Journal of Lie Theory
Volume **17** (2007) 357–377

On Algebraic Torsion Forms and their Spin Holonomy Algebras

Niels Bernhardt and Paul-Andi Nagy

Communicated by A. Valette

Abstract. We study holonomy algebras generated by an algebraic element of the Clifford algebra, or equivalently, the holonomy algebras of certain spin connections in flat space. We provide series of examples in arbitrary dimensions and establish general properties of the holonomy algebras under some mild conditions on the generating element.
Mathematical Subject Classification 2000: 53C10, 53C27, 53C29.
Key Words and Phrases: Spin connection, spin holonomy algebra.

1. Introduction

Let (M^n, g) be a Riemannian spin manifold, with spinor bundle to be denoted by $\not{S}$. For any differential form T on M, not necessarily of pure degree, one can form the linear connection ∇^T on $\not{S}$ by setting

$$\nabla^T_X \psi = \nabla_X \psi + (X \lrcorner T)\psi$$

whenever ψ belongs to $\Gamma(\not{S})$ and X is in TM. Here ∇ is the connection induced by the Levi-Civita connection on the spinor bundle $\not{S}$. This can be thought of as the spin analogue of a connection with non-trivial torsion on the tangent bundle of M. A special case is when T is actually a 3-form in which case ∇^T is the induced spin connection of a connection with torsion on TM. In low dimensions, ranging from 6 to 8, parallel spinors w.r.t to a connection with 3-form torsion are nowadays rather well understood in terms of geometric structures on the tangent bundle to the manifold [4, 11] and extensive effort toward their classification has been made [9, 6, 8]. This has been also studied in connection with the so-called Strominger's type II string equations [14, 2]. Another special case, which no longer reflects the presence of a particular connection at the level of the tangent bundle of M, is when T consists of forms of degree 3 and 4, the latter being termed fluxes in physics literature (see [7] and references therein). In all the above mentioned cases one of the issues to understand is under which conditions ∇^T admits parallel spinors, therefore one looks, more generally, at the holonomy representation of ∇^T.

ISSN 0949–5932 / \$2.50

This is because of the well-known fact [3] that a spinor is parallel if and only if it is fixed by the holonomy representation at a point.

In this paper we shall study the holonomy of the connection ∇^T in the flat case, when moreover T is assumed to have constant coefficients. This is the simplest geometric case one could think of but already raises some interesting and quite difficult algebraic questions. We set

Definition 1.1. *Let $(V^n, \langle\cdot,\cdot\rangle)$ be an Euclidean vector space and let $Cl_n(V)$ be its Clifford algebra. Then:*

(i) the fix algebra of T in $Cl_n(V)$ is the Lie sub-algebra $\mathfrak{g}_T^*$ of $Cl_n(V)$ generated by $\{X \lrcorner T : X \in V\}$.

(ii) the holonomy algebra of some T in $Cl_n(V)$ is given as $\mathfrak{h}_T^* = [\mathfrak{g}_T^*, \mathfrak{g}_T^*]$.

This is motivated by the observation [1] that in the flat case the holonomy algebra of the spin connection ∇^T equals $\mathfrak{h}_T^*$. When $\mathfrak{g}_T^*$ is *perfect*, that is $\mathfrak{g}_T^* = [\mathfrak{g}_T^*, \mathfrak{g}_T^*]$, the two algebras above coincide and in this respect the fix algebra $\mathfrak{g}_T^*$ appears to be a very useful intermediary object for establishing structure results, although it seems to lack of further geometric content. For 3-forms complete structure results concerning the fix and holonomy algebras have been obtained in [1].

Our paper is organised as follows. In section 2 we review a number of elementary facts concerning Clifford algebras and their representations, with accent put on the different phenomena appearing in some arithmetic series of dimensions. In section 3 we start our study of holonomy algebras by determining - under some mild assumptions on the generating element - the model algebra those are contained in. We also establish a number of useful general properties, like semisimplicity. Further on, we investigate the space of the so-called fixed spinors which, for some T in $Cl_n(V)$, is defined as

$$Z_T = \{\psi \in \$: (X \lrcorner T)\psi = 0 \text{ for all } X \text{ in } V\} \tag{1}$$

where $\$$ is an irreducible $Cl_n(V)$ module. We provide first order information about these spaces and also discuss some simple examples. The section ends with giving a necessary condition for certain holonomy algebras to be perfect, namely

Theorem 1.1. *Let T in $Cl_n^0 \cap Cl_n^+$, where $n \equiv 0 \pmod 4$ satisfy $T^t = T$. If $Z_T = \{0\}$ then $\mathfrak{g}_T^*$ is a semisimple Lie algebra, in particular it is perfect.*

Section 4 describes situations where the holonomy algebras can be directly computed and provides series of useful, in hindsight, examples. Elements of the Clifford algebra which are being looked at are unipotent and squares of spinors, which actually give idempotents. In the latter situation, the dimension (mod 8) of the underlying vector space appears to lead to very different results. More precisely

Theorem 1.2. *Let V be an Euclidean vector space with volume form ν and let T belong to $Cl_n(V)$. Then:*

(i) if $n \equiv 0 \pmod 4$ and T in $Cl_n^0 \cap Cl_n^+$ is unipotent, that is $T^2 = 1 + \nu$ and $T^t = T$, then its holonomy algebra is isomorphic to $\mathfrak{so}(n,1)$ and $\mathfrak{g}_T^*$ is perfect.

(ii) if T is the square of a positive spinor when $n \equiv 0 \pmod 8$, then the fix algebra of T is perfect and its holonomy algebra is isomorphic to $\mathfrak{so}(n,1)$.

(iii) if T is the square of a spinor when $n \equiv 7 \pmod 8$, then $\mathfrak{g}_T^*$ is abelian, in particular the holonomy algebra vanishes.

This essentially exploits specific features of the powerful squaring construction for spinors [13, 5]. Note that the perfectness of the fix algebra in (i) of Theorem 1.2 follows directly from Theorem 1.1 whereas in the case of (ii) it does not. This is based on the facts that unipotent elements cannot fix non-zero spinors while squares of spinors admit non-trivial fixed spinors, which are proved in section 3 (Theorem 4.1) and section 4 (Theorems 4.2 and 4.3) of this paper.

2. Preliminaries

This section is mainly intended to recall a number of facts concerning Clifford algebras and spinors, which we shall constantly use in what follows. A thorough account of all these notions can be found in [13].

2.1. Clifford algebras.

Let V be an n-dimensional vector space over $\mathbb{R}$ equipped with a positive definite (or Euclidean) scalar product, to be denoted by $\langle\cdot,\cdot\rangle$. We shall denote by $Cl_n(V)$ the Clifford algebra associated with $(V, \langle\cdot,\cdot\rangle)$, and if there is no ambiguity on the vector space used we shall simply write Cl_n for $Cl_n(V)$. There is a canonical isomorphism of vector spaces between the space $\Lambda^*(V)$ of forms in V and the Clifford algebra $Cl_n(V)$. Then $Cl_n(V)$ can be given the structure of an algebra, with multiplication denoted by „ $\cdot$ “ $: Cl_n(V) \to Cl_n(V)$ which satisfies

$$e \cdot \varphi = e \wedge \varphi - e \lrcorner \varphi, \qquad \varphi \cdot e = (-1)^k (e \wedge \varphi + e \lrcorner \varphi) \tag{2}$$

whenever e belongs to $\Lambda^1(V)$ and φ is in $\Lambda^k(V)$, although this notation will no longer be used in what follows. Here and henceforth we will identify 1-forms and vectors via the given scalar product $\langle\cdot,\cdot\rangle$. Let $L : Cl_n(V) \to Cl_n(V)$ be defined by

$$L(\varphi) = \sum_{i=1}^{n} e_i \varphi e_i, \tag{3}$$

whenever $\varphi \in Cl_n(V)$ and for some orthonormal basis $\{e_i\}$, $1 \le i \le p$ in V. Then the eigenspaces of L are the canonical images of $\Lambda^k(V)$:

$$L|_{\Lambda^k} = (-1)^k (2k - n) 1_{\Lambda^k}. \tag{4}$$

Any Clifford algebra comes with two involutions, the first being the transposition map $()^t : Cl_n(V) \to Cl_n(V)$ defined by

$$(e_1 e_2 \ldots e_{k-1} e_k)^t = e_k e_{k-1} \ldots e_2 e_1. \tag{5}$$

for some orthonormal frame $\{e_i, 1 \le i \le n\}$. Note however that the transpose is frame independent and therefore extends to an anti-automorphism of $Cl_n(V)$, i.e.

$$(\varphi_1 \varphi_2)^t = \varphi_2^t \varphi_1^t \tag{6}$$

for all φ_1, φ_2 in $Cl_n(V)$. The second involution $\alpha : Cl_n(V) \to Cl_n(V)$ results from extending -1_V to an automorphism of the algebra $Cl_n(V)$, in the sense that

$$\alpha(\varphi_1 \varphi_2) = \alpha(\varphi_1)\, \alpha(\varphi_2),$$

where φ_1, φ_2 are in $Cl_n(V)$. The multiplication in Cl_n is then subject, in accordance with (2), to

$$\begin{aligned} \alpha(\varphi)X &= X \wedge \varphi + X \lrcorner \varphi \\ \alpha(\varphi)X - X\varphi &= 2X \lrcorner \varphi \end{aligned} \tag{7}$$

whenever X belongs to V and φ is in Cl_n. These relations will be constantly used in subsequent calculations. Since α is an involution it can also be used to obtain a splitting

$$Cl_n(V) = Cl_n^0(V) \oplus Cl_n^1(V)$$

into the $\pm$-eigenspaces of α. Note this corresponds to the splitting of $\Lambda^*(V)$ into even and respectively odd degree forms. The vector space $Cl_n(V)$ inherits from V a scalar product, still to be denoted by $\langle \cdot, \cdot \rangle$ and having the property that

$$\begin{aligned} \langle \varphi\varphi_1, \varphi_2 \rangle &= \langle \varphi_1, \alpha(\varphi^t)\varphi_2 \rangle \\ \langle \varphi_1\varphi, \varphi_2 \rangle &= \langle \varphi_1, \varphi_2\alpha(\varphi^t) \rangle \end{aligned} \tag{8}$$

whenever $\varphi, \varphi_1, \varphi_2$ belong to $Cl_n(V)$. Note that the scalar product constructed on $Cl_n(V)$ corresponds, via the canonical isomorphism, to the scalar product on $\Lambda^*(V)$ induced by that on V.

