Department of Mathematics
Maths 260 *Differential Equations*

An introduction to software used in Maths 260

This handout provides information about using the software package Matlab to investigate the behaviour of solutions to differential equations. Matlab allows you to enter your own differential equations; it will plot slope fields and find numerical approximations to solutions of your equations.

The course will make use of various functions in addition to the standard Matlab package. These functions are called: *analyzer*, *dfield*, *numerical* and *pplane*. Before you can use these functions you must download copies of the files from the course website:

`http://www.math.auckland.ac.nz/wiki/MATHS_260_Semester_2_2010_Website`

and put them in your own directory. Instructions on how to do this are given on the handout "Installing the software needed for Maths 260".

To use one of these functions, move into the directory containing the function files and open Matlab. Then type the name of the function you wish to use. For example, to start *analyzer*, type

```
analyzer
```

at the command prompt.

The purposes of the functions we will use in this course can be summarised as follows:

- *analyzer* will plot a function that you specify. You will use this function when you have already solved a differential equation and you want to see what the solution looks like.
- *dfield* computes and plots approximate (numerical) solutions to a differential equation that you provide. You will use this function when you want to find out about solutions to a differential equation when you cannot (or do not want to) solve the differential equation analytically.
- *numerical* calculates numerical approximations to a solution of a differential equation. You will use this function when you want to investigate details of the numerical methods (for instance, the effect of using different step-sizes or different methods).
- *pplane* computes and plots approximate (numerical) solutions to a system of two, first order differential equations that you provide. You will use this function when you want to find out about solutions to a system of differential equations when you cannot (or do not want to) solve the system of equations analytically.

An easy way to learn how to use these functions is to work through the following exercises.

1 analyzer

A simple way to plot the graph of one or more functions is to use to function *analyzer*.

Exercise 1: Use *analyzer* to draw graphs (all on one set of axes) of the function

$$x(t) = c_1 \sin t + c_2 t$$

for $-8 < t < 8$, and with the following values of c_1 and c_2: (a) $c_1 = 1$, $c_2 = 0$; (b) $c_1 = 0$, $c_2 = 1$; (c) $c_1 = 1$, $c_2 = 1$; (d) $c_1 = 0.8$, $c_2 = 0.2$. Print out the graphs.

Procedure

1. Start *analyzer*. The choice $c_1 = 1$, $c_2 = 0$ gives $x(t) = \sin t$. To plot the graph corresponding to this choice, enter the expression `sin(t)` into the box after '`x(t)=`'

   Note: You need to be careful with the way you type your functions. For example, to input the equation $x(t) = \exp(t^2) + \dfrac{3}{2t}$, in the box after '`x(t)=`' enter

   ```
   exp(t^2)+3/(2*t)
   ```

   Remember to put the symbol * between 2 and t to represent the multiplication!

2. In the "Axis limits" box, set 'lower t' to -8 and 'upper t' to 8, as specified above. You will also usually need to choose your own limits on the x values, but the default values are fine for this example.

3. Click $\boxed{\text{Proceed}}$

4. To plot the next graph, return to the *ANALYZER PLOT DETAILS* window and enter the next equation in place of the old.

5. Again click $\boxed{\text{Proceed}}$. Repeat this process for the remaining choices of c_1 and c_2 asked for in Exercise 1.

6. If you wish to give the graph a title, enter the following into the *MATLAB* command window:

   ```
   title('Exercise 1: graph of x(t)')
   ```

7. To print the graph, choose *Print...* in *Menu* from the *ANALYZER PLOT* window.

8. To quit *analyzer*, click $\boxed{\text{Quit}}$ in the *ANALYZER PLOT DETAILS* window.

Notes:

In the ANALYZER PLOT DETAILS window

i. Uncheck the “Overlay on current plot” box if you want to plot each new graph in a new plot window.

ii. Type different numbers in the “No. of points to be plotted” box to change the accuracy of the plot. The greater the number of points, the better the graph looks but it takes a longer time to plot.

In the ANALYZER PLOT window

iii. To take a copy of the graph, choose *Copy to clipboard* from *Menu*. This takes the current plot window as an image which you can then paste into Microsoft Word, or any other program.

iv. You can put the grid on the current plot window by clicking the “Grid on/off” menu item.

v. You can zoom in by first choosing *Zoom in* from the “Zoom” menu then left-clicking on the plot area, or click and drag to set an area to be zoomed into. You can zoom out by right-clicking on the **zoomed in** plot area, or by selecting *Zoom to original size* from the “Zoom” menu.

