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On the Ricci tensor in the common sector of type II string theory*

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Abstract

Let ∇ be a metric connection with totally skew-symmetric torsion T on a Riemannian manifold. Given a spinor field Ψ and a dilaton function Φ, the basic equations in the common sector of type II string theory are

$$\nabla\Psi = 0, \qquad \delta(\mathrm{T}) = a \cdot (\mathrm{d}\Phi \lrcorner \mathrm{T}), \qquad \mathrm{T}\cdot\Psi = b\cdot \mathrm{d}\Phi\cdot\Psi + \mu\cdot\Psi$$

for some auxiliary parameters a, b, μ. We derive some relations between the length $\|\mathrm{T}\|^2$ of the torsion form, the scalar curvature of ∇, the dilaton function Φ and the parameters a, b, μ. We show that for constant dilaton and $\mu = 0$ (the physically relevant case), there cannot be even local solutions to this system of equations with vanishing scalar curvature. The main results deal with the divergence of the Ricci tensor Ric^∇ of the connection. In particular, if the supersymmetry Ψ is non-trivial and if the conditions

$$(\mathrm{d}\Phi \lrcorner \mathrm{T}) \lrcorner \mathrm{T} = 0, \qquad \delta^\nabla(\mathrm{dT})\cdot\Psi = 0$$

hold, then the energy–momentum tensor is divergence free. We show that the latter condition is satisfied in many examples constructed out of special geometries. A special case is $a = b$. Then the divergence of the energy–momentum tensor vanishes if and only if one condition $\delta^\nabla(\mathrm{dT})\cdot\Psi = 0$ holds. Strong models $(\mathrm{dT} = 0)$ have this property, but there are examples with $\delta^\nabla(\mathrm{dT}) \neq 0$ and $\delta^\nabla(\mathrm{dT})\cdot\Psi = 0$.

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1. Type II B string theory with constant dilaton

The mathematical model discussed in the common sector of type II superstring theory (also sometimes referred to as type I supergravity) consists of a Riemannian manifold (M^n, g), a

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metric connection ∇ with totally skew-symmetric torsion T and a non-trivial spinor field Ψ. Putting the full Ricci tensor aside for starters, there are three equations relating these objects:

$$(*)\qquad \nabla\Psi = 0, \qquad \delta(\mathrm{T}) = 0, \qquad \mathrm{T}\cdot\Psi = \mu\cdot\Psi.$$

The spinor field describes the supersymmetry of the model. The first equation means that the spinor field Ψ is parallel with respect to the metric connection ∇. The second equation is a conservation law for the 3-form T. Since ∇ is a metric connection with totally skew-symmetric torsion, the divergences $\delta^{\nabla}(\mathrm{T}) = \delta^{g}(\mathrm{T})$ of the torsion form coincide (see [2, 8]). We denote this unique 2-form simply by $\delta(\mathrm{T})$. The third equation is an algebraic link between the torsion form T and the spinor field Ψ. Indeed, the 3-form T acts as an endomorphism in the spinor bundle and the last equation requires that Ψ is an eigenspinor for this endomorphism. Generically, $\mu = 0$ in the physical model; but the mathematical analysis becomes more transparent if we first include this parameter. *A priori*, μ may be an arbitrary function. Since T acts on spinors as a symmetric endomorphism, μ has to be real. Moreover, we will see that only real, constant parameters μ are possible. It is well known (see [8]) that the conservation law $\delta(\mathrm{T}) = 0$ implies that the Ricci tensor Ric^{∇} of the connection ∇ is symmetric. Denote by Scal^{∇} the ∇-scalar curvature and by Scal^{g} the scalar curvature of the Riemannian metric. The existence of the ∇-parallel spinor field yields the so-called integrability conditions (see [6]), i.e. relations between μ, T and the curvature tensor of the connection ∇.