Let us assume now that V is oriented by ν in $\Lambda^n(V)$ such that for an orthonormal oriented frame $\{e_k, 1 \le k \le n\}$ this is given as $\nu = e_1 \dots e_n$. Then it is easy to check that

$$\nu^2 = (-1)^{\frac{n(n+1)}{2}}, \qquad \nu^t = (-1)^{\frac{n(n-1)}{2}} \nu. \tag{9}$$

Now the Hodge star operator $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ is defined by $\omega_1 \wedge *\omega_2 = \langle \omega_1, \omega_2 \rangle \nu$, for all ω_1, ω_2 in $\Lambda^*(V)$ and relates to Clifford multiplication with ν by

$$*\varphi = (-1)^{\frac{k}{2}(k+1)} \varphi\nu = (-1)^{\frac{k}{2}(2n-k+1)} \nu\varphi, \tag{10}$$

for φ in $\Lambda^k(V) \subset Cl_n(V)$. Moreover, we have

$$\begin{aligned} \varphi\,\nu &= \nu\,\varphi, && \text{for all } \varphi \text{ in } Cl_n^0, \\ \varphi\,\nu &= (-1)^{n+1} \nu\,\varphi, && \text{for all } \varphi \text{ in } Cl_n^1. \end{aligned} \tag{11}$$

In particular, when $n \equiv 1 \pmod 2$, the volume element ν belongs to the center of the Clifford algebra Cl_n.

If $n \equiv 0, 3 \pmod 4$ then $\nu^2 = 1$ whence the Hodge star operator, realised as in (10), provides a decomposition of the Clifford algebra into self-dual and anti-self-dual elements:

$$Cl_n(V) = Cl_n^+(V) \oplus Cl_n^-(V). \tag{12}$$

where $\nu\varphi = \pm\varphi$ whenever φ belongs to $Cl_n^\pm(V)$.

We end this section by recalling two more facts, to be used later on in this paper. If γ_k belong to $\Lambda^2(V)$ for $k = 1, 2$ with associated skew-symmetric endomorphisms F_k, that is $\gamma_k = \langle F_k \cdot, \cdot \rangle$ for $k = 1, 2$, we have

$$[\gamma_1, \gamma_2] = 2\,\langle [F_1, F_2] \cdot, \cdot \rangle. \tag{13}$$

This is because the map $\lambda_* : \mathfrak{spin}(n) \to \mathfrak{so}(n)$, given by $\lambda_*(\gamma) = 2F$ for all $\gamma \in \Lambda^2(V)$ with associated skew endomorphism F, is a Lie algebra isomorphism [13, Chap. I, Prop. 6.2].

Let $\hat{V} = V \oplus \mathbb{R}$ where $\mathbb{R}$ is spanned by some non-zero vector e. For any non-zero ε in $\mathbb{R}$ we equip $\hat{V}$ with the (possibly Lorentzian) scalar product $g_\varepsilon = g + \varepsilon e \otimes e$, where g denotes the scalar product on V. Consider now the vector space $\Lambda^2 V \oplus V$ equipped with the Lie bracket defined by

$$\begin{aligned} [\gamma_1, \gamma_2]_\varepsilon &= \langle [F_1, F_2] \cdot, \cdot \rangle = \tfrac{1}{2}[\gamma_1, \gamma_2] \\ [\gamma, v]_\varepsilon &= v \lrcorner \gamma, \text{ if } \gamma \text{ is in } \Lambda^2(V) \text{ and } v \text{ in } V \\ [v, w]_\varepsilon &= \varepsilon v \wedge w, \text{ for } v, w \text{ in } V. \end{aligned} \tag{14}$$

Then the Lie algebra $(\Lambda^2 V \oplus V, [\cdot,\cdot]_\varepsilon)$ is isomorphic to $\mathfrak{so}(\hat{V}, g_\varepsilon)$, in particular to $\mathfrak{so}(n+1)$ if $\varepsilon = 1$ and to $\mathfrak{so}(n,1)$ for $\varepsilon = -1$. These follow by elementary considerations using essentially the block structure of an endomorphism of $\hat{V}$ w.r.t. the splitting $\hat{V} = V \oplus \mathbb{R}$ (see e.g. [10, page 238ff]).

2.2. The space of spinors.

We need to recall some elementary facts about spinors. Let $\$$ be an irreducible $Cl_n(V)$-module. We shall call elements of $\$$ spinors and denote by $\mu : Cl_n(V) \to \mathrm{End}(\$)$ the Clifford multiplication acting on $\$$. On $\$$ we have a scalar product $\langle \cdot, \cdot \rangle$ which has the following property

$$\langle \varphi\psi_1, \psi_2 \rangle = \langle \psi_1, \alpha(\varphi^t)\psi_2 \rangle, \tag{15}$$

for all φ in Cl_n and ψ_1, ψ_2 in $\$$. Recalling that

$$\alpha(\varphi^t) = (-1)^{\frac{k}{2}(k+1)} \varphi, \qquad \varphi \in \Lambda^k(V) \subset Cl_n(V). \tag{16}$$

it follows that the Clifford multiplication operator $\mu_\varphi : \$ \to \$$, with an element φ of $\Lambda^k(V)$, is symmetric when $k \equiv 0, 3 \pmod 4$ and skew-symmetric when $k \equiv 1, 2 \pmod 4$. Now when $n \equiv 0, 3 \pmod 4$ the volume form ν squares to 1.
If $n \equiv 0 \pmod 4$ this allows splitting the irreducible, real Clifford module $\$$ as $\$ = \$^+ \oplus \$^-$, where ν acts as ± 1 on $\$^\pm$. Note that $\$^\pm$ are not Cl_n modules although they have the same dimension.

Peculiar to the case when $n \equiv 3 \pmod 4$ is the fact that any irreducible, real Clifford module $\$$ has either $\nu\psi = -\psi$ for all ψ in $\$$ or $\nu\psi = \psi$ for all ψ in $\$$. Both possibilities can occur and produce different Cl_n-representations. As a convention, in what follows we shall always work with the latter representation. Let us also mention that when $n \equiv 3 \pmod 4$ we have that $\nu\varphi = \varphi\nu$ for all φ in Cl_n and that α interchanges Cl_n^+ and Cl_n^-, that is realises an isomorphism $\alpha : Cl_n^+ \to Cl_n^-$. For it will be used constantly in what follows we also recall the following stability Lemma.

Lemma 2.1. *Let $\$$ be any real Clifford module. Then the following stability conditions hold:*

(i) when $n \equiv 0 \pmod 4$,

$$\begin{array}{ll} \varphi\$^{\pm} \subseteq \$^{\pm}, & \text{for all } \varphi \in Cl_n^0(V) \cap Cl_n^{\pm}(V) \\ \varphi\$^{\pm} = \{0\}, & \text{for all } \varphi \in Cl_n^0(V) \cap Cl_n^{\mp}(V) \text{ or } Cl_n^1(V) \cap Cl_n^{\pm}(V) \\ \varphi\$^{\pm} \subseteq \$^{\mp}, & \text{for all } \varphi \in Cl_n^1(V) \cap Cl_n^{\mp}(V), \end{array}$$

(ii) while for $n \equiv 3 \pmod 4$ we have $Cl_n^+\$ \subseteq \$$, $Cl_n^-\$ = 0$, provided that $\mu_\nu = 1_\$$.

The proof, which is left to the reader, follows from the above properties of Clifford multiplication with ν. We end this section by recalling two more well known facts, with proofs given for the sake of completeness.

Lemma 2.2. *If ζ in Cl_n satisfies $[\zeta, \Lambda^2(V)] = 0$ then ζ is a linear combination of 1 and ν.*

Proof. Since $\zeta XY = XY\zeta$ for all X, Y in V it follows that $Y(X\zeta X)Y = |X|^2|Y|^2\zeta$ and further, after a double tracing $L^2\zeta = n^2\zeta$. The eigenvalues of L^2 being $(2p-n)^2$ it follows that the only degrees present in ζ are 0 and n. Therefore ζ is a linear combination of 1 and ν. ∎

Lemma 2.3. *Let $\$$ be any real, Cl_n-module where $n \equiv 0, 3 \pmod 4$. Then:*

(i) $Tr(\mu_\varphi) = dim_{\mathbb{R}}\$ \langle \varphi, 1 \rangle$, if $n \equiv 0 \pmod 4$,

(ii) $Tr(\mu_\varphi) = dim_{\mathbb{R}}\$ \langle \varphi, 1+\nu \rangle$, if $n \equiv 3 \pmod 4$, where $\$$ is the irreducible Cl_n-module with $\mu_\nu = 1_\$$.

