\}, \{\beta_j\}$ are L^2-orthonormal basis in $\mathcal{H}^p(M, g)$ and $\mathcal{H}^q(M, g)$, respectively, then at each point of M we obtain orthonormal basis in $\Lambda^p M$ and $\Lambda^q M$, respectively. It follows that $\Lambda^{p+q} M$ is spanned by forms of the type $\alpha_i \wedge \beta_j$ which are harmonic because the metric g is formal. Since scalar products between harmonic forms are constant after Gramm-Schmidt orthonormalisation we obtain a basis in $\mathcal{H}^{p+q}(M, g)$.

(ii) By Hodge duality $b_{n-p}(M)$ is maximal and hence by (i) so is $b_{n-p+q}(M) = b_{q-p}(M)$ whence the claim.

(iii) If $b_p(M)$ is maximal then for any integers q and k, $1 \leqslant k \leqslant n$ such that $pq \equiv k (\mathrm{mod}\ n)$, $b_k(M)$ is also maximal by using (i). Since $(p, n) = 1$, we arrive by means of (ii) at $b_1(M)$ maximal, and it follows that g is flat by Theorem 1.1, (iii). □

Hence, when n is odd and $b_2(M)$ is maximal, $b_1(M)$ is maximal too and the metric g is flat. Therefore, we need only to consider the case when n is even.

5.1. *Reduction to the symplectic case*

As an immediate consequence of Proposition 4.2 we have the following.

Proposition 5.2. *Let M^n be a geometrically formal manifold with formal metric g such that $b_2(M)$ is maximal and n is even. Then g admits a compatible almost-Kähler structure, that is, an almost-complex structure J, which is compatible with g and such that the 2-form $g(J\cdot, \cdot)$ is closed.*

Proof. We first claim that there exists a harmonic 2-form α which is non-degenerate, that is, $\alpha^k \neq 0, n = 2k$ at some point x of M. Indeed if $\varphi^k = 0$ on M for any φ in $\mathcal{H}^2(M, g)$ then after polarisation we find $\varphi_1 \wedge \ldots \wedge \varphi_k = 0$ whenever $\varphi_i, 1 \leqslant i \leqslant k$ belong to $\mathcal{H}^2(M, g)$. Since frames in $\mathcal{H}^2(M, g)$ give frames in the $\Lambda^2 M$ it is easy to obtain a contradiction and the existence of α as above follows. The claim is now proved by using (iii) in Proposition 4.2. □

5.2. *Proof of flatness*

We consider hereafter a compact almost-Kähler manifold (M^n, g, J) $(n = 2k)$ such that g is a formal metric and, moreover, $b_2(M) = \binom{n}{2}$. Let $\omega = g(J\cdot, \cdot)$ be the so-called Kähler form of the almost-Kähler structure. We first remark that the bi-type splitting of $\Lambda^2 M$ is preserved at the level of harmonic forms (note, by contrast with the Kähler case that this needs no longer be true in the case of an arbitrary almost-Kähler manifold).

LEMMA 5.1. *Any harmonic 2-form splits as $\alpha = \alpha_1 + \alpha_2$, where the harmonic α_1, α_2 are in $\lambda^{1,1}M$ and $\lambda^2 M$, respectively.*

Proof. Choose α in $\Lambda^2 M$, which splits as $\alpha = \alpha_1 + \alpha_2$ with α_1 in $\lambda^{1,1}M$ and α_2 in $\lambda^2 M$. Because of formality we can assume without loss of generality that α is primitive. Again the formality tells us that $L_\alpha^\star(\omega \wedge \omega)$ is harmonic and from the proof of Proposition 4.1 it follows that it is actually proportional to $\alpha_1 - \alpha_2$. This eventually proves the Lemma. $\square$