Theorem 1.1. *Let $(M^n, g, \nabla, \mathrm{T}, \Psi, \mu)$ be a solution of $(*)$ and assume that the spinor field Ψ is non-trivial. Then the function μ is constant and we have*

$$\|\mathrm{T}\|^2 = \mu^2 - \frac{\mathrm{Scal}^{\nabla}}{2} \geqslant 0, \qquad \mathrm{Scal}^{g} = \frac{3}{2}\mu^2 + \frac{\mathrm{Scal}^{\nabla}}{4}.$$

Moreover, the spinor field Ψ is an eigenspinor of the endomorphism defined by the 4-form dT*,*

$$\mathrm{dT}\cdot\Psi = -\frac{\mathrm{Scal}^{\nabla}}{2}\cdot\Psi.$$

Proof. Let us associate with the 3-form T the following 4-form σ_{T}:

$$\sigma_{\mathrm{T}} := \frac{1}{2}\sum_{k=1}^{n}(e_k \lrcorner \mathrm{T}) \wedge (e_k \lrcorner \mathrm{T}).$$

The square T^2 of the 3-form T in the Clifford algebra is the sum of a scalar and a 4-form (see [2]),

$$\mathrm{T}^2 - \|\mathrm{T}\|^2 = -2\cdot\sigma_{\mathrm{T}}.$$

The existence of a ∇-parallel spinor yields the following condition (see [8]):

$$3\cdot\mathrm{dT}\cdot\Psi + 2\cdot\delta(\mathrm{T})\cdot\Psi - 2\cdot\sigma_{\mathrm{T}}\cdot\Psi + \mathrm{Scal}^{\nabla}\cdot\Psi = 0.$$

Finally, there is a formula for the anti-commutator of the ∇-Dirac operator D_{T} and the endomorphism T (see [8]),

$$\mathrm{D}_{\mathrm{T}}\circ\mathrm{T} + \mathrm{T}\circ\mathrm{D}_{\mathrm{T}} = \mathrm{dT} + \delta(\mathrm{T}) - 2\cdot\sigma_{\mathrm{T}} - 2\sum_{i=1}^{n}(e_i \lrcorner \mathrm{T})\cdot\nabla_{e_i}.$$

Combining these formulae we obtain, for example,

$$\mathrm{grad}(\mu)\cdot\Psi = \mathrm{Scal}^{\nabla}\cdot\Psi + 2\cdot(\|\mathrm{T}\|^2 - \mu^2)\cdot\Psi$$

and the result follows immediately. □

Since μ has to be constant, the equation $\mathrm{T} \cdot \Psi = \mu \cdot \Psi$ yields

Corollary 1. *For all vectors X, one has*

$$(\nabla_X \mathrm{T}) \cdot \Psi = 0.$$

Corollary 2. *Assume that there exists a spinor field* $\Psi \neq 0$ *satisfying equations* $(*)$. *If* $\mu = 0$ *and* $\mathrm{Scal}^\nabla = 0$, *the torsion form* T *has to vanish.*

Proof. The inequality $\mathrm{Scal}^\nabla \leqslant 2\mu^2$ holds whenever there exists a spinor field $\Psi \neq 0$ satisfying the equations; hence $\mathrm{Scal}^\nabla = 2\mu^2$ implies $\mathrm{T} = 0$. □

Remark 1.1. This generalizes the observation (see [1]) that the existence of a non-trivial solution of $\nabla\Psi = 0$, $\mathrm{Ric}^\nabla = 0$, $\mathrm{T} \cdot \Psi = 0$ implies $\mathrm{T} = 0$ on *compact* manifolds. It underlines the strength of the algebraic identities in theorem 1.1. Note that no assumption on the full Ricci tensor is needed, only the vanishing of its trace!

Remark 1.2. In the common sector of type II string theories, the 'Bianchi identity' $\mathrm{dT} = 0$ is usually additionally required. It does not affect the mathematical structure of equations $(*)$; hence we do not include it in our discussion.

The last equation in type II string theory deals with the Ricci tensor Ric^∇ of the connection. Usually one requires for constant dilaton that the Ricci tensor has to vanish (see [11]). The result above, however, indicates that this condition may be too strong. Understanding this tensor as an energy–momentum tensor, it seems to be more convenient to impose a weaker condition, namely

$$\mathrm{div}(\mathrm{Ric}^\nabla) = 0.$$

A subtle point is, however, the fact that there are *a priori* two different divergence operators. The first operator div^g is defined by the Levi-Civita connection of the Riemannian metric, while the second operator div^∇ is defined by the connection ∇. It turns out that this difference does not play a role in the formulation of the field equation under discussion. Moreover, under the assumption that a ∇-parallel spinor exists, we can reformulate the condition $\mathrm{div}(\mathrm{Ric}^\nabla) = 0$ in such a way that only the spinor Ψ and the torsion form T are involved. The next lemma, although simple to prove, is crucial.

