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On Nearly-Kähler Geometry

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Abstract. We consider complete nearly-Kähler manifolds with a canonical Hermitian connection. We prove some metric properties of strict nearly-Kähler manifolds and give a sufficient condition for the reducibility of the canonical Hermitian connection. A holonomic condition for a nearly-Kähler manifold to be a twistor space over a quaternionic-Kähler manifold is given. This enables us to give classification results in 10-dimensions.

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1. Introduction

Nearly-Kähler (briefly NK) geometry is related to the concept of weak holonomy introduced by Gray [7] in 1971. He proved that, among those groups acting transitively on the sphere, there are only three groups, namely

$$\mathrm{U}(n) \text{ in dimension } 2n, \quad G_2 \text{ in dimension } 7, \quad \mathrm{Spin}(9) \text{ in dimension } 16,$$

that can occur as weak holonomy groups and produce geometries other than the classical holonomy approach. NK geometry corresponds to weak holonomy $U(n)$ and was intensively studied in the seventies by Gray [8, 9]. Also note that the class of NK manifolds appears naturally as one of the 16 classes of almost Hermitian manifolds described by the Gray–Hervella classification [10].

Recent interest in the study of such manifolds can be justified by the fact that, in dimension 6, NK manifolds are related to the existence of a Killing spinor (see [11]). Furthermore, NK manifolds provide a natural example of almost-Hermitian manifolds admitting a Hermitian connection with totally skew symmetric torsion. From this point of view, they are of interest in string theory (see [5]).

The aim of this paper is to investigate a number of properties of NK manifolds related to the reducibility of the canonical Hermitian connection. We begin by proving a decomposition result which allows us to restrict our attention to strict NK manifolds (see Section 1). Our first main result is the following:

THEOREM 1.1. *Let (M^{2n}, g, J) a complete, strict NK manifold. Then the following hold:*

(i) *If g is not an Einstein metric, then the canonical Hermitian connection has reduced holonomy.*
(ii) *The metric g has positive Ricci curvature, hence M is compact with a finite fundamental group.*
(iii) *The scalar curvature of the metric g is a strictly positive constant.*

The previous theorem is a synthesis of the results contained in Section 2.

Let us recall now that one main class of examples of NK manifolds is formed by the so-called 3-symmetric spaces [8]. Other examples are provided by total spaces of Riemannian submersions with totally geodesic fibers admitting a compatible Kähler structure. These manifolds admit a canonical NK structure such that the canonical Hermitian connection has reduced holonomy (see Section 3). In particular, twistor spaces over positive quaternion-Kähler manifolds (here positive means of positive scalar curvature) have canonical NK structures, a result already proven in [1]. See also [15] for the case of twistor bundles over 4-manifolds.

In the second part of this paper, we are concerned with the study of the most simple case of reducible NK geometry which is the following:

THEOREM 1.2. *Let (M^{2n}, g, J) be a complete, simply connected, strict NK manifold. If the holonomy group of the canonical Hermitian connection is contained in $U(1) \times U(n-1)$, then M is a Riemannian product $M_1 \times M_2$, where M_1 is a strict NK manifold and M_2 is the twistor space of a positive quaternionic-Kähler manifold endowed with its canonical NK structure.*

Theorem 1.2 was already proven in 6-dimensions by a different method in [2]. Our approach consists in showing that the torsion of the canonical Hermitian connection has to be of a special algebraic type with respect to the holonomy decomposition. This will be done in Section 4. Then, using standard arguments, one can show that *M* carries a complex contact structure and a Kähler–Einstein metric. The conclusion follows from a theorem of LeBrun (see Section 5).

As a corollary of Theorem 1.2, we obtain a structure result in 10-dimensions. Note that in 8-dimensions it has already been shown by Gray [9] that there are no strict NK manifolds.