Therefore, if $b_2(M)$ is maximal, both $\lambda^{1,1}M$ and $\lambda^2 M$ are spanned by harmonic forms. We need now to see which geometric properties a harmonic 2-form in $\lambda^2 M$ must have. To do so, recall that the first-canonical Hermitian connection $\overline{\nabla}$ of the almost-Kähler (g, J) is given by

$$\overline{\nabla}_X = \nabla_X + \eta_X$$

for all X in TM. Here ∇ is the Levi–Civita connection of g and $\eta_X = (1/2)(\nabla_X J)J$ for all X in TM gives the intrinsic torsion of the $U(n)$-structure induced by (g, J). The connection $\overline{\nabla}$ is metric and Hermitian, that is, it preserves both the metric and the almost-complex structure. The almost-Kähler condition, that is, $d\omega = 0$, when formulated in terms of the intrinsic torsion tensor η reads

$$\langle \eta_X Y, Z \rangle + \langle \eta_Y Z, X \rangle + \langle \eta_Z X, Y \rangle = 0 \tag{5.1}$$

for all X, Y, Z in TM. The latter also implies that (g, J) is quasi-Kähler:

$$\eta_{JX} = \eta_X J \tag{5.2}$$

for all X in TM. Moreover, we have

$$\eta_X J = -J \eta_X \tag{5.3}$$

in other words η belongs to $\lambda^1 M \otimes_1 \lambda^2 M$. The relations (5.1), (5.2), and (5.3) will be used implicitly in subsequent computations.

LEMMA 5.2. *Let (M^{2k}, g, J) be an almost-Kähler manifold and let $\alpha = g(F\cdot, \cdot)$ be harmonic in $\lambda^2 M$. Then*

$$(\overline{\nabla}_{JX} F)JY + (\overline{\nabla}_X F)Y = -2\eta_{FX} Y \tag{5.4}$$

for all X, Y in TM.

Proof. From $d\alpha = 0$ we have that $a(\nabla \alpha) = 0$. However, $\nabla_X \alpha = \overline{\nabla}_X \alpha + \langle [F, \eta_X]\cdot, \cdot \rangle$ for all X in TM and, moreover, a simple computation based on (5.1) shows that

$$a((X, Y, Z) \to \langle [F, \eta_X]Y, Z \rangle) = a((X, Y, Z) \to \langle \eta_{FX} Y, Z \rangle).$$

Therefore $a(\overline{\nabla}\alpha + \eta_{F\cdot}) = 0$ and since the tensor under alternation belongs to $\lambda^1 M \otimes \lambda^2 M$ we use Proposition 2.1, (ii) to conclude that it is actually in $\lambda^1 M \otimes_2 \lambda^2 M$ and the proof of the claim follows by using the relations (5.2) and (5.3). $\square$

If Q is an endomorphism of M, let us define the tensor $Q \bullet \eta$ by

$$(Q \bullet \eta)(X, Y, Z) = \sigma_{X,Y,Z} \langle \eta_{QX} Y, Z \rangle$$

for all X, Y, Z in TM, where σ stands for the cyclic sum. Note that this is different from the usual action of $End(TM)$.

LEMMA 5.3. *Let (M^{2k}, g, J) be an almost-Kähler manifold and let $\alpha = g(F\cdot, \cdot)$ be harmonic in $\lambda^2 M$ with harmonic square. Then*

$$F^2 \bullet \eta = 0. \tag{5.5}$$

Proof. That $d^\star(\alpha \wedge \alpha) = 0$ translates after a calculation which parallels that in the proof of Proposition 3.1 into

$$\sigma_{X,Y,Z} \langle (\nabla_{FX} F)Y, Z \rangle = 0$$

for all X, Y, Z in TM. Rewritten by means of the canonical Hermitian connection and using (5.1) it yields

$$\begin{aligned} &\langle (\overline{\nabla}_{FX} F)Y, Z \rangle + \langle (\overline{\nabla}_{FY} F)Z, X \rangle + \langle (\overline{\nabla}_{FZ} F)X, Y \rangle \\ &\quad + \langle \eta_X FY, FZ \rangle + \langle \eta_Y FZ, FX \rangle + \langle \eta_Z FX, FY \rangle = 0. \end{aligned} \tag{5.6}$$

We shall exploit now the algebraic symmetries of the above. Changing (Y, Z) in (JY, JZ) and subtracting from the original equation implies

$$\begin{aligned} &2\langle (\overline{\nabla}_{FX} F)Y, Z \rangle - 2\langle \eta_X FZ, FY \rangle \\ &\quad + \langle (\overline{\nabla}_{FY} F)Z + (\overline{\nabla}_{JFY} F)JZ, X \rangle - \langle (\overline{\nabla}_{FZ} F)Y + (\overline{\nabla}_{JFZ} F)JY, X \rangle = 0 \end{aligned}$$

or further, after using the relation (5.4)

$$\langle (\overline{\nabla}_{FX} F)Y, Z \rangle - \langle \eta_X FZ, FY \rangle - \langle \eta_{F^2Y} Z, X \rangle + \langle X, \eta_{F^2Z} Y \rangle = 0. \tag{5.7}$$