Proof. Let now $t : Cl_n \to \mathbb{R}$ be given as $t(\varphi) = Tr(\mu_\varphi)$ for all φ in Cl_n. Since this is linear, it can be written as $t = \langle \cdot, T \rangle$ for some T in Cl_n. We now pick some orthonormal basis $\{e_i : 1 \le i \le n\}$ in V and write $e^{ij} = e^i \wedge e^j$ for all $1 \le i, j \le n$. Because $\mu_{e^{ij}}, i \ne j$, is an isometry of $\$$, we get from the independence of the trace of some orthonormal basis in Cl_n and (15) that $t(e^{ij}\varphi e^{ij}) = -t(\varphi)$, for all φ in Cl_n. Using (8) this results in having $e^{ij}Te^{ij} = -T$ or further $[T, e^{ij}] = 0$ for all $1 \le i \ne j \le n$, where we have used that $(e^{ij})^2 = -1$. Hence $[T, \Lambda^2] = 0$, leading by Lemma 2.2 to $T = \lambda_1 + \lambda_2\nu$ for some λ_1, λ_2 in $\mathbb{R}$ which can be computed as $\lambda_1 = Tr(\mu_1) = dim_{\mathbb{R}}\$$ and $\lambda_2 = Tr(\mu_\nu)$.
To prove (i) we use that $Tr(\mu_\nu) = dim_{\mathbb{R}}\$^+ - dim_{\mathbb{R}}\$^- = 0$ for $n \equiv 0 \pmod 4$ whereas (ii) follows from $Tr(\mu_\nu) = dim_{\mathbb{R}}\$$, a consequence of $\mu_\nu = 1_\$$. ∎

3. Structure results

3.1. The general setup.

In this section our aim is mainly to locate some classes of holonomy algebras inside the Clifford algebra and derive a number of general properties they must satisfy. Let A be the subset of Cl_n given by

$$A = \{\varphi \in Cl_n : \varphi^t = -\varphi\}.$$

This is meant to be the model algebra for most classes of holonomy algebras we will be looking at, in a sense to be made precise below.

Lemma 3.1. *The following hold :*

(i) A is a Lie sub-algebra of $(Cl_n, [\cdot,\cdot])$.

(ii) The symmetric bilinear form $\beta(\varphi_1, \varphi_2) = \langle \varphi_1, \alpha(\varphi_2) \rangle$ is non-degenerate on A and invariant, that is

$$\beta([\varphi_1, \varphi_2], \varphi_3) = -\beta([\varphi_1, \varphi_3], \varphi_2)$$

whenever $\varphi_k, 1 \le k \le 3$ belong to A.

Proof. (i) Follows immediately from anti-symmetrising that $(\varphi_1\varphi_2)^t = \varphi_2^t\varphi_1^t = \varphi_2\varphi_1$ whenever φ_1, φ_2 belong to A.
(ii) at first, β is symmetric since α is an isometric involution of Cl_n. Next, the non-degeneracy of β follows from A being preserved by the involution α. Now

$$\begin{aligned} \beta([\varphi_1, \varphi_2], \varphi_3) &= \langle [\varphi_1, \varphi_2], \alpha(\varphi_3) \rangle = \langle \varphi_1\varphi_2 - \varphi_2\varphi_1, \alpha(\varphi_3) \rangle \\ &= \langle \varphi_2, \alpha(\varphi_1^t)\alpha(\varphi_3) - \alpha(\varphi_3)\alpha(\varphi_1^t) \rangle \\ &= -\langle \varphi_2, \alpha([\varphi_1, \varphi_3]) \rangle = -\beta(\varphi_2, [\varphi_1, \varphi_3]). \end{aligned}$$

This is what we had to show. ∎

Since A is stable under α, it inherits from Cl_n a bi-grading $A = A^0 \oplus A^1$ into its even respectively its odd degree components, where we have set $A^i := A \cap Cl^i$ for $i = 0, 1$. The usual rules $[A^0, A^0] \subseteq A^0$, $[A^0, A^1] \subseteq A^1$, $[A^1, A^1] \subseteq A^0$ apply, in particular A^0 is a Lie sub-algebra of A. We can obtain now first order information about some of the holonomy algebras, by assuming the generating element to be well related to the standard decompositions of Cl_n.

Proposition 3.1. *Let T belong to Cl_n^0 and satisfy $T^t = T$. Then:*

(i) $\mathfrak{g}_T^*$ is a Lie sub-algebra of A.

(ii) $\alpha(\mathfrak{g}_T^*) = \mathfrak{g}_T^*$.

Proof. (i) follows eventually after checking that the generating set $\{X \lrcorner T :$ $X \in V\}$ is contained in A as

$$(X \lrcorner T)^t = -X \lrcorner T^t = -X \lrcorner T$$

or all X in V. To prove (ii) we notice that $\alpha(X \lrcorner T) = -X \lrcorner T$ for all X in V, in other words α preserves the generating set. Since α is a Lie algebra automorphism preserving A, the result follows. ■

Therefore, for any T in Cl_n^0 such that $T^t = T$, the Lie algebra $\mathfrak{g}_T^*$ splits as

$$\mathfrak{g}_T^* = \mathfrak{g}_T^{*,0} \oplus \mathfrak{g}_T^{*,1},$$

where the obvious notation applies. Getting closer to the specific features of the algebras $\mathfrak{g}_T^*$ requires some Lie algebra background we shall now briefly outline. For our setup the most convenient is to start from

Definition 3.1. *Let $\mathfrak{g}$ be a real Lie algebra. It is called quadratic if it admits a symmetric bilinear form β which is non-degenerate and satisfies*

$$\beta([\varphi_1, \varphi_2], \varphi_3) = -\beta([\varphi_1, \varphi_3], \varphi_2)$$

for all $\varphi_1, \varphi_2, \varphi_3$ in $\mathfrak{g}$.

This essentially ensures that any non-degenerate ideal $\mathfrak{i}$ of $\mathfrak{g}$ has trivial extension, that is there exists an ideal $\mathfrak{i}^{\perp}$ such that $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^{\perp}$, where $\mathfrak{i}^{\perp}$ denotes the orthogonal complement of $\mathfrak{i}$ w.r.t the non-degenerate form β. Note that any semisimple Lie algebra is quadratic when the bilinear form β is set to the Killing form.

Proposition 3.2. *Let $\mathfrak{g}$ be a real Lie algebra. If $\mathfrak{g}$ is quadratic then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ if and only if it has trivial center. Here the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is given as $Z(\mathfrak{g}) =$ $\{\zeta \in \mathfrak{g} : [\zeta, \mathfrak{g}] = 0\}$.*

Proof. That $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ implies the vanishing of the center is an easy exercise. If $Z(\mathfrak{g}) = \{0\}$, the invariance and non-degeneracy of β ensure that the ideal $[\mathfrak{g}, \mathfrak{g}]$ is non-degenerate and the considerations above apply. ■

Real Lie algebras $\mathfrak{g}$ satisfying $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ are called *perfect* and Proposition 3.2 provides a tool for checking this, to be used later on. In analogy with the classical notion in the context of semisimple Lie algebras we have

Definition 3.2. *Let $(\mathfrak{g}, \beta)$ be a real quadratic Lie algebra. An endomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution if and only if*

(i) α is an involution, that is $\alpha^2 = 1_{\mathfrak{g}}$,

(ii) α is a Lie algebra automorphism,

(iii) the bilinear form $\beta(\cdot, \alpha \cdot)$ is symmetric and positive definite.

As in the semisimple case, a quadratic Lie algebra $(\mathfrak{g}, \beta)$ equipped with a Cartan involution α admits a Cartan decomposition

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$$

where $\mathfrak{g}^0$ and $\mathfrak{g}^1$ are the ± 1-eigenspaces of α. That α is an automorphism of $\mathfrak{g}$ ensures the validity of the inclusions $[\mathfrak{g}^0, \mathfrak{g}^0] \subseteq \mathfrak{g}^0, [\mathfrak{g}^0, \mathfrak{g}^1] \subseteq \mathfrak{g}^1$ and $[\mathfrak{g}^1, \mathfrak{g}^1] \subseteq \mathfrak{g}^0$. Moreover, from (iii) in the Definition 3.2 the bilinear form β is seen to be positive definite on $\mathfrak{g}^0$, negative definite on $\mathfrak{g}^1$ and subject to $\beta(\mathfrak{g}^0, \mathfrak{g}^1) = 0$.

We shall provide now a simple criterion for a quadratic Lie algebra to be semisimple.