Lemma 1.1. *If* ∇ *is a metric connection with totally skew-symmetric torsion and S a symmetric 2-tensor, then*

$$\mathrm{div}^g(S) = \mathrm{div}^\nabla(S).$$

Proof. The difference

$$\mathrm{div}^g(S)(X) - \mathrm{div}^\nabla(S)(X) = -\frac{1}{2}\sum_{i,j=1}^{n} S(e_i, e_j)\mathrm{T}(e_i, X, e_j) = 0$$

vanishes, since S is symmetric and T is skew-symmetric. □

Theorem 1.2. *Let* $(M^n, g, \nabla, \mathrm{T}, \Psi, \mu)$ *be a solution of* $(*)$,

$$\nabla\Psi = 0, \qquad \delta(\mathrm{T}) = 0, \qquad \mathrm{T} \cdot \Psi = \mu \cdot \Psi$$

and assume that the spinor field Ψ *is non-trivial. Then the Riemannian and the* ∇*-divergence of the Ricci tensor* Ric^∇ *coincide,* $\mathrm{div}^g(\mathrm{Ric}^\nabla) = \mathrm{div}^\nabla(\mathrm{Ric}^\nabla)$. *Moreover,* $\mathrm{div}(\mathrm{Ric}^\nabla)$ *vanishes if and only if* $\delta^\nabla(\mathrm{dT}) \cdot \Psi = 0$ *holds.*

Proof. The assumption $\delta(\mathrm{T}) = 0$ implies that the Ricci tensor Ric^∇ is symmetric (see [8]). Therefore, the vectors $\mathrm{div}^g(\mathrm{Ric}^\nabla) = \mathrm{div}^\nabla(\mathrm{Ric}^\nabla)$ coincide by lemma 1.1. Any ∇-parallel spinor satisfies the condition (see [8])

$$(X \lrcorner \mathrm{dT} + 2\nabla_X \mathrm{T}) \cdot \Psi - 2\,\mathrm{Ric}^\nabla(X) \cdot \Psi = 0.$$

Since we already know that $(\nabla_X \mathrm{T}) \cdot \Psi = 0$, the condition simplifies,

$$(X \lrcorner \mathrm{dT}) \cdot \Psi - 2\,\mathrm{Ric}^\nabla(X) \cdot \Psi = 0.$$

First we differentiate this equation with respect to ∇ and compute the trace,

$$\sum_{k=1}^{n} \nabla_{e_k}(e_k \lrcorner \mathrm{dT}) \cdot \Psi - 2\sum_{k=1}^{n} \nabla_{e_k}(\mathrm{Ric}^\nabla(e_k)) \cdot \Psi = 0.$$

The latter equation is equivalent to

$$2\,\mathrm{div}^g(\mathrm{Ric}^\nabla) \cdot \Psi = 2\,\mathrm{div}^\nabla(\mathrm{Ric}^\nabla) \cdot \Psi = \delta^\nabla(\mathrm{dT}) \cdot \Psi.$$

□

Tuples $(M^n, g, \nabla, \mathrm{T}, \Psi, \mu)$ with a ∇-parallel torsion form, $\nabla\mathrm{T} = 0$, are particularly interesting. This condition implies automatically the conservation law $\delta(\mathrm{T}) = 0$. Nearly Kähler manifolds, Sasakian manifolds or nearly parallel G_2-manifolds in dimension $n = 7$, all equipped with their unique characteristic connection, are examples of metric connections with this property (see [8]). In dimension $n = 6$ we constructed several Hermitian manifolds of that type [4]. Moreover, the canonical connection of any naturally reductive space satisfies $\nabla\mathrm{T} = 0$ (see [1]). The assumption $\nabla\mathrm{T} = 0$ implies that the length $\|\mathrm{T}\|^2$ is constant. If, moreover, there exists a spinor field Ψ such that $\nabla\Psi = 0$, $\mathrm{T} \cdot \Psi = \mu \cdot \Psi$, then by theorem 1.1 the scalar curvatures Scal^g and Scal^∇ are constant. On the other hand, we use the formula

$$0 = \mathrm{d}^\nabla\mathrm{T} = \sum_{k=1}^{n} e_k \wedge \nabla_{e_k}\mathrm{T} = \sum_{k=1}^{n} e_k \wedge \nabla^g_{e_k}\mathrm{T} + \Sigma(\mathrm{T},\mathrm{T}) = \mathrm{dT} + \Sigma(\mathrm{T},\mathrm{T}),$$

where $\Sigma(\mathrm{T},\mathrm{T})$ is a quadratic expression in T. Then we conclude that

$$\nabla(\mathrm{dT}) = 0 \qquad \text{and} \qquad \delta^\nabla(\mathrm{dT}) = 0,$$

i.e., we can apply theorem 1.2.