COROLLARY 1.1. *Let (M^{10}, g, J) be a complete NK manifold. Then either the universal cover of M is a Riemannian product of a Kähler surface with a six-dimensional NK manifold, or M is the twistor space of a positive, eight-dimensional quaternionic-Kähler manifold equipped with a canonical NK structure.*

Using results from [16] (see also [13]) we know that the only positive quaternionic-Kähler manifolds of 8-dimensions are the symmetric spaces $P\mathbb{H}^2$, $\mathbb{G}r_2(\mathbb{C}^4)$, $G_2/\mathrm{SO}(4)$ with canonical metrics. Hence, their twistor spaces, which are described in [13], equipped with the canonical NK structure, exhaust the list of complete, strict NK manifolds of dimension 10.

2. Nearly-Kähler Geometry

A NK manifold is an almost-Hermitian manifold (M^{2n}, g, J) such that $(\nabla_X J)X = 0$ for every vector field X on M (here ∇ denotes the Levi-Civita connection associated with the metric g). A NK manifold is called *strict* if $\nabla_X J \neq 0$ for every $X \in TM$, $X \neq 0$.

Recall that the tensor ∇J has a number of important algebraic properties that can be summarized as follows: the tensors A and B defined for X, Y, Z in TM by

$$A(X, Y, Z) = \langle(\nabla_X J)Y, Z\rangle \quad \text{and} \quad B(X, Y, Z) = \langle(\nabla_X J)Y, JZ\rangle$$

are skew-symmetric and have type $(0, 3) + (3, 0)$ as real 3-forms. Denote by Ric the Ricci tensor of the metric g and by $\mathrm{Ric}^\star$ its star version, i.e. the operator defined by

$$\langle \mathrm{Ric}^\star(X), Y\rangle = \frac{1}{2}\sum_{i=1}^{2n} R(X, JY, e_i, Je_i),$$

where R is the curvature tensor of (M, g) and $\{e_1, \dots, e_{2n}\}$ a local frame field. The difference of these tensors, to be denoted by r, is given by the formula (see [9])

$$\langle rX, Y\rangle = \sum_{i=1}^{2n}\langle(\nabla_{e_i} J)X, (\nabla_{e_i} J)Y\rangle.$$

Obviously, r is symmetric, positive and commutes with J. Another object of particular importance is the canonical Hermitian connection defined by

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY.$$

It is easy to see that $\overline{\nabla}$ is the unique Hermitian connection on M with totally skew-symmetric torsion (see, for example, [5]). Note that the torsion of $\overline{\nabla}$ given by $T(X, Y) = (\nabla_X J)JY$ vanishes iff (M, g, J) is a Kähler manifold.

The tensor r has strong geometric properties. To begin, we have

$$\overline{\nabla} r = 0. \tag{2.1}$$

In fact, Gray proved in [9] that, for all X, Y, Z in TM, we have

$$2\langle(\nabla_X r)Y, Z\rangle = \langle r(\nabla_X J)Y, JZ)\rangle + \langle r(JY), (\nabla_X J)Z\rangle.$$

But this is nothing else than (2.1)!

PROPOSITION 2.1. *Let (M^{2n}, g, J) be a complete, simply connected, NK manifold. Then M is a riemannian product $M_1 \times M_2$ where M_1 is a Kähler manifold and M_2 a strict NK manifold.*

Proof. Set $E_1 = \mathrm{Ker}(r)$ and let E_2 be the orthogonal complement of E_1 in TM. By (2.1) both E_1 and E_2 are $\overline{\nabla}$-parallel. Since $\nabla_X J$ vanishes whenever X is in E_1, the distribution E_1 is in fact ∇-parallel. Now, if X is in TM and Y in E_2, we have $(\nabla_X J)Y \in E_1^{\perp} = E_2$, hence E_2 is ∇-parallel. It is now easy to conclude by a theorem of de Rham. □

Remark 2.1. (i) Proposition 2.1 was already proven in [9] under the assumption that the tensor r is ∇-parallel.