Now taking the cyclic sum and using (5.6) we get the desired result. □

REMARK 5.1. On an almost-Kähler manifold (M^{2k}, g, J) a harmonic form α in $\lambda^2 M$ with harmonic exterior powers needs not to be parallel with respect to to the Levi–Civita connection of the metric g. This happens for instance when $\alpha = g(I\cdot, \cdot)$ for a g-compatible almost-complex structure I with $IJ + JI = 0$, which actually induces a complex-symplectic structure on M. Examples in this direction, which are not hyper-Kähler, can be constructed on certain classes of nilmanifolds [**4**].

From the Lemma above we find by J-polarisation that

$$[F, G] \bullet \eta = 0$$

for all F, G dual to harmonic forms in $\lambda^2 M$. It is well known that the splitting $\mathfrak{so}(2k) = \mathfrak{u}(k) \oplus \mathfrak{m}$, where $\mathfrak{m}$ consists in elements of $\mathfrak{so}(2k)$ anti-commuting with J, is such that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{u}(k)$ for $k \geqslant 2$. Therefore, if g is a formal metric on M^{2k} and $b_2(M)$ is maximal, we get that $F \bullet \eta = 0$ for all F dual to forms in $\lambda^{1,1} M$ provided that dim $M \geqslant 6$.

LEMMA 5.4. *If $\dim M \geqslant 6$, the intrinsic torsion tensor η must vanish identically.*

Proof. It is enough to prove the statement at an arbitrary point m of M. Choose an arbitrary unit vector V in $T_m M$ and let F be the skew-symmetric, J-invariant endomorphism of TM which is J on $E = \langle \{V, JV\} \rangle$ and vanishes on $H = E^{\perp}$. That $F \bullet \eta = 0$ says

$$\langle \eta_{FX} Y, Z\rangle + \langle \eta_{FY} Z, X\rangle + \langle \eta_{FZ} X, Y\rangle = 0$$

for all X, Y, Z in TM. It follows that $\langle \eta_V X, Y\rangle = 0$ for all X, Y in H, and hence $\eta_V X$ is in E for any $X \in H$. Moreover, since dim $M \geqslant 6$, there exists a unit vector $U \in TM$ so that (V, JV, U, JU, X, JX) is an orthogonal system. Let us consider the skew-symmetric, J-invariant endomorphism G of TM defined by $GV = U$, $GJV = JU$, $GU = -V$, $GJU = -JV$ and G vanishes on $E'^{\perp}$, where $E' = \langle \{V, JV, U, JU\} \rangle$. Then

$$\langle \eta_{GU} X, V\rangle + \langle \eta_{GX} V, U\rangle + \langle \eta_{GV} U, X\rangle = 0.$$

This implies that $\langle \eta_V X, V\rangle = -\langle \eta_U X, U\rangle$. Changing V in JV and using the J-anti-invariance of η we get $\langle \eta_V X, V\rangle = 0$. Then

$$\eta_V X = 0$$

for all $X \in H$ and $\eta_V X = \langle X, V\rangle \eta_V V + \langle X, JV\rangle \eta_V JV$ for all $X \in TM$. However from (5.2) it follows that $\eta_V V = \eta_V JV = 0$ and $\eta_V X = 0$ for all $X \in TM$. $\square$

In other words (g, J) is a Kähler structure and the flatness of the metric follows now from [**8**]. To complete the proof of Theorem 1.5, it remains to treat the case when $n = 4$. In this situation, we notice that the bundles $\Lambda^{\pm} M$ of (anti) self-dual forms are trivialised by almost-Kähler structure satisfying the quaternionic identities and using the well-known Hitchin lemma [**6**] we obtain that $\Lambda^{\pm} M$ both contain a hyper-Kähler structure and this leads routineously to the flatness of the metric.

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J.-F. Grosjean
Institut Élie Cartan
Université H. Poincaré
Nancy I, B.P.239
F-54506 Vandoeuvre-Lès-Nancy
France

grosjean@iecn.u-nancy.fr

P.-A. Nagy
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland
New Zealand

nagy@math.auckland.ac.nz