Corollary 1.3. *Let* $(M^n, g, \nabla, \mathrm{T}, \Psi, \mu)$ *be a tuple satisfying*

$$\nabla\Psi = 0, \qquad \nabla(\mathrm{T}) = 0, \qquad \mathrm{T} \cdot \Psi = \mu \cdot \Psi$$

and assume that the spinor field Ψ *is non-trivial. Then the scalar curvatures are constant and the divergence of the Ricci tensor vanishes,* $\mathrm{div}(\mathrm{Ric}^\nabla) = 0$.

1.1. Five-dimensional examples

Let $(M^5, g, \eta, \xi, \varphi)$ be a five-dimensional quasi-Sasakian manifold. Its Nijenhuis tensor N vanishes and the fundamental form F is a closed 2-form,

$$\mathrm{N} = 0, \qquad \mathrm{dF} = 0.$$

There exists a unique connection ∇ preserving the contact structure with totally skew-symmetric torsion, the *characteristic connection* of $(M^5, g, \eta, \xi, \varphi)$. Its torsion form is given by (see [8, 9])

$$\mathrm{T} = \eta \wedge \mathrm{d}\eta.$$

If the differential $dT = d\eta \wedge d\eta$ is proportional to $F \wedge F$ with a constant factor, the characteristic connection ∇ (see [4 , 7]) of the 5-manifold solves the equation

$$\delta^{\nabla}(dT) = 0.$$

Indeed, ∇ preserves the contact structure and we conclude that under this assumption the 'volume form' $F \wedge F$ of the four-dimensional bundle consisting of all vectors in TM^5 orthogonal to ξ is ∇-parallel. In particular, $\delta^{\nabla}(dT) = 0$ holds. In general, a quasi-Sasakian 5-manifold does not have to admit any ∇-parallel spinor field. However, such examples are known and have been thoroughly investigated. Let us first consider the case of a Sasakian manifold, $d\eta = 2F$. There are Sasakian 5-manifolds admitting a ∇-parallel spinor Ψ such that the following equations are satisfied (see [8]):

$$\nabla\Psi = 0, \qquad \delta(T) = 0, \qquad T \cdot \Psi = \pm 4\Psi, \qquad \mathrm{div}^g(\mathrm{Ric}^{\nabla}) = 0.$$

The geometric data in these examples are

$$\|T\|^2 = 8, \qquad \mathrm{Scal}^{\nabla} = 16, \qquad \mathrm{Scal}^g = 28.$$

Moreover, there is a (locally) unique Sasakian 5-manifold admitting a ∇-parallel spinor field Ψ such that $T \cdot \Psi = 0$ holds (see [9]). It is the five-dimensional Heisenberg group equipped with its canonical Sasakian structure. In this case we have

$$\nabla\Psi = 0, \qquad \delta(T) = 0, \qquad T \cdot \Psi = 0, \qquad \mathrm{div}^g(\mathrm{Ric}^{\nabla}) = 0$$

and the geometric data are

$$\|T\|^2 = 8, \qquad \mathrm{Scal}^{\nabla} = -16, \qquad \mathrm{Scal}^g = -4.$$

In the paper [9], we constructed a family $M^5(a, b, c, d)$ depending on four real numbers a, b, c, d of quasi-Sasakian manifolds with ∇-parallel spinor field Ψ. In this case we have

$$\nabla\Psi = 0, \qquad \delta(T) = 0, \qquad T \cdot \Psi = \pm\sqrt{(a-d)^2 + 4b^2 + 4c^2} \cdot \Psi, \qquad \mathrm{div}^g(\mathrm{Ric}^{\nabla}) = 0$$

and the geometric data are

$$\|T\|^2 = a^2 + 2b^2 + 2c^2 + d^2, \qquad \mathrm{Scal}^{\nabla} = 4(b^2 + c^2 - ad).$$