(ii) Proposition 2.1 was first proven by different means in [12], where it was also proved that the Nijenhuis tensor of a NK manifold is $\overline{\nabla}$-parallel. We thank the referee for pointing out this paper.

Therefore, we can restrict our attention to the class of strict NK manifolds.

PROPOSITION 2.2. *Let (M^{2n}, g, J) a strict NK manifold.*

(i) *Suppose that r has more than one eigenvalue. Then the canonical Hermitian connection has reduced holonomy.*
(ii) *If the tensor r has exactly one eigenvalue, then M is a positive Einstein manifold. Furthermore, the first Chern class of (M, J) vanishes.*

Proof. (i) If $\lambda_i > 0, i = \overline{1, p}$ are the eigenvalues of r we have a $\overline{\nabla}$-parallel decomposition

$$TM = \bigoplus_{i=1}^{p} E_i, \tag{2.2}$$

where E_i is the eigenbundle corresponding to the eigenvalue λ_i. Hence, each factor is preserved by the holonomy group, which is thus reducible.

(ii) The proof can be found in [9, p. 242] but let us give it for the sake of completeness. We recall the following formula:

$$\sum_{i,j=1}^{2n} \langle re_i, e_j\rangle (R(X, e_i, Y, e_j) - 5R(X, e_i, JY, Je_j)) = 0 \tag{2.3}$$

(see [9]), where $\{e_i\}_{i=\overline{1,2n}}$ is a local orthonormal frame field and X, Y are in TM. If $r = \lambda 1_{TM}, \lambda > 0$ this formula becomes $\mathrm{Ric} - 5\mathrm{Ric}^{\star} = 0$, hence, $\mathrm{Ric} = 5\lambda/4$ as $\mathrm{Ric} - \mathrm{Ric}^{\star} = r$. The second assertion follows from the description of the first Chern class of (M, J) given in [9]. □

The first part of Theorem 1.1 now follows from the previous proposition. We will now compute the Ricci tensor of a NK manifold and show that it is completely determined by the spectral decomposition of the tensor r. This computation will be equally used in Section 5.

LEMMA 2.1. *We have, by respect to the decomposition* (2.2)*:*

(i) $\mathrm{Ric}(X, Y) = 0$ *if* X *and* Y *are vector fields belonging to* E_i *and* E_j, *respectively, and* $i \neq j$.
(ii) *If* X, Y *are vector fields in* E_i*:*

$$\mathrm{Ric}(X, Y) = \frac{\lambda_i}{4}\langle X, Y\rangle + \frac{1}{\lambda_i}\sum_{s=1}^{p}\lambda_s\langle r^s(X), Y\rangle,$$

where the tensors $r^s\colon TM \to TM$, $1 \leq s \leq p$ *are defined by* $\langle r^s(X), Y\rangle = -Tr_{E_s}(\nabla_X J)(\nabla_Y J)$ *whenever* X, Y *are in* TM.

Proof. (i) Let us denote by $\overline{R}$ the curvature tensor of the connection $\overline{\nabla}$. We have (see [9, p. 237]):

$$\overline{R}(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{2}\langle(\nabla_X J)Y, (\nabla_Z J)T\rangle + + \tfrac{1}{4}[\langle(\nabla_X J)Z, (\nabla_Y J)T\rangle - \langle(\nabla_X J)T, (\nabla_Y J)Z\rangle]. \quad (2.4)$$

Let $\{e_k\}_{k=\overline{1,2n}}$ be an orthonormal base of TM which gives orthonormal bases in E_s for $1 \leq s \leq p$. We get

$$\mathrm{Ric}(X, Y) = \sum_{s=1}^{p}\sum_{e_k \in E_s} R(X, e_k, Y, e_k).$$

If $s \neq j$, we have $\overline{R}(X, e_k, Y, e_k) = 0$, hence

$$R(X, e_k, Y, e_k) = \tfrac{1}{4}\langle((\nabla_{e_k} J))X, (\nabla_{e_k} J)Y\rangle$$

by (2.4). If $s = j$, then $s \neq i$ and, as before, we get

$$R(X, e_k, Y, e_k) = R(Y, e_k, X, e_k) = \tfrac{1}{4}\langle((\nabla_{e_k} J))X, (\nabla_{e_k} J)Y\rangle.$$

It follows that $\mathrm{Ric}(X, Y) = (1/4)\langle rX, Y\rangle = 0$.