Proposition 3.3. *Let $(\mathfrak{g}, \beta)$ be a quadratic Lie algebra such that $Z(\mathfrak{g}) = \{0\}$. If $\mathfrak{g}$ admits a Cartan involution $\alpha : \mathfrak{g} \to \mathfrak{g}$ then the Lie algebra $\mathfrak{g}$ is semisimple.*

To prove this we need first to establish

Lemma 3.2. *Under the assumptions in Proposition 3.3, if $\mathfrak{i} \subseteq \mathfrak{g}$ is an abelian ideal such that $\alpha(\mathfrak{i}) = \mathfrak{i}$ then $\mathfrak{i} = 0$.*

Proof. Since $\mathfrak{i}$ is preserved by α we have the splitting $\mathfrak{i} = \mathfrak{i}^0 \oplus \mathfrak{i}^1$ where $\mathfrak{i}^p = \mathfrak{i} \cap \mathfrak{g}^p, p = 0, 1$. Because $\mathfrak{i}$ is an ideal, $[\mathfrak{i}^0, \mathfrak{g}] \subseteq \mathfrak{i}$ and further, using the properties of the Cartan decomposition of $\mathfrak{g}$, $[\mathfrak{i}^0, \mathfrak{g}^0] \subseteq \mathfrak{i}^0$ and $[\mathfrak{i}^0, \mathfrak{g}^1] \subseteq \mathfrak{i}^1$. That $\mathfrak{i}$ is abelian yields now $[\mathfrak{i}^p, \mathfrak{i}^q] = \{0\}$ for all $0 \leq p, q \leq 1$. Therefore the invariance of β tells us that $[\mathfrak{i}^0, \mathfrak{g}^0]$ is orthogonal (w.r.t. β) to $\mathfrak{i}^0$ hence we must have $[\mathfrak{i}^0, \mathfrak{g}^0] = \{0\}$ by using that β is positive definite on $\mathfrak{g}^0$. Similarly, one finds that $[\mathfrak{i}^0, \mathfrak{g}^1] = \{0\}$ and this leads to $\mathfrak{i}^0 = \{0\}$ when taking into account that $\mathfrak{g}$ has trivial center. The vanishing of $\mathfrak{i}^1$ is proved in a completely analogous manner, thus details are left to the reader. ■

Proof of Proposition 3.3:
Let us see first that $\mathfrak{g}$ does not contain non-zero, abelian ideals. Indeed, if $\mathfrak{i}$ is an abelian ideal in $\mathfrak{g}$ we have that $\alpha(\mathfrak{i})$ is an abelian ideal as well, by using (i) and (ii) in Definition 3.2. It follows that $\mathfrak{i} \cap \alpha(\mathfrak{i})$ is an abelian ideal in $\mathfrak{g}$, which is preserved by α. ¿From Lemma 3.2 we get $\mathfrak{i} \cap \alpha(\mathfrak{i}) = \{0\}$ hence $[\mathfrak{i}, \alpha(\mathfrak{i})] \subseteq \mathfrak{i} \cap \alpha(\mathfrak{i})$ vanishes too. So $\mathfrak{i} \oplus \alpha(\mathfrak{i})$ is an abelian ideal preserved by α, which must then vanish, again by Lemma 3.2. It follows that $\mathfrak{i} = \{0\}$.
It is now easy to see, by an induction argument on the commutator series and using the absence of non-zero abelian ideals, that any solvable ideal of $\mathfrak{g}$ must vanish. Therefore $\mathfrak{g}$ is semisimple and the proof is finished. ■

Lemma 3.3. *Suppose that $n \equiv 0$ (mod 8). The following hold:*

(i) A is a semisimple Lie algebra.

(ii) A is isomorphic to $\mathfrak{so}(d, d)$ where $d = \frac{1}{2} dim_{\mathbb{R}} \$$ and $\$$ is the irreducible real Cl_n module.

(iii) the adjoint representation of A^0 on A^1 is irreducible.

Proof. (i) That A is quadratic follows directly from Lemma 3.1, (ii). Moreover A has no center by Lemma 2.3 thus the claim follows from Proposition 3.3 when taking into account that α is a Cartan involution of A.
(ii) Let us equip $\$$ with the indefinite scalar product $\hat{\beta}$ which leaves $\$^+$ and $\$^-$ orthogonal and equals $\pm\langle\cdot,\cdot\rangle$ on $\$^\pm$. In short, $\hat{\beta}(x,y) = \langle \nu x, y\rangle$ for all x,y in $\$$. If φ is in A it is easy to check that $\hat{\beta}(\mu_\varphi x, y) + \hat{\beta}(\mu_\varphi y, x) = 0$ for all x,y in $\$$, that is μ_φ belongs to $so(\$,\hat{\beta}) \cong so(d,d)$. But when $n \equiv 0 \pmod 8$ the Clifford multiplication gives a linear isomorphism $\mu : Cl_n \to \mathrm{End}(\$)$ which is also a Lie algebra isomorphism and our claim follows.
(iii) from (ii) it follows that $(A,\alpha) \cong (so(d,d),\alpha)$ is an orthogonal symmetric Lie algebra in the sense of [10, page 213], that is a semisimple Lie algebra with equipped with a Cartan involution. Here the Cartan involution is given by α, and the corresponding Cartan decomposition is $A = A^0 \oplus A^1$. It follows that our claim is equivalent with $(so(d,d),\alpha)$ being an irreducible orthogonal symmetric Lie algebra as defined in [10, page 377]. The proof is finished by making use of Table V of [10, page 518]. ∎

Similar results can be proved in the remaining series of dimensions but this is somewhat beyond the scope of the present paper. In the same vein

Proposition 3.4. *Let T in Cl_n^0 with $T^t = T$ be such that $Z(\mathfrak{g}_T^*) = \{0\}$. Then the Lie algebra $\mathfrak{g}_T^*$ is semisimple.*

Proof. At first, $\mathfrak{g}_T^*$ is a quadratic Lie algebra when equipped with the restriction of β to $\mathfrak{g}_T^*$. Indeed, we need only see that the restriction of β to $\mathfrak{g}_T^*$ is non-degenerate. But this follows easily from Proposition 3.1, (ii). By Proposition 3.1, (ii) we have that α preserves $\mathfrak{g}_T^*$ so the latter also admits a Cartan involution. We conclude now by using Proposition 3.3. ∎

Recall now that the adjoint representation of the Clifford algebra, Ad: $Cl_n^\times \to \mathrm{End}(Cl_n)$, is given by $\mathrm{Ad}(\varphi)x = \varphi x \varphi^{-1}$ for all φ in $Cl_n^\times$ and for all x in Cl_n. Here $Cl_n^\times$ denotes the group of invertible elements in Cl_n.

Lemma 3.4. *Let T be in Cl_n^0. Then*

$$\mathfrak{g}^*_{\mathrm{Ad}(e)T} = \mathrm{Ad}(e)\mathfrak{g}_T^*$$

for any unit vector e in V.

Proof. At first we notice that $\mathrm{Ad}(e)T = -eTe$ still belongs to Cl_n^0. We have

$$-2X \lrcorner (eTe) = X(eTe) - (eTe)X = -e(XT)e + e(TX)e = 2e(X \lrcorner T)e$$

for all X in $(e)^\perp$. Similarly, $e \lrcorner (eTe) = e(e \lrcorner T)e$ hence $X \lrcorner (eTe) = e(F_e X \lrcorner T)e$ for all X in V, where F_e is the invertible endomorphism of V which equals -1 on $(e)^\perp$ and 1 on (e). Therefore $\mathrm{Ad}(e)$ maps the generating set of $\mathfrak{g}_T^*$ onto that of $\mathrm{Ad}(e)\mathfrak{g}_T^* = e\mathfrak{g}_T^* e$. Since $\mathrm{Ad}(e) : Cl_n \to Cl_n$ is a Lie algebra isomorphism it is now easy to conclude. ∎

When $n \equiv 0 \pmod 4$ the map $\mathrm{Ad}(e)$ where e is some unit vector in V, intertwines, say $Cl_n^+ \cap Cl_n^0$ and $Cl_n^- \cap Cl_n^0$, therefore from the Lemma above we see that the fix algebra does not distinguish between the generating elements being in Cl_n^+ or Cl_n^-. Hence all results obtained for fix algebras generated by elements in Cl_n^+ extend automatically to generating elements in Cl_n^-. We end this section by an example of forms when the holonomy algebras can be easily computed.

Proposition 3.5. *Let $(V^n, \langle\cdot,\cdot\rangle)$ be an Euclidean vector space oriented by ν in $\Lambda^n(V)$. Then:*

(i) $\mathfrak{g}_\nu^* = so(n,1)$ for $n \equiv 0, 1 \pmod 4$,

(ii) $\mathfrak{g}_\nu^* = so(n+1)$ for $n \equiv 2, 3 \pmod 4$,

(iii) in both cases $\mathfrak{g}_\nu^* \psi = 0$ if and only if $\psi = 0$, for any $\psi \in \$$.