1.2. Six-dimensional examples

Let (M^6, g, J) be a six-dimensional nearly Kähler manifold. It admits a unique connection ∇ with totally skew-symmetric torsion preserving the nearly Kähler structure (see [4, 8]), which was first investigated by Gray [12]. Moreover, there are two ∇-parallel spinor fields $\Psi^{\pm}$, and there exists a positive number a such that

$$\nabla\Psi^{\pm} = 0, \qquad \delta(T) = 0, \qquad T \cdot \Psi^{\pm} = \pm 2\sqrt{2}a\Psi^{\pm}, \qquad \mathrm{div}^g(\mathrm{Ric}^{\nabla}) = 0.$$

The geometric data are

$$\|T\|^2 = 2a, \qquad \mathrm{Scal}^{\nabla} = 12a, \qquad \mathrm{Scal}^g = 15a.$$

The Ricci tensors Ric^g and Ric^{∇} are proportional to the metric,

$$\mathrm{Ric}^g = \tfrac{5}{2}a\,\mathrm{Id}, \qquad \mathrm{Ric}^{\nabla} = 2a\,\mathrm{Id}.$$

There is another interesting example. The paper [4] contains the construction of a Hermitian 6-manifold (M^6, g, J) of type $\mathcal{W}_3$ such that its characteristic connection ∇ has a three-dimensional, complex irreducible holonomy representation $\mathrm{Hol}(\nabla) \subset \mathrm{U}(3) \subset \mathrm{SO}(6)$. There exist two ∇-parallel spinor fields $\Psi^{\pm}$ and we have

$$\nabla\Psi^{\pm} = 0, \qquad \delta(T) = 0, \qquad T \cdot \Psi^{\pm} = 0, \qquad \mathrm{div}^g(\mathrm{Ric}^{\nabla}) = 0.$$

The Ricci tensors are again proportional to the metric,

$$\mathrm{Ric}^{\nabla} = -\tfrac{1}{3}\|T\|^2\,\mathrm{Id}, \qquad \mathrm{Scal}^{\nabla} = -2\|T\|^2, \qquad \mathrm{Scal}^g = -\tfrac{1}{2}\|T\|^2.$$

1.3. Seven-dimensional examples

Let (M^7, g, ω^3) be a seven-dimensional nearly parallel G_2-manifold. The equation $d\omega^3 = -a(*\omega^3)$, $a = \text{constant} \neq 0$, characterizes this class of G_2-manifolds. The torsion form of the characteristic connection is given by the formula (see [8])

$$\mathrm{T} = -\frac{a}{6}\omega^3.$$

There always exists a ∇-parallel spinor field Ψ and we have

$$\nabla\Psi = 0, \qquad \delta(\mathrm{T}) = 0, \qquad \mathrm{T}\cdot\Psi = \tfrac{7}{6}a\Psi, \qquad \mathrm{div}^g(\mathrm{Ric}^\nabla) = 0.$$

The Ricci tensors are again proportional to the metric (see [3, 8]),

$$\mathrm{Ric}^g = \tfrac{3}{8}a^2\,\mathrm{Id}, \qquad \mathrm{Scal}^g = \tfrac{21}{8}a^2, \qquad \mathrm{Scal}^\nabla = \tfrac{7}{3}a^2, \qquad \|\mathrm{T}\|^2 = \tfrac{7}{36}a^2.$$

If the nearly parallel G_2-structure is induced by an underlying 3-Sasakian structure, we can construct a two-parameter family of torsion forms T satisfying $\nabla\mathrm{T} = 0$ and admitting parallel spinors (see [2]). Corollary 1.3 applies to this family, too.

Let us next consider cocalibrated G_2-manifolds such that the scalar product $(d\omega^3, *\omega^3)$ is constant. G_2-manifolds of that type are characterized by the conditions (see [5])

$$d*\omega^3 = 0, \qquad (d\omega^3, *\omega^3) = \text{const}.$$

The torsion form T of its characteristic connection is given by the formula (see [8])

$$\mathrm{T} = -*d\omega^3 + \tfrac{1}{6}(d\omega^3, *\omega^3)\cdot\omega^3.$$

There exists a ∇-parallel spinor field Ψ, and for any considered G_2-manifold we have

$$\nabla\Psi = 0, \qquad \delta(\mathrm{T}) = 0, \qquad \mathrm{T}\cdot\Psi = -\tfrac{1}{6}(d\omega^3, *\omega^3)\Psi.$$

The geometric data are given by (see [3, 10])

$$\mathrm{Scal}^g = -\tfrac{1}{2}\|\mathrm{T}\|^2 + \tfrac{1}{18}(d\omega^3, *\omega^3)^2, \qquad \mathrm{Scal}^\nabla = -2\|\mathrm{T}\|^2 + \tfrac{1}{18}(d\omega^3, *\omega^3)^2.$$