(ii) Using (2.3), we obtain

$$\sum_{s=1}^{p}\lambda_s\left(\sum_{e_k \in E_s} R(X, e_k, Y, e_k) - 5R(X, e_k, JY, Je_k)\right) = 0.$$

Reasoning as in the proof of (i), for $s \neq i$ we get that

$$R(X, e_k, JY, Je_k) = -3R(X, e_k, Y, e_k) = -\tfrac{3}{4}\langle(\nabla_{e_k} J)X, (\nabla_{e_k} J)Y\rangle.$$

It follows that

$$4\sum_{\substack{s=1\\ s\neq i}}\lambda_s\langle r^s X, Y\rangle + \lambda_i\left(\sum_{e_k \in E_s} R(X, e_k, Y, e_k) - 5R(X, e_k, JY, Je_k)\right) = 0$$

and, further,

$$4\sum_{\substack{s=1\\ s\neq i}}(\lambda_s-\lambda_i)\langle r^s X, Y\rangle + \lambda_i\langle(\mathrm{Ric}-5\mathrm{Ric}^\star)X, Y\rangle = 0.$$

We conclude by using that $\mathrm{Ric}-\mathrm{Ric}^\star = r$ and $\sum\limits_{s=1}^{p} r^s = r$. □

Note that by definition, the tensors r^s, $1 \leq s \leq p$ are positive. Setting $\lambda = \min\{\lambda_i : 1 \leq i \leq p\}$, Proposition 2.1 obviously implies that $\mathrm{Ric} \geq \lambda g$. This, together with Myer's theorem, proves the second assertion of Theorem 1.1.

Another result we will use in the next section is

LEMMA 2.2. *The tensors* r^s, $1 \leq s \leq p$ *are* $\overline{\nabla}$*-parallel.*

The proof is analogous to that of the $\overline{\nabla}$-parallelism of r so it will be left to the reader. Thus, using Lemma 2.1 we obtain the following corollary:

COROLLARY 2.1. *The Ricci tensor and the Ricci* $\star$ *tensor of a compact NK manifold are* $\overline{\nabla}$*-parallel.*

It follows that the scalar curvature and more, the $\star$-scalar curvature of (M, g, J), are strictly positive constants. The proof of Theorem 1.1 is now finished.

3. Examples of NK Manifolds

Let us consider a Riemannian submersion with totally geodesic fibers

$$F \hookrightarrow (M, g) \to N$$

and let $TM = \mathcal{V} \oplus H$ be the corresponding splitting of TM. We will suppose that M admits a complex structure J compatible with g and preserving $\mathcal{V}$ and H such that (M, g, J) is a Kähler manifold. Consider now the Riemannian metric on M defined by

$$\hat{g}(X, Y) = \tfrac{1}{2}g(X, Y) \quad \text{if } X, Y \in \mathcal{V}, \quad \hat{g}(X, Y) = g(X, Y) \quad \text{for } X, Y \text{ in } H.$$

The metric $\hat{g}$ admits a compatible almost complex structure $\hat{J}$ given by $\hat{J}_{|\mathcal{V}} = -J$ and $\hat{J}_{|H} = J$. This almost complex structure was introduced in [4] for the case of twistor spaces over 4-manifolds.