Proof. Let us first notice that the generating set $\{X \lrcorner \nu : X \in V\}$ is isomorphic to V, since the volume form ν is non-degenerate. Keeping in mind that by (2) we have $X \lrcorner \nu = -X\nu$ and using (9), (11) this yields

$$[X \lrcorner \nu, Y \lrcorner \nu] = (-1)^{\frac{1}{2}(n+1)(n+2)}[X,Y] = 2(-1)^{\frac{1}{2}(n+1)(n+2)} X \wedge Y$$

for all X,Y in V. Similarly we get for the triple commutators

$$[\gamma, X \lrcorner \nu] = [X,\gamma]\nu = 2FX \lrcorner \nu$$

for all X in V, where $\gamma = \langle F\cdot,\cdot\rangle$ belongs to $\Lambda^2(V)$. Let now $\iota : V \oplus \Lambda^2 V \to \mathfrak{g}_\nu^*$ be given by $\iota(X,\gamma) = \frac{1}{2}(X \lrcorner \nu, \gamma)$. This is a linear isomorphism and moreover from the commutators above we see, by making use of (14), that $\iota : (V \oplus \Lambda^2 V, [\cdot,\cdot]_\varepsilon) \to \mathfrak{g}_\nu^*$ is a Lie algebra isomorphism, where $\varepsilon = (-1)^{\frac{1}{2}(n+1)(n+2)}$.
The claims in (i) and (ii) now follow. (iii) follows easily from the invertibility of ν in Cl_n, as defined in Def. 3.3. ∎

3.2. The set of fixed spinors.

As it will appear below the holonomy algebra of some element T in Cl_n is intimately related to the space of spinors fixed by T, which we recall to be defined as

$$Z_T = \{\psi \in \$: (X \lrcorner T)\psi = 0, \text{ for all } X \in V\}. \tag{17}$$

Notice that if (the non-zero) T is of degree 1 or 2 the set Z_T is obviously reduced to zero and moreover the latter holds for forms of degree 3 (see [1]). We now gather a number of basic facts concerning the set Z_T. If $n \equiv 0 \pmod 4$ and T belongs to $Cl_n^i, i = 0, 1$ we split Z_T along the splitting $\$ = \$^+ \oplus \$^-$ and get

$$Z_T = Z_T^+ \oplus Z_T^-$$

where the obvious notation applies.

Lemma 3.5. *Let T belong to $Cl_n^0 \cap Cl_n^+$ where $n \equiv 0 \pmod 4$. Then*

(i)

$$\begin{aligned} Z_T^+ &= \{\psi \in \not{S}^+ : T\psi = 0\} \\ Z_T^- &= \{\psi \in \not{S}^- : TV\psi = 0\} \end{aligned}$$

(ii) if $n = 8$, then $Z_T^- = \{0\}$ provided T does not vanish.

Proof. (i) follows directly from the stability conditions.
(ii) If $Z_T^- \neq \{0\}$ there exists a non-zero ψ in $\not{S}^-$ with $TV\psi = 0$. Since ψ is not zero, the linear map from V to $\not{S}^+$ given by $v \mapsto v\psi$ for all v in V is injective. It is surjective as well because in dimension 8 the space $\not{S}^+$ is eight dimensional. Then $V\psi = \not{S}^+$ hence $T\not{S}^+ = 0$ which leads to $T = 0$, a contradiction. ■

A very simple observation, dealing with the case when the form T is of pure degree, is

Lemma 3.6. *Let T belong to $\Lambda^k(V), k \neq 0$. Then:*

(i) $TZ_T = 0$,

(ii) $Z_T = Z_{\nu T}$.

Proof. (i) If ψ belongs to Z_T we have $(X \lrcorner T)\psi = 0$ for all X in V. Therefore $\sum_{i=1}^{n} e_i(e_i \lrcorner T)\psi = 0$ for some orthonormal frame $\{e_i, 1 \leq i \leq n\}$ leading to $kT\psi = 0$ and the claim follows.
(ii) From (i) we get that $Z_T = \{\psi \in \not{S} : TV\psi = 0\}$ for any pure degree form T and the claim follows easily. ■

Definition 3.3. *An element T of $Cl_n^0 \cap Cl_n^+$ is invertible in $Cl_n/(1,\nu)$ if there exists T^{-1} in $Cl_n^0 \cap Cl_n^+$ with $T^{-1}T = TT^{-1} = 1 + \nu$. Here we have denoted by $(1,\nu)$ the vector sub-space of Cl_n generated by 1 and ν.*

Actually a necessary and sufficient condition for Z_T to vanish is

Proposition 3.6. *Let T belong to $Cl_n^0 \cap Cl_n^+, n \equiv 0 \pmod 4$. Then $Z_T = \{0\}$ if and only if T is invertible in $Cl_n/(1,\nu)$.*

Proof. If T is invertible Lemma 3.5 yields immediately the vanishing of $Z_T^{\pm}$ hence that of T.
For any $\zeta \in Cl_n^0 \cap Cl_n^+$ let $L_\zeta : Cl_n^+ \cap Cl_n^0 \to Cl_n^+ \cap Cl_n^0$ be left multiplication with ζ, which is obviously well-defined (see Lemma 2.1) and linear. Suppose now that $Z_T = \{0\}$. If φ is in the kernel of L_T it follows that $T(\varphi\not{S}^+) = 0$ and moreover, since $\varphi\not{S}^+ \subseteq \not{S}^+$ Lemma 3.5 tells us that $\varphi\not{S}^+ \subseteq Z_T^+ \subseteq Z_T$ hence $\varphi\not{S}^+ = 0$. We recall now that μ is faithful when $n \equiv 0 \pmod 4$, essentially because Cl_n is a simple algebra in the considered dimensions [13, Chap. I, Thm 5.6]. Therefore φ vanishes and it follows that L_T is injective, hence an isomorphism. Thus there exists a linear map $L_T^{-1} : Cl_n^0 \cap Cl_n^+ \to Cl_n^0 \cap Cl_n^+$ such

that $L_T \circ L_T^{-1} = L_T^{-1} \circ L_T = 1_{Cl_n^0 \cap Cl_n^+}$. Setting $T^{-1} = L_T^{-1}(1+\nu)$ we get from $L_T \circ L_T^{-1} = 1_{Cl_n^0 \cap Cl_n^+}$ that T^{-1} in $Cl_n^0 \cap Cl_n^+$ satisfies $TT^{-1} = 1 + \nu$. Now we compose $L_T^{-1} \circ L_T = 1_{Cl_n^0 \cap Cl_n^+}$ at right with $L_{T^{-1}}$ and get that $L_T^{-1} = \frac{1}{2} L_{T^{-1}}$. It follows that $\frac{1}{2} L_{T^{-1}} \circ L_T = 1_{Cl_n^0 \cap Cl_n^+}$ which leads easily to $T^{-1}T = 1 + \nu$ when using that $1_{Cl_n^+ \cap Cl_n^0} = \frac{1}{2}(1+\nu)$. So T is invertible in the sense of Definition 3.3 and the proof is finished. ■

In the rest of this section we shall present examples of situations when the set of fixed spinors can be seen directly to be trivial.

Proposition 3.7. *Let γ in $\Lambda^2(V)$ be a two-form such that $T = \gamma \wedge \gamma \neq 0$. Then $Z_T = \{0\}$.*

Proof. Let F be the skew-symmetric endomorphism associated to γ via the metric g, that is $\gamma = \langle F\cdot, \cdot\rangle$. We have $X \lrcorner (\gamma \wedge \gamma) = 2(X \lrcorner \gamma) \wedge \gamma = 2FX \wedge \gamma$ for all X in V. Let now ψ be in Z_T and let us set $r = \mathrm{rank}(F)$. From

$$(X \lrcorner T)\psi = 2\,(FX \wedge \gamma)\psi = 0$$

follows $(X \wedge \gamma)\psi = 0$ for all X in $Im(F)$. By Clifford contraction we get immediately

$$\sum_{e_i \in Im(F)} e_i(e_i \wedge \gamma)\psi = -\sum_{e_i \in Im(F)} (e_i \lrcorner (e_i \wedge \gamma))\psi = (2-r)\,\gamma\psi = 0,$$

which leads to $\gamma\psi = 0$ since having $r = 2$ would imply $T = \gamma \wedge \gamma = 0$, a contradiction. Therefore

$$0 = X\gamma\psi = (X \wedge \gamma - X \lrcorner \gamma)\psi$$

for all X in V and since $(X \wedge \gamma)\psi = 0$ for all X in $Im(F)$ we are lead to $(X \lrcorner \gamma)\psi = 0$ for all X in $Im(F)$. It follows that $\psi = 0$ as $\gamma \neq 0$. ■

Proposition 3.8. *For any T in $\Lambda^k(\mathbb{R}^n), n \leq 7$ the set Z_T is trivial.*

Proof. This is obvious when $k = 1, 2$ and when $k = 3$ it was proved in [1, Thm. 4.2]. Now if $k \geq 4$ we have $Z_T = Z_{\nu T}$ and since νT has degree $n - k \leq 3$ we conclude by the Lemma 3.6, (ii). ■

Therefore the first case of interest is that of dimension 8.

3.3. Perfect fix algebras.

In this section we shall examine situations when the fix algebra $\mathfrak{g}_T^*$ for some T in Cl_n is perfect. We will see that this is not always the case since those can be abelian by the examples in the next section. However we will show that it is possible to give necessary conditions to that extent. Let us set first a preparatory Lemma.