The Ricci tensor Ric^∇ of the characteristic connection is in general not divergence free, but both possible divergences coincide, $\mathrm{div}^g(\mathrm{Ric}^\nabla) = \mathrm{div}^\nabla(\mathrm{Ric}^\nabla)$. This vector is computable using the spinor field Ψ and the torsion form T only. On the other hand, a 3-form π^3 vanishes on the special parallel spinor Ψ ($\pi^3\cdot\Psi = 0$) if and only if the 3-form satisfies the following two algebraic equations (see [8]):

$$\pi^3 \wedge \omega^3 = 0, \qquad \pi^3 \wedge (*\omega^3) = 0.$$

This algebraic observation yields the following result.

Corollary 1.4. *Let* (M^7, g, ω^3) *be a cocalibrated* G_2*-manifold such that the scalar product* $(d\omega^3, *\omega^3)$ *is constant. Then the divergence of the Ricci tensor* Ric^∇ *vanishes if and only if*

$$\delta^\nabla(d\mathrm{T}) \wedge \omega^3 = 0, \qquad \delta^\nabla(d\mathrm{T}) \wedge (*\omega^3) = 0.$$

Example 1.1. There exist G_2-structures of pure type $\mathcal{W}_3$ in the Fernandez/Gray classification (see [5]) on the product of $\mathbb{R}^1$ by the six-dimensional Heisenberg group and on the product of $\mathbb{R}^1$ by the three-dimensional complex solvable Lie group. The torsion form of its characteristic connection has been investigated in the paper [8]. Using these formulae, one computes directly that these examples satisfy the conditions $\delta^\nabla(d\mathrm{T}) \wedge \omega^3 = 0$, $\delta^\nabla(d\mathrm{T}) \wedge (*\omega^3) = 0$, but $\delta^\nabla(d\mathrm{T}) \neq 0$.

2. Type II B string theory with a dilaton function

In the first part of the paper, we discussed the Ricci tensor in the model of type II B string theory. In fact, the model is much more flexible; it contains an additional function Φ. In the second part of the paper, we study the corresponding results in this more general situation. We use basically the same arguments (although computationally more involved) as in the proofs of theorems 1.1 and 1.2; hence we shall not repeat them all. Again, the integrability conditions following from $\nabla\Psi = 0$ as derived in [8, 10] are the key ingredient.

The equations now read

$$(**) \qquad \nabla\Psi = 0, \qquad \delta(\mathrm{T}) = a \cdot (\mathrm{d}\Phi \lrcorner \mathrm{T}), \qquad \mathrm{T}\cdot\Psi = b\cdot \mathrm{d}\Phi\cdot\Psi + \mu\cdot\Psi.$$

Usually the constant a has a precise value, namely $a = \pm 2$. In order to understand the role of the parameters in the equations, we slightly generalized them and allow for two arbitrary parameters a, b.

Theorem 2.1. *Let $(M^n, g, \nabla, \mathrm{T}, \Psi, \Phi, \mu, a)$ be a tuple satisfying $(**)$ and assume that the spinor field Ψ is non-trivial. Then*

$$(b-a)\cdot\delta(\mathrm{T})\cdot\Psi = 0, \qquad \mathrm{dT}\cdot\Psi = -\frac{\mathrm{Scal}^{\nabla}}{2}\cdot\Psi - \frac{b}{2}\Delta(\Phi)\cdot\Psi,$$

$$\|\mathrm{T}\|^2 = \mu^2 - \frac{\mathrm{Scal}^{\nabla}}{2} + b^2\|\mathrm{d}\Phi\|^2 - \frac{3b}{2}\Delta(\Phi),$$

and the Riemannian scalar curvature is given by the formula

$$\mathrm{Scal}^g = \frac{3}{2}\mu^2 + \frac{3b^2}{2}\|\mathrm{d}\Phi\|^2 + \frac{\mathrm{Scal}^{\nabla}}{4} - \frac{9b}{4}\Delta(\Phi).$$