PROPOSITION 3.1. *The manifold* $(M, \hat{g}, \hat{J})$ *is nearly Kähler. The distributions* $\mathcal{V}$ *and* H *are parallel with respect to the canonical Hermitian connection of* $(M, \hat{g}, \hat{J})$ *which thus has reduced holonomy.*

Proof. Let A: $TM \times TM \to TM$ be the O'Neill tensor of the Riemannian submersion (M, g). As g is Kähler, we must have $A_X J = J A_X$ for all X in TM. Using the relations between the Levi-Civita connections of $\hat{g}$ and g given in [3], after a standard computation, we obtain

$$(\hat{\nabla}_X \hat{J})V = -(\hat{\nabla}_V \hat{J})X = -A_X(JV),$$

$$(\hat{\nabla}_V \hat{J})W = 0, \ (\hat{\nabla}_X \hat{J})Y = 2A_X(JY),$$

for every X, Y in $\mathcal{V}$ and V, W in H. It is now straightforward to conclude. □

COROLLARY 3.1. *The twistor space of a positive quaternionic-Kähler manifold of dimension* $4k$ *admits a canonical NK structure with reducible holonomy, contained in* $\mathrm{U}(1) \times \mathrm{U}(2k)$.

Proof. We have only to recall [17] that such a twistor space is the total space of a Riemannian submersion with totally geodesic fibers of dimension 2 and that it admits a compatible Kähler structure. □

4. Reducible NK Manifolds

In this section, we consider strict NK manifolds (M^{2n}, g, J) such that the holonomy of the canonical Hermitian connection is contained in $\mathrm{U}(1) \times \mathrm{U}(n-1)$. This leads to a $\overline{\nabla}$-parallel decomposition of TM, orthogonal with respect to g and stable by J, $TM = L \oplus E$ with L of rank two. Note that the torsion of $\overline{\nabla}$ vanishes on L and $T(L, E) \subseteq E$.

LEMMA 4.1. *We have*

(i) $\overline{R}(X, Y, V, JV) = -2\langle(\nabla_V J)^2 X, JY\rangle$ *for every vector fields* X, Y *on* E *and* V *on* L.
(ii) $\overline{R}(X, V, V, JV) = 0$ *if* X *belongs to* E *and* V *to* L.

Proof. (i) Using (2.4) we get

$$\overline{R}(X, Y, V, JV) = R(X, Y, V, JV) - \tfrac{1}{2}\langle(\nabla_V J)^2 X, JY\rangle.$$

Now the first Bianchi identity gives

$$R(X, Y, V, JV) = -R(Y, V, X, JV) + R(X, V, Y, JV).$$

As E is $\overline{\nabla}$-parallel, we must have $\overline{R}(Y, V, X, JV) = 0$ so we find by (2.4) that

$$R(Y, V, X, JV) = \tfrac{3}{4}\langle(\nabla_V J)^2 X, JY\rangle.$$

In the same way we have

$$R(X, V, Y, JV) = -\tfrac{3}{4}\langle(\nabla_V J)^2 X, JY\rangle$$

and the result easily follows.

(ii) Using (2.4) twice we get

$$\overline{R}(X, V, V, JV) = R(X, V, V, JV) = R(V, JV, X, V) = \overline{R}(V, JV, X, V)$$

and we conclude by the fact that E is $\overline{\nabla}$-parallel. □

Let us denote by Ω the curvature form of the line bundle L. Then we have

$$\overline{R}(X, Y)V = \Omega(X, Y)JV$$

for X, Y in TM and V in L. We denote by ω^L the restriction of the Kähler form ω to L. Let F be the endomorphism of TM defined by

$$\langle FX, Y\rangle = -\tfrac{1}{2}\mathrm{Tr}_L(\nabla_X J)(\nabla_Y J)$$

whenever X, Y are in TM.

Remark 4.1. If V is a local vector field on L of norm 1 we have $F = -(\nabla_V J)^2$. Hence, F is symmetric and positive, with $[F, J] = 0$. By Lemma 2.2, F is $\overline{\nabla}$-parallel and it easily follows that $\nabla_V F = 0$ for every vector field V in L.