Lemma 3.7. *If ζ_1, ζ_2 in Cl_n satisfy $\zeta_1 X = X\zeta_2$ for all X in V then ζ_1 and ζ_2 belong to $(1,\nu)$.*

Proof. It follows that $-|X|^2\zeta_1 = X\zeta_2 X$ for all X in V and further

$$|X|^2 Y\zeta_2 Y = |Y|^2 X\zeta_2 X$$

for all X, Y in V. By left multiplication with some non-zero X we get $(XY)\zeta_2 Y = -|Y|^2\zeta_2 X$ and now right multiplication with a non-zero Y yields $(XY)\zeta_2 = \zeta_2(XY)$ for all X, Y in V. Now ζ_2 is in $(1,\nu)$ by Lemma 2.2 and it is easy to see this implies the claim for ζ_1 as well. ∎

Proposition 3.9. *Suppose that $n \equiv 0 \pmod 4$ and let T satisfy $\nu T = T\nu = T$ and $T^t = T$. If $Z_T = \{0\}$ then $\mathfrak{g}_T^*$ has trivial center.*

Proof. If ζ in $Z(\mathfrak{g}_T^*)$ we must clearly have

$$[\zeta, XT - TX] = 0 \tag{18}$$

for all X in V. Since $XT - TX$ belongs to Cl_n^1 for all X in V by applying α to the equation above we get that $[\alpha(\zeta), XT - TX] = 0$ for all X in V, hence after splitting ζ into its even resp. odd components it is enough to treat (18) when ζ belongs to Cl_n^0 resp. Cl_n^1.
Case I: ζ belongs to Cl_n^0.
Since $\nu(XT - TX) = -(XT + TX)$ for all X in V and $\nu\zeta = \zeta\nu$ after left multiplication of (18) with the volume form we get $\zeta(XT + TX) = (XT + TX)\zeta$ whenever X belongs to V. Taking linear combinations with (18) gives further

$$\begin{aligned} \zeta XT &= XT\zeta \\ \zeta TX &= TX\zeta \end{aligned}$$

for all X in V. Since $Z_T = \{0\}$ we know that T must be invertible in $Cl_n/(1,\nu)$ (see Proposition 3.6) hence using the second equation above we have

$$(T^{-1}\zeta T)X = T^{-1}TX\zeta = (1+\nu)X\zeta = X(1-\nu)\zeta$$

for all X in V. But from Lemma 3.7, (ii) we get that $(1-\nu)\zeta$ belongs to $(1,\nu)$ and hence vanishes since $((1-\nu)\zeta)^t = -(1-\nu)\zeta$ as of ζ being an element of A by Proposition 3.1,(i). From the vanishing of $(1-\nu)\zeta$ it follows that $T^{-1}\zeta T = 0$ and this leads after right multiplication with T^{-1} resp. left multiplication with T to $(1+\nu)\zeta = 0$. Thus $\zeta = 0$ in this case.
Case II: ζ belongs to Cl_n^1.
We have as before $\zeta(XT - TX) = (XT - TX)\zeta$ for all X in V. But $\nu(XT - TX) = -(XT + TX)$ for all X in V because $\nu T = T$ and since $\nu\zeta = -\zeta\nu$ after left multiplication with the volume form we obtain

$$\zeta(XT + TX) = -(XT + TX)\zeta$$

for all X in V. Taking linear combinations with the original equation gives now

$$\begin{aligned} (\zeta T)X &= -X(T\zeta) \\ \zeta XT &= -TX\zeta \end{aligned}$$

for all X in V. Using Lemma 3.7, (ii) we then get that ζT and $T\zeta$ belong to $(1,\nu)$ and therefore must vanish since they are elements of Cl_n^1. The invertibility of T in $Cl_n/(1,\nu)$ leads then to $(1+\nu)\zeta = 0$ whereas left multiplication by $1+\nu$ in the second equation above gives $TX\zeta = 0$ for all X in V. Again the invertibility of T implies that $(1+\nu)X\zeta = X(1-\nu)\zeta = 0$ for all X in V and we conclude that $(1-\nu)\zeta = 0$ hence $\zeta = 0$ and the proof is finished. ∎

Summarising, after making use of the fact that $(\mathfrak{g}_T^*, \beta)$ is a quadratic Lie algebra and of Propositions 3.2 and 3.9, we obtain

Theorem 3.1. *Let T belong to $Cl_n^0 \cap Cl_n^+$ where $n \equiv 0 \pmod 4$ and satisfy $T^t = T$. If moreover $Z_T = \{0\}$, the algebra $\mathfrak{g}_T^*$ is perfect, that is*

$$\mathfrak{g}_T^* = [\mathfrak{g}_T^*, \mathfrak{g}_T^*].$$

The proof of Theorem 1.1 follows while making use of the fact that, under the above conditions on T, the fix algebra $\mathfrak{g}_T^*$ is semisimple when its center vanishes by Proposition 3.4.

4. Holonomy algebras from distinguished Clifford algebra elements

4.1. Unipotent elements.

In this section we shall compute directly the fix and holonomy algebras of a unipotent element T of $Cl_n^+, n \equiv 0 \pmod 4$ as introduced below.

Definition 4.1. *Let T belong to Cl_n^+ where $n \equiv 0 \pmod 4$. It is called unipotent if it satisfies $T^t = T$ and $T^2 = 1 + \nu$.*

In particular any unipotent element T belongs to Cl_n^0. We need first to state and prove the following preliminary result, to be used later on as well.

Lemma 4.1. *Let T belong to $Cl_n^+ \cap Cl_n^0$ where $n \equiv 0 \pmod 4$. Then:*

$$4[X \lrcorner T, Y \lrcorner T] = -T[X,Y]T + YT^2X - XT^2Y$$

whenever X, Y belong to V.

Proof. Follows directly from the stability relations of Lemma 2.1 under the form $TXT = 0$ for all X in V. Details are left to the reader. ∎

Theorem 4.1. *Let T be a unipotent element of Cl_n^+ where $n \equiv 0 \pmod 4$. Then:*

(i) $Z_T = \{0\}$

(ii) the fix algebra of T is perfect

(iii) the holonomy algebra of T is isomorphic to $\mathfrak{so}(n,1)$.

Proof. (i) Since any unipotent element T of Cl_n^+ is clearly invertible in $Cl_n/(1,\nu)$ Proposition 3.6 implies that $Z_T = \{0\}$.
(ii) follows from (i) and Theorem 3.1.
(iii) For notational convenience let $E_T = \{X \lrcorner T : X \in V\}$ be the generating set of $\mathfrak{g}_T^*$. It is isomorphic to V under the map $\iota^1 : V \to E_T, \iota^1(X) = X \lrcorner T$. Here only the injectivity of ι^1 has to be proved, and indeed, if $\iota^1(X) = 0$ it follows that $XT = TX$ and further $0 = TXT = T^2X$ leading to the vanishing of X. Now by Lemma 4.1 combined with the unipotency of T the space $[E_T, E_T]$ equals $\{T\gamma T + (1-\nu)\gamma : \gamma \in \Lambda^2(V)\}$. This is isomorphic to $\Lambda^2(V)$ under $\iota^2 : \Lambda^2(V) \to [E_T, E_T], \iota^2(\gamma) = T\gamma T + (1-\nu)\gamma$. Indeed, if $\iota^2(\gamma) = 0$ we find that $T\gamma T + (1-\nu)\gamma = 0$ but then both summands vanish as the first is in Cl_n^+ and the second in Cl_n^-. Hence $\gamma = 0$ and so ι^2 is injective. Note also that Lemma 4.1 yields equally the commutator identity

$$2[\iota^1 X, \iota^1 Y] = -\iota^2(X \wedge Y)$$

for all X, Y in V. Now

$$\begin{aligned}[T\gamma_1 T + (1-\nu)\gamma_1, T\gamma_2 T + (1-\nu)\gamma_2] &= [T\gamma_1 T, T\gamma_2 T] + [(1-\nu)\gamma_1, (1-\nu)\gamma_2] \\ &+ [(1-\nu)\gamma_1, T\gamma_2 T] + [T\gamma_1 T, (1-\nu)\gamma_2] \\ &= [T\gamma_1 T, T\gamma_2 T] + [(1-\nu)\gamma_1, (1-\nu)\gamma_2]\end{aligned}$$

for all γ_1, γ_2 in $\Lambda^2(V)$ after using that T belongs to Cl_n^+. Obviously

$$[(1-\nu)\gamma_1, (1-\nu)\gamma_2] = (1-\nu)^2[\gamma_1, \gamma_2] = 2(1-\nu)[\gamma_1, \gamma_2]$$

and moreover the unipotency of T leads easily to

$$[T\gamma_1 T, T\gamma_2 T] = (1+\nu)T[\gamma_1, \gamma_2]T = 2T[\gamma_1, \gamma_2]T.$$

Altogether this yields

$$2\iota^2[\gamma_1, \gamma_2] = [\iota^2(\gamma_1), \iota^2(\gamma_2)]$$

for all γ_1, γ_2 in $\Lambda^2(V)$, in other words $\frac{1}{2}\iota^2 : \Lambda^2(V) \to [E_T, E_T]$ is a Lie algebra isomorphism. Now, we compute

$$\begin{aligned}-2[T\gamma T + (1-\nu)\gamma, X \lrcorner T] &= [T\gamma T + (1-\nu)\gamma, XT - TX] \\ &= [T\gamma T, XT] + [(1-\nu)\gamma, XT] \\ &\quad -[T\gamma T, TX] - [(1-\nu)\gamma, TX].\end{aligned}$$

We now examine each term separately. We have

$$[T\gamma T, XT] = -XT^2\gamma T = -X(1+\nu)\gamma T = -2X\gamma T$$

after using that $TXT = 0$. Similarly, $[T\gamma T, TX] = 2T\gamma X$ and using furthermore that T belongs to Cl_n^+ we finally obtain

$$-2[T\gamma T + (1-\nu)\gamma, X \lrcorner T] = -2X\gamma T + 2\gamma XT - 2T\gamma X + 2TX\gamma.$$

This ends up by saying that $[\iota^2(\gamma), \iota^1(X)] = 4\,\iota^1(X \lrcorner \gamma)$ whenever γ belongs to $\Lambda^2(V)$ and X in V. We have showed that $\mathfrak{g}_T^* = E_T + [E_T, E_T]$ and moreover the sum is direct since $E_T \subseteq Cl_n^1$ whilst $[E_T, E_T] \subseteq Cl_n^0$. Also

$$\frac{1}{\sqrt{2}}\,\iota^1 \oplus \frac{1}{4}\,\iota^2 : V \oplus \Lambda^2(V) \to \mathfrak{g}_T^*$$

is a linear isomorphism. From the commutation rules between ι^1 and ι^2 proved above it is easy to check, using (14), that $\frac{1}{\sqrt{2}}\iota^1 \oplus \frac{1}{4}\iota^2$ is a Lie algebra isomorphism between $(V \oplus \Lambda^2(V), [\cdot,\cdot]_{-1})$ and $\mathfrak{g}_T^*$. The desired Lie algebra isomorphism between $\mathfrak{g}_T^*$ and $so(n,1)$ then follows. ■

We have therefore proved the claim in (i) of Theorem 1.2.