In particular, if $b \neq a$, we obtain $\delta(\mathrm{T})\cdot\Psi = 0$. In this case, the endomorphism T^2 acts on the spinor by scalar multiplication,

$$\mathrm{T}^2\Psi = (b^2\|\mathrm{d}\Phi\|^2 + \mu^2)\cdot\Psi.$$

The differential $\mathrm{d}\Phi$ of the dilaton Φ is a 1-form. Its differentials $\nabla^g\mathrm{d}\Phi, \nabla\mathrm{d}\Phi$ with respect to the Levi-Civita connection ∇^g and with respect to the connection ∇, respectively, are bilinear forms. Since the Levi-Civita connection is torsion free, $\nabla^g\mathrm{d}\Phi$ is symmetric, $\nabla^g\mathrm{d}\Phi(X, Y) = \nabla^g\mathrm{d}\Phi(Y, X)$. By lemma 1.1, one has

$$\mathrm{div}^g(\nabla^g\mathrm{d}\Phi) = \mathrm{div}^{\nabla}(\nabla^g\mathrm{d}\Phi).$$

The difference between the two bilinear forms is given by the torsion,

$$\nabla\mathrm{d}\Phi = \nabla^g\mathrm{d}\Phi - \tfrac{1}{2}\cdot(\mathrm{d}\Phi\lrcorner\mathrm{T}).$$

Now we generalize theorem 1.2.

Theorem 2.2. *Let $(M^n, g, \nabla, \mathrm{T}, \Psi, \Phi, a, b, \mu)$ be a tuple satisfying $(**)$. Then we have*

$$2\,\mathrm{div}^g(\mathrm{Ric}^{\nabla} - b\cdot\nabla^g\mathrm{d}\Phi)\cdot\Psi = \delta^{\nabla}(\mathrm{dT})\cdot\Psi + (a-b)\cdot\delta^{\nabla}(\mathrm{d}\Phi\lrcorner\mathrm{T})\cdot\Psi$$
$$2\,\mathrm{div}^{\nabla}(\mathrm{Ric}^{\nabla} - b\cdot\nabla^g\mathrm{d}\Phi)\cdot\Psi = \delta^{\nabla}(\mathrm{dT})\cdot\Psi - b\cdot\delta^{\nabla}(\mathrm{d}\Phi\lrcorner\mathrm{T})\cdot\Psi,$$
$$2\,\mathrm{div}^g(\mathrm{Ric}^{\nabla} - b\cdot\nabla\mathrm{d}\Phi)\cdot\Psi = \delta^{\nabla}(\mathrm{dT})\cdot\Psi + (a-b)\cdot\delta^{\nabla}(\mathrm{d}\Phi\lrcorner\mathrm{T})\cdot\Psi,$$
$$2\,\mathrm{div}^{\nabla}(\mathrm{Ric}^{\nabla} - b\cdot\nabla\mathrm{d}\Phi)\cdot\Psi = \delta^{\nabla}(\mathrm{dT})\cdot\Psi.$$

In particular, the differences are given by ($\lambda \in \mathbb{R}$ is an arbitrary parameter)

$$2(\mathrm{div}^g - \mathrm{div}^{\nabla})(\mathrm{Ric}^{\nabla} - \lambda\cdot\nabla^g\mathrm{d}\Phi) = a\cdot\delta^{\nabla}(\mathrm{d}\Phi\lrcorner\mathrm{T}) = \delta^{\nabla}\delta(\mathrm{T}),$$
$$2(\mathrm{div}^g - \mathrm{div}^{\nabla})(\mathrm{Ric}^{\nabla} - b\cdot\nabla\mathrm{d}\Phi) = (a-b)\cdot\delta^{\nabla}(\mathrm{d}\Phi\lrcorner\mathrm{T}).$$

Proof. $2\,\mathrm{Ric}^\nabla - 2\lambda\nabla^g d\Phi + \delta(T)$ is a symmetric tensor. Hence, lemma 1.1 yields

$$2(\mathrm{div}^g - \mathrm{div}^\nabla)(\mathrm{Ric}^\nabla - \lambda\cdot\nabla^g d\Phi) = \delta^\nabla\delta(T) - \delta^g\delta^g(T) = \delta^\nabla\delta(T).$$

$2\,\mathrm{Ric}^\nabla + \delta(T) - 2b\nabla d\Phi - b(d\Phi \lrcorner T)$ is symmetric, too. Consequently,

$$\begin{aligned}2(\mathrm{div}^g - \mathrm{div}^\nabla)(\mathrm{Ric}^\nabla - b\cdot\nabla d\Phi) &= \delta^\nabla\delta(T) - b\delta^\nabla(d\Phi \lrcorner T) - \delta^g\delta^g(T) + b\delta^g(d\Phi \lrcorner T)\\ &= (a-b)\cdot\delta^\nabla(d\Phi \lrcorner T).\end{aligned}$$