If q^E is the 2-form on E defined by $q^E(X, Y) = \langle FX, JY\rangle$ for X, Y in E, by Lemma 4.1 we obtain that $\Omega = f\omega^L + 2q^E$, where f is a smooth function on M.

LEMMA 4.2. *We have:*

(i) $\mathrm{d}\omega^L(X, V, JV) = \mathrm{d}q^E(X, V, JV) = 0$ *if V is in L and X in E.*
(ii) $\mathrm{d}\omega^L(V, X, Y) = -\langle(\nabla_V J)X, Y\rangle$, $\mathrm{d}q^E(V, X, Y) = -2\langle F(\nabla_V J)X, Y\rangle$, *where V, X, Y are vector fields belonging to L resp. E.*

Proof. The proof of (i) is straightforward. We leave it to the reader and concentrate on (ii). We have

$$\mathrm{d}\omega^L(V, X, Y) = \nabla_V\omega^L(X, Y) - \nabla_X\omega^L(V, Y) + \nabla_Y\omega^L(V, X).$$

The fact that ω^L vanishes as soon as we take a direction in E gives us

$$\nabla_V\omega^L(X, Y) = 0, \quad \nabla_X\omega^L(V, Y) = -\omega^L(V, \nabla_X Y)$$

and

$$\nabla_Y\omega^L(V, X) = -\omega^L(V, \nabla_Y X).$$

The claimed formula for $\mathrm{d}\omega^L(V, X, Y)$ follows using the fact that $\overline{\nabla}_X Y$ and $\overline{\nabla}_Y X$ belong to E. Next, we have

$$\mathrm{d}q^E(V, X, Y) = (\nabla_V q^E)(X, Y) - (\nabla_X q^E)(V, Y) + (\nabla_Y q^E)(V, X).$$

The vanishing of q^E on $L \times E$ implies that

$$(\nabla_V q^E)(X, Y) = \langle (\nabla_V F)X, JY \rangle + \langle FX, (\nabla_V J)Y \rangle = \langle FX, (\nabla_V J)Y \rangle$$

(see Remark 4.1) and

$$(\nabla_X q^E)(V, Y) = \tfrac{1}{2}\langle F(\nabla_V J)X, Y \rangle, \quad (\nabla_Y q^E)(V, X) = \tfrac{1}{2}\langle F(\nabla_V J)Y, X \rangle.$$

We conclude by using the fact that F commutes with $\nabla_V J$. □

Let ω^E be the restriction of the form ω to E. Let us define the subbundle E_1 of E by setting $E_1 = Ker(F)$, and let E_2 be his orthogonal complement in E. Note that the $\overline{\nabla}$-parallelism of F implies that the distributions E_1 and E_2 are $\overline{\nabla}$-parallel. We can now have a complete description of the curvature form of our line bundle L as follows.

PROPOSITION 4.1. (i) *There exists a constant* $k > 0$ *such that* $F_{|E_2} = (k/4)1_{E_2}$. *Moreover, the curvature form of the line bundle* L *is* $(k/2)(-2\omega^L + \omega^{E_2})$.

(ii) *We have that* $(\nabla_X J)Y$ *belongs to* L *whenever* X, Y *are in* E_2*, that* $(\nabla_X J)Y = 0$ *if* X *is in* E_1 *and* Y *in* E_2 *and that* $\nabla_X J)Y$ *belongs to* E_1 *if* X, Y *belong to* E_1.