Remark 4.1. *Explicit examples of unipotent elements are easy to make. When the dimension of our vector space V satisfies $\dim_{\mathbb{R}} V \equiv 0 \pmod 4$ we see that $\frac{1}{\sqrt{2}}(1+\nu)$ is unipotent and therefore Theorem 4.1 recovers the computation of $\mathfrak{g}_\nu^*$ in (ii) of Proposition 3.5 when $n \equiv 0 \pmod 4$. Moreover, if we take V_1, V_2 to be Euclidean vector spaces of dimensions $\equiv 0 \pmod 4$ oriented by volume forms $\nu_k, k = 1, 2$ then $\frac{1}{\sqrt{2}}(\nu_1 + \nu_2)$ is an unipotent element of the direct product space $V_1 \times V_2$.*

4.2. Squares of spinors.

We shall first recall in what follows some facts about the squaring construction in two series of dimensions. To begin with, let $(V^n, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space which furthermore is supposed to be oriented, with orientation given by ν in $\Lambda^n(V)$. A peculiar property of the Clifford multiplication when $n \equiv 8 \pmod 8$ is then to give an isomorphism (see [13, Chap. IV, Prop. 10.17]):

$$\mu : Cl_n \to \mathrm{Hom}_{\mathbb{R}}(\$, \$) \tag{19}$$

where $\$$ is the irreducible real Cl_n module. When $n \equiv 7 \pmod 8$ this still holds provided Cl_n is replaced by Cl_n^+. Let us now fix a spinor $x \in \$^+$ (or in $\$$ if $n \equiv 7 \pmod 8$), which we normalise to $|x| = 1$. Then the isomorphism (19) gives rise to an element $x \otimes x \in Cl_n$ (or Cl_n^+ when $n \equiv 7 \pmod 8$) such that:

$$(x \otimes x)\,\psi = \langle \psi, x\rangle\, x \tag{20}$$

for all ψ in $\$$. From the definition, Clifford multiplication with $x \otimes x$ is a symmetric endomorphism of $\$^+$ (or $\$$ if $n \equiv 7 \pmod 8$), therefore (15) implies

$$\alpha(x \otimes x)^t = x \otimes x. \tag{21}$$

The element $x \otimes x$ is customarily called the square of the spinor x. Below we list some of the properties of $x \otimes x$, of relevance for our study.

Lemma 4.2. *Let x be a unit length spinor in $\$^+$ if $n \equiv 0 \pmod 8$ of in $\$$ if $n \equiv 7 \pmod 8$. The following hold:*

(i) The spinor square $x \otimes x$ is an idempotent of Cl_n, that is $(x \otimes x)^2 = x \otimes x$.

(ii) We have $\nu\,(x \otimes x) = (x \otimes x)\,\nu = x \otimes x$.

(iii) For all $\varphi \in Cl_n$ we have

$$(x \otimes x)\,\varphi\,(x \otimes x) = \kappa\,\langle \varphi, x \otimes x\rangle\,(x \otimes x),$$

where $\kappa = 2^{\frac{n}{2}}$ when $n \equiv 0 \pmod 8$ and $\kappa = 2^{\frac{n+1}{2}}$ when $n \equiv 7 \pmod 8$.

Proof. (i) We use (20) for $\psi = x$ which gives $(x \otimes x)\, x = x$. Therefore left multiplication of (20) with $x \otimes x$ gives:

$$(x \otimes x)^2\, \psi = \langle \psi, x \rangle\, (x \otimes x)\, x = \langle \psi, x \rangle\, x = (x \otimes x)\, \psi,$$

for all ψ in $\$$ and the claim follows.
(ii) if $n \equiv 7 \pmod 8$ this follows automatically from the definition of the spinor square. When $n \equiv 0 \pmod 8$, we use the definition (20) to obtain after recalling that $x \in \$^+$

$$\nu\, (x \otimes x)\, \psi = \langle \psi, x \rangle\, \nu x = \langle \psi, x \rangle\, x = (x \otimes x)\, \psi,$$

for all $\psi \in \$$. Since $x \otimes x \in Cl_n^0$ and $n \equiv 0 \pmod 8$, we further have $[\nu, x \otimes x] = 0$.
(iii) Let φ belong to Cl_n and $n \equiv 0 \pmod 8$. Using again (20) we compute

$$\begin{aligned}(x \otimes x)\, \varphi\, (x \otimes x)\, \psi &= \langle \psi, x \rangle\, (x \otimes x)\, \varphi\, x = \langle \psi, x \rangle \langle \varphi x, x \rangle x \\ &= \langle \varphi x, x \rangle (x \otimes x) \psi\end{aligned}$$

for all ψ in $\$$ hence $(x \otimes x)\varphi(x \otimes x) = \langle \varphi x, x \rangle x \otimes x$ for all φ in Cl_n. Recall now that $\langle 1, x \otimes x \rangle = 2^{-\frac{n}{2}}$ fact which follows essentially by taking traces and using Lemma 2.3. Therefore

$$\begin{aligned}2^{-\frac{n}{2}} \langle \varphi x, x \rangle &= \langle 1, (x \otimes x)\varphi(x \otimes x) \rangle = \langle \alpha(x \otimes x)^t \cdot 1, \varphi(x \otimes x) \rangle \\ &= \langle x \otimes x, \varphi(x \otimes x) \rangle = \langle (x \otimes x)\alpha(x \otimes x)^t, \varphi \rangle \\ &= \langle (x \otimes x)^2, \varphi \rangle = \langle x \otimes x, \varphi \rangle\end{aligned}$$

and the claim follows. For $n \equiv 7 \pmod 8$ this is proved analogously. ∎

Remark 4.2. *For any unit length spinor in $\$^+$ if $n \equiv 0 \pmod 8$, or in $\$$ if $n \equiv 7 \pmod 8$ one uses (21), (16) to see that its square $x \otimes x$ has the form*

$$x \otimes x = \sum_{k \equiv 0,3 \,(\mathrm{mod}\, 4)}^{n} (x \otimes x)_k, \tag{22}$$

where $(x \otimes x)_k$ denotes the projection of $x \otimes x$ onto $\Lambda^k(V)$. Note that when $n \equiv 0 \pmod 8$ the odd degrees are not present since then $x \otimes x$ is in Cl_n^0 by Lemma 4.2, (ii).

Based on the technical Lemma above we shall compute now the holonomy algebras of the square of a spinor. Let us begin with the case of $n \equiv 0 \pmod 8$.

Theorem 4.2. *Let $n \equiv 0 \pmod 8$ and x be a unit length spinor in $\$^+$ and $x \otimes x$ be its square. Then:*

(i) $\mathfrak{g}^*_{x \otimes x} = \mathfrak{h}^*_{x \otimes x} \cong \mathfrak{so}(n, 1)$,

(ii) $Z_{x \otimes x} = \{\psi \in \$^+ : \psi \perp x\} \oplus \{\psi \in \$^- : \psi \perp Vx\}$.