Here we used once again the equation $\delta(T) = a\cdot(d\Phi \lrcorner T)$. The equation $T\cdot\Psi = b\cdot d\Phi\cdot\Psi + \mu\cdot\Psi$ yields

$$(\nabla_X T)\cdot\Psi = b\cdot(\nabla_X d\Phi)\cdot\Psi.$$

Next we differentiate the integrability condition

$$\big(X \lrcorner dT + 2\nabla_X T\big)\cdot\Psi - 2\,\mathrm{Ric}^\nabla(X)\cdot\Psi = 0$$

and proceed as in the proof of theorem 1.2. The result is a similar one,

$$2\,\mathrm{div}^\nabla(\mathrm{Ric}^\nabla - b\cdot\nabla d\Phi)\cdot\Psi = \delta^\nabla(dT)\cdot\Psi.$$

Finally, we have

$$\begin{aligned}2\,\mathrm{div}^\nabla(\mathrm{Ric}^\nabla - b\cdot\nabla^g d\Phi)\cdot\Psi &= 2\,\mathrm{div}^\nabla(\mathrm{Ric}^\nabla - b\cdot\nabla d\Phi)\cdot\Psi + 2b\,\mathrm{div}^\nabla(\nabla d\Phi - \nabla^g d\Phi)\cdot\Psi\\ &= \delta^\nabla(dT)\cdot\Psi - b\cdot\delta^\nabla(d\Phi \lrcorner T)\cdot\Psi.\end{aligned}$$

The remaining formulae now follow directly from what has already been shown. □

Using the equation $\delta(T) = a\cdot(d\Phi \lrcorner T)$, the formula $\delta^g\delta(T) = \delta^g\delta^g(T) = 0$ as well as the formulae comparing δ^g and δ^∇ on differential forms (see [2]), we compute that the 1-form $\delta^\nabla\delta(T)$ is proportional to the 1-form $(d\Phi \lrcorner T) \lrcorner T$. Consequently, we obtain a necessary and sufficient algebraic condition under which the different divergences coincide.

Corollary 2.1. *If, in addition,* $(d\Phi \lrcorner T) \lrcorner T = 0$ *holds, then the following divergences coincide:*

$$\begin{aligned}&\mathrm{div}^g(\mathrm{Ric}^\nabla - \lambda\cdot\nabla^g d\Phi) = \mathrm{div}^\nabla(\mathrm{Ric}^\nabla - \lambda\cdot\nabla^g d\Phi) \qquad \textit{for any} \quad \lambda\in\mathbb{R},\\ &\mathrm{div}^g(\mathrm{Ric}^\nabla - b\cdot\nabla d\Phi) = \mathrm{div}^\nabla(\mathrm{Ric}^\nabla - b\cdot\nabla d\Phi).\end{aligned}$$

Corollary 2.2. *If* $(d\Phi \lrcorner T) \lrcorner T = 0$ *and* $\delta^\nabla(dT)\cdot\Psi = 0$ *hold, then all the divergences vanish.*

The previous discussion shows that the case of $a = b$ is a special one. Normalizing the constants, we assume that $a = b = -2$. Then equations $(**)$ read

$$\nabla\Psi = 0, \qquad \delta(T) = -2\cdot(d\Phi \lrcorner T), \qquad T\cdot\Psi = -2\cdot d\Phi\cdot\Psi + \mu\cdot\Psi.$$

In this case, the condition $\delta(T)\cdot\Psi = 0$ is *not* an integrability condition and the correct formula for $T^2\cdot\Psi$ is different,

$$T^2\cdot\Psi = (4\|d\Phi\|^2 + \mu^2)\cdot\Psi - 2\cdot\delta(T)\cdot\Psi.$$

The divergence of the energy–momentum tensor is given by

$$2\,\mathrm{div}^g(\mathrm{Ric}^\nabla + 2\cdot\nabla^g d\Phi)\cdot\Psi = \delta^\nabla(dT)\cdot\Psi.$$

In strong models ($dT = 0$) the divergence of the energy–momentum tensor vanishes, but there are models with $\delta^\nabla(dT) \neq 0$ and $\delta^\nabla(dT)\cdot\Psi = 0$. The last example in section 1.3 is one of them.

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