Proof. (i) As Ω is closed, we get $f\,\mathrm{d}\omega^L + \mathrm{d}f \wedge \omega^L = -2\,\mathrm{d}q^E$. If X, resp. V, are vector fields in E, resp. L, it follows by Lemma 4.2(i) that $X.f = 0$, hence $\mathrm{d}f_{|E} = 0$. This implies $[X, Y].f = 0$ whenever X, Y are vector fields in E and, further, that $(\nabla_X J)Y.f = 0$ (here we used that E is $\overline{\nabla}$-parallel and

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y] + (\nabla_X J)JY).$$

But the map u: $E \times E \to L$ defined at $(v, w) \in E \times E$ as the orthogonal projection of $(\nabla_v J)w$ on L is surjective (otherwise it should be identically zero, and this is a contradiction with the fact that M is strict). Hence, $\mathrm{d}f$ vanishes on L and thus $\mathrm{d}f = 0$, that is f is constant, equal to c.

Let now X, Y, resp. V, be vector fields in E resp. L. As $\mathrm{d}\Omega(V, X, Y) = 0$ (Ω is closed form) we get from Lemma 4.2(ii):

$$-c\langle(\nabla_V J)X, Y\rangle - 4\langle F(\nabla_V J)X, Y\rangle = 0.$$

We deduce that $(\nabla_V J)(4F + c) = 0$ and, further, $F(4F + c) = 0$ on E. As the restriction of F to E_2 is injective it follows that $F = (-c/4)\mathrm{id}$ on E_2. We set $k = -c$.

(ii) Let X, Y, Z be vector fields on E. As we obviously have $\mathrm{d}\omega^L(X, Y, Z) = 0$, it follows by (i) that $\mathrm{d}\omega^{E_2}(X, Y, Z) = 0$. If X, Y, Z are in E_2, a straightforward computation gives

$$(\nabla_X \omega^{E_2})(Y, Z) = -\langle(\nabla_X J)Y, Z\rangle$$

from which we deduce that $\mathrm{d}\omega^{E_2}(X, Y, Z) = -\langle(\nabla_X J)Y, Z\rangle$, hence the first affirmation follows. The two others are proved in a similar way. □

Hence, we obtain the following decomposition result:

COROLLARY 4.1. *If M is simply connected, it is a Riemannian product $M_1 \times M_2$ where M_1 is a strict NK manifold and M_2 is a strict NK manifold whose tangent bundle admits a parallel (with respect to the Hermitian connection) line bundle such that the corresponding tensor F has no kernel.*

Proof. We have a $\overline{\nabla}$-parallel decomposition $TM = E_1 \oplus (L \oplus E_2)$ that is in fact $\overline{\nabla}$-parallel by the algebraic properties of Proposition 4.1(ii). Hence, we may apply the de Rham decomposition theorem as M is simply connected. □

Hence, from now on we will make the assumption that our NK manifold M satisfies $F_{|E} = (k/4)1_E$. For proving Theorem 1.2, all we need to see is that such a NK manifold is in fact a twistor space.

COROLLARY 4.2. (i) *The tensor r has exactly two eigenvalues: $(k(n-1))/2$, resp. k, with eigenbundles L resp. E.*

(ii) *The Ricci tensor of (M, g) has exactly two eigenvalues: $(k(n+7))/8$ and $(k(n+2))/4$ with eigenbundles L, resp. E.*

Proof. (i) The fact that $r_{|L} = (k(n-1))/2$ easily follows by the fact that F is constant on E. If x is in E, let v in L be unitary, and $\{e_i\}_{1 \le i \le 2(n-1)}$ be an orthogonal basis of E. Then we have

$$\langle rx, x\rangle = 2\|(\nabla_v J)x\|^2 + \sum_{i=1}^{2(n-1)} \|(\nabla_{e_i} J)x\|^2.$$

As $(\nabla_{e_i} J)x$ belongs to L, the last sum equals $2\|(\nabla_v J)x\|^2$ and we use $F_{|E} = k/4$.