Proof. (i) Let $E_T = \{X \lrcorner (x \otimes x) : X \in V\}$ be the generating set of V, which is easily seen to be isomorphic to V under the map $\iota^1 : V \to E_T, \iota^1(X) = X \lrcorner (x \otimes x)$. Further on, let us define $\iota^2 : \Lambda^2(V) \to Cl_n(V)$ by $\iota^2(\gamma) = \sum_{i=1}^{n} e_i\, (x \otimes x)\, (e_i \lrcorner \gamma)$ for some orthonormal frame $\{e_i, 1 \le i \le n\}$ on V and some $\gamma \in \Lambda^2(V)$. Let us show now that ι^2 is injective. Indeed, if γ in Λ^2 satisfies $\iota^2(\gamma) = 0$ we get by right multiplication with $X(x \otimes x)$, where X is in V, that $\sum_{i=1}^{n} e_i\, (x \otimes x)\, (e_i \lrcorner \gamma)\, X\, (x \otimes x) = 0$. Since $(e_i \lrcorner \gamma)\, X = (e_i \lrcorner \gamma) \wedge X - \gamma(e_i, X)$ for all $1 \le i \le n$ and $(x \otimes x)\Lambda^2(x \otimes x) = 0$ from Lemma 4.2, (iii) this further yields $(X \lrcorner \gamma)(x \otimes x)^2 = 0$ for all X in V. Using the idempotency of $x \otimes x$, as of Lemma 4.2, (i) gives further $(X \lrcorner \gamma)(x \otimes x) = 0$ for all X in V. It follows that $\gamma = 0$, proving the injectivity of ι^2.
Let now γ_1 and γ_2 belong to Λ^2 with associated skew-symmetric endomorphisms F_1 and F_2. We compute

$$\begin{aligned}(\iota^2(\gamma_1))(\iota^2(\gamma_2)) &= \sum_{i,j} e_i(x \otimes x)(F_1 e_i) e_j (x \otimes x) F_2 e_j = -\sum_{i,j} \langle F_1 e_i, e_j \rangle e_i (x \otimes x) F_2 e_j \\ &= -\sum_i e_i (x \otimes x) F_2 F_1 e_i\end{aligned}$$

where we have used the expansion of the Clifford product in (2) together with Lemma 4.2, (iii) under the form $(x \otimes x)\Lambda^2(x \otimes x) = 0$, as well as the idempotency of the spinor square $x \otimes x$ from (i) of the same Lemma. Skew-symmetrising the above equation in γ_1 and γ_2 leads now to

$$2[\iota^2(\gamma_1), \iota^2(\gamma_2)] = \iota^2[\gamma_1, \gamma_2]$$

by making use of $[\gamma_1, \gamma_2] = 2\langle [F_1, F_2]\cdot, \cdot \rangle$, see (13). Now,

$$\begin{aligned}4\,[X \lrcorner (x \otimes x), Y \lrcorner (x \otimes x)] &= -X(x \otimes x)Y + Y(x \otimes x)X \\ &\quad -(x \otimes x)XY(x \otimes x) + (x \otimes x)XY(x \otimes x) \\ &= -\iota^2(X \wedge Y),\end{aligned}$$

whenever X, Y belong to V. Here we have made once more extensive use of the stability conditions and of Lemma 4.2, under the form $(x \otimes x)\Lambda^2(x \otimes x) = 0$. Hence the even commutators span $\mathfrak{so}(n)$. The triple commutator is similarly computed:

$$\begin{aligned}[\iota^2(\gamma), X \lrcorner (x \otimes x)] &= -\tfrac{1}{2} \sum_{i=1}^{n} \Big(e_i(x \otimes x) F e_i X (x \otimes x) + (x \otimes x) X e_i (x \otimes x) F e_i \\ &\qquad -X(x \otimes x) e_i (x \otimes x) F e_i - e_i (x \otimes x) F e_i (x \otimes x) X \Big) \\ &= FX \lrcorner (x \otimes x) = \iota^1(X \lrcorner \gamma),\end{aligned}$$

where we have made use of Lemma 4.2. By inspecting the commutator relations between ι^1 and ι^2 proved above and keeping in mind (14), it is easy to see that $2\iota^1 \oplus \iota^2 : (V \oplus \Lambda^2 V, [\cdot,\cdot]_{-1}) \to \mathfrak{g}^*_{x \otimes x}$ is a Lie algebra isomorphism. Therefore $\mathfrak{g}^*_{x \otimes x}$ is isomorphic to $\mathfrak{so}(n, 1)$. Since $\mathfrak{so}(n, 1)$ is semisimple for $n \ge 2$ [12, page 59], hence perfect, it follows that $\mathfrak{h}^*_{x \otimes x}$ is isomorphic with $\mathfrak{so}(n, 1)$ as well.
(ii) Let ψ belong to $Z_{x \otimes x}$. Then by Lemma 3.5 this is equivalent with $(x \otimes x)\psi^+ = 0$ and $(x \otimes x)V\psi^- = 0$ and the claim follows now from the definition of $x \otimes x$, where $\psi^\pm \in \$^\pm$ are the components of ψ w.r.t the splitting $\$ = \$^+ \oplus \$^-$. ∎

The proof of (ii) in Theorem 1.2 is now complete

Remark 4.3. *Squares of spinors provide examples of unipotent elements other than those coming from volume forms. Indeed, if x belongs to $\$^+$ with $|x| = 1$ it is easy to see by making use of (21) and Lemma 4.2 (i) and (ii), that $2\sqrt{2}(x \otimes x - \frac{1}{4}(1+\nu))$ is a unipotent element in Cl_n^+. In spite of the absence of fixed spinors in this case, fact which follows from Theorem 4.1, (i), the holonomy algebra remains isomorphic to $\mathfrak{so}(n,1)$ by (iii) of Theorem 4.1.*

When $n \equiv 7 \pmod 8$ we get fix and holonomy algebras of quite different nature than those seen before. In particular, those appear not to be perfect.

Theorem 4.3. *Let $n \equiv 7 \pmod 8$ and x belong to $\$$ such that $|x| = 1$. Then:*

(i) $\mathfrak{g}^*_{x\otimes x}$ *is abelian, isomorphic to* V *hence* $\mathfrak{h}^*_{x\otimes x} = \{0\}$,

(ii) $Z_{x\otimes x} = (x)^\perp$.

Proof. (i) For any X, Y in V we compute

$$\begin{aligned} 4(X \lrcorner (x\otimes x))(Y \lrcorner (x\otimes x)) &= (\alpha(x\otimes x)X - Xx\otimes x)(\alpha(x\otimes x)Y - Yx\otimes x) \\ &= [\alpha(x\otimes x)X\alpha(x\otimes x)]Y - \alpha(x\otimes x)XY(x\otimes x) \\ &\quad -X(x\otimes x)\alpha(x\otimes x)Y + X[(x\otimes x)Y(x\otimes x)]. \end{aligned}$$

But $x \otimes x$ belongs to Cl_n^+, hence $\alpha(x\otimes x)$ is in Cl_n^- leading to the vanishing of the second and third term above in view of the stability conditions in Lemma 2.1. Now the first and the last terms vanish too by Lemma 4.2, (iii) and since $x\otimes x$ does not contain degree 1 forms, therefore $(X \lrcorner (x\otimes x))(Y \lrcorner (x\otimes x)) = 0$ for all X, Y in V. It follows that the fix algebra $\mathfrak{g}^*_{x\otimes x}$ is abelian, in particular the holonomy algebra satisfies $\mathfrak{h}^*_{x\otimes x} = \{0\}$. But $x\otimes x$ is non-degenerate, as it contains a non-zero multiple of the volume form whence $\mathfrak{g}^*_{x\otimes x}$ is isomorphic with V.
(ii) follows easily from the construction of $x \otimes x$. ∎

We have therefore proved (iii) of Theorem 1.2 whose proof is now complete.

Acknowledgements

During the preparation of this paper both authors were partially supported by University of Auckland grants. They warmly acknowledge the Centro di Giorgi in Pisa and S. Salamon for giving them the opportunity to attend the meeting "Geometry and Topology" where parts of this work were completed. During an early stage of this paper the research of P.-A. N. has been supported through the VW-foundation at the HU-Berlin, while the work of N. B. was funded by the German Research Foundation (DFG). Special thanks go to I. Agricola, S. Chiossi, Th. Friedrich and A. R. Gover for a number of useful discussions. We are grateful to the referee for valuable suggestions on how to improve this work.

References

[1] Agricola, I., and Th. Friedrich, *On the holonomy of connections with skew-symmetric torsion*, Math. Ann. **328** (2004), 711–748.

[2] Agricola, I., Th. Friedrich, P.-A. Nagy and C.Puhle, *On the Ricci tensor in the common sector of type II string theory*, Class. Quantum Grav. **22** (2005), 2569–2577.

[3] Besse, A., "Einstein manifolds," Springer Verlag, 1986.

[4] Bryant, R. L., *Metrics with exceptional holonomy*, Ann. of Math. **126** (1986), 525–576.

[5] Dadok, J., and R. Harvey, *Calibrations and spinors*, Acta Math. **170** (1993), 83–120.

[6] Fernández, M., *A classification of Riemannian manifolds with structure group Spin(7)*, Ann. Mat. Pura Appl. **143** (1986), 101–122.

[7] Figueroa-O'Farrill, J. F., and G. Papadopoulos, *Maximally supersymmetric solutions of ten- and eleven-dimensional supergravities*, J. High Energy Phys. **2003** (2003), no. 3, 048, 1–25.

[8] Friedrich, Th., and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math.**6** (2002), 303–335.

[9] Gray, A., and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) **123** (1980), 35–58.

[10] Helgason, S., "Differential geometry, Lie groups, and symmetric spaces," Graduate Studies in Math., **34**, Amer. Math. Soc., Providence, RI, 2001.

[11] Ivanov, S., *Connections with torsion, parallel spinors and the geometry of Spin(7) manifolds*, Math. Res. Lett. **11** (2004), 171–186.

[12] Knapp, A. W., "Lie Groups Beyond an Introduction," Birkhäuser, 2002.

[13] Lawson, H. B. Jr., and M. L. Michelsohn, "Spin geometry," Princeton University Press, 1989.

[14] Strominger, A., *Superstrings with torsion*, Nucl. Phys. B **274** (1986), 253-281.

Niels Bernhardt
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland, New Zealand
niels.bernhardt@math.auckland.ac.nz

Paul-Andi Nagy
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland, New Zealand
nagy@math.auckland.ac.nz

Received September 11, 2006
and in final form December 12, 2006