(ii) Follows from Lemma 2.1 and (i). □

5. The Twistor Structure

Let us define a new Riemannian metric on M, called $\overline{g}$, as follows:

$$\overline{g}(X, Y) = g(X, Y) \quad \text{if } X, Y \in E,$$

$$\overline{g}(X, Y) = 2g(X, Y) \quad \text{for } X, Y \text{in } L.$$

The *reversing* almost complex structure defined by $\overline{J}_{|L} = -J$ and $\overline{J}_{|E} = J$ is in fact integrable, the proof being identical to that given in six dimensions in [2]. The Kähler form of $(M, \overline{g}, \overline{J})$ is exactly $-2\omega^L + \omega^E$ and, hence, it is closed by Proposition 4.1(i). Thus, $(M, \overline{g}, \overline{J})$ is a Kähler manifold.

LEMMA 5.1. *$(M, \overline{g})$ is an Einstein manifold, with Einstein constant $(n + 1/4)k$.*

Proof. This is a computation very similar to that of [3, p. 232], where the Ricci tensor of the canonical variation of a Riemannian submersion is computed. Let $\tilde{\nabla}$

be the Levi-Civita connection of the metric $\overline{g}$. If V, resp. X, Y, are vector fields in L, resp. E, we have

$$\widetilde{\nabla}_V X = \overline{\nabla}_V X, \quad \widetilde{\nabla}_X V = \overline{\nabla}_X V - (\nabla_X J)JV$$

and

$$\widetilde{\nabla}_X Y = \nabla_X Y.$$

Moreover, $\widetilde{\nabla}_V W = \nabla_V W$ whenever V, W are in L. This follows from the definition of the Levi-Civita connection and by the fact that the $\overline{\nabla}$-parallelism of L and E allows us to identify the projections on L, resp. E, of brackets of the type $[V, X]$ and $[X, Y]$.

Let $\widetilde{R}$ be the curvature tensor of $\widetilde{\nabla}$. Using the above formulas, after a standard computation, we get

$$\begin{aligned}\widetilde{R}(V, X, V, X) &= \langle FX, X\rangle \|V\|^2 = \frac{k}{4}\|V\|^2\|X\|^2,\\ \widetilde{R}(X, Y, X, Y) &= \overline{R}(X, Y, X, Y) - \tfrac{1}{2}\|(\nabla_X J)Y\|^2\\ &= R(X, Y, X, Y) - \tfrac{3}{4}\|(\nabla_X J)Y\|^2,\end{aligned}$$

by (2.4). The result now follows by Corollary 4.2. □

Thus, $(M, \overline{g}, \overline{J})$ is a Kähler–Einstein manifold which is also Fano. Moreover, the distribution E defines a complex contact structure on the complex manifold $(M, \overline{J})$ as it is $\overline{J}$-holomorphic and the map $(X, Y) \in E \times E \to (\nabla_X J)Y$ which gives the Frobenius obstruction is everywhere nondegenerate. By a result of LeBrun (see [14]), $(M, \overline{g})$ is the twistor space of a positive quaternionic-Kähler manifold. Moreover, from the construction of the metric $\overline{g}$ we deduce that (M, g) is the twistor space of a positive quaternionic-Kähler manifold endowed with its canonical NK structure. This proves Theorem 1.2.

Remark 5.1. If M is of dimension 6, it has constant type and Proposition 4.1 is automatically satisfied. Corollary 4.2 follows from the fact that every six-dimensional NK manifold is Einstein [9]. Thus, all we need to prove Theorem 1.1 in this case is Lemma 5.1.

Let us now prove Corollary 1.1. It is well known (see [9]) that in 10-dimensions the eigenvalues of r are $4(\alpha^2+\beta^2)$ with multiplicity 2, $4\alpha^2$ and $4\beta^2$ each of multiplicity 4, where $\alpha \geq \beta \geq 0$. If $\beta = 0$ then it follows by [9] that the universal cover of M is a Riemannian product as stated. If $\beta > 0$ then M is strict and let L be the line bundle given by the eigenspaces of r corresponding to the eigenvalue $4(\alpha^2 + \beta^2)$. Again by computations from [9] we see that the tensor F is injective and the previous discussion applies.

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