2 dfield

In the previous example, we used *analyzer* to plot graphs of functions; these functions might be solutions to a differential equation that we had already calculated analytically. In contrast, *dfield* can be used to compute numerical approximations to solutions of a differential equation, and then to plot these approximate solutions.

Exercise 2: Use *dfield* to draw a direction field for the differential equation

$$\frac{dx}{dt} = 0.5xt$$

and plot the solution which satisfies the initial condition

$$x(2) = 5$$

for $-5 < t < 5$ and $-10 < x < 10$. Print the output.

Procedure

1. Start *dfield*. Input `0.5*x*t` after ‘dx/dt=’ in ‘The differential equation’ box.

2. Set the minimum values of x and t to the values specified above. Click $\boxed{\text{Proceed}}$.

3. There are two ways to find the particular solution.

   - Mouse click To draw the particular solution requested, click on the point (2,5) in the $t - x$ plane. The solution through that point will be drawn. Watch the text in the lower left of the *PLOT* window to see if you have got the right starting point.
   - Keyboard input In the 'dfield Display' window choose *Keyboard Input* from *Options*. A small window will appear. Enter in the initial condition $(t, x) = (2, 5)$ and click $\boxed{\text{Compute}}$

   If you are not satisfied with your solution, choose *Erase all solutions* from the *Edit* menu and try again.

4. Plot some other solutions to the differential equation by clicking on other initial condition points of your choice in the $t - x$ plane.

5. To print the graph, choose *Print ...* from *Menu*.

Notes:

i. The solution to our differential equation is of the following form:

$$x(t) = k \exp(t^2/4)$$

for $-5 < t < 5$ and $-10 < x < 10$ for various values of k. We can plot this equation using *analyzer* with values of $k = \pm1, \pm0.5, 0.01$, and see if they match our result from *dfield* above.

ii. To see which numerical method has been used to plot the solution or to alter the numerical method, choose *Solver* from the *Options* menu in the 'dfield Display' window. To see which step-size has been used or to alter the step-size, choose *Settings* from the *Options* menu. More information will be given about these items later in the course.

3 numerical

The *numerical* tool can be used to calculate an approximate solution to an initial value problem using one of three numerical methods: Euler's method, Improved Euler's Method, and the 4th order Runge-Kutta method. It will calculate the approximate value of the solution at a specified point for various choices of step-size. If you provide a final value for the exact solution, the program will also calculate the error in each approximate solution and the effective order of the numerical method.

Exercise 3: Use *numerical* to calculate an approximate solution to the initial value problem (IVP)

$$\frac{dy}{dt} = 2ty, \quad y(1) = 2.5$$

at $t = 2$, using the Improved Euler method. What is the effective order of the method at step-size $h = 0.125$?

Procedure

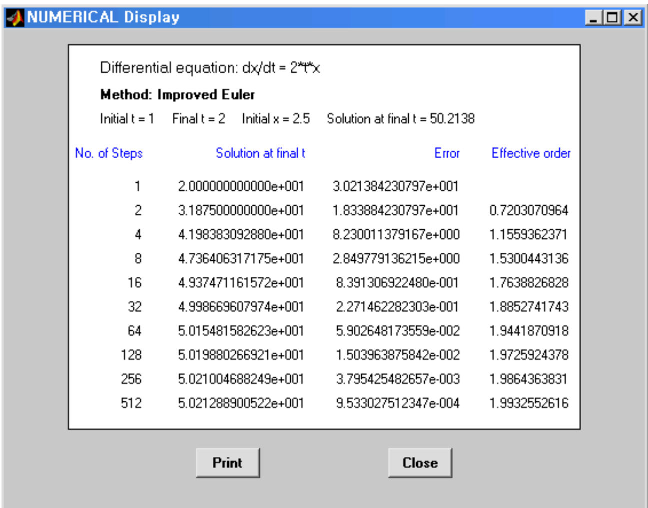
1. Start *numerical*. When the dialogue box appears, enter: `2*t*x` after 'dx/dt='. (Note: you will have to use x in place of y for this example).

2. Enter the initial and end values of t (i.e., 1 and 2 respectively) and the initial value of x (i.e., 2.5).

3. The formula for the exact solution to the IVP is $x(t) = 2.5\exp(t^2 - 1)$. (You can obtain this solution by the method of separation of variables). Thus, the exact value of x at $t = 2$ is $x(2) = 2.5\exp((2)^2 - 1) = 2.5\exp(3) = 50.2138$. Type $2.5 * \exp(3)$ in the box labelled 'solution at final t'.

4. Choose the numerical method Improved Euler. A step-size of 0.125 means that to get from $t = 1$ to $t = 2$, 8 steps are needed (i.e., $(2-1)/0.125 = 8$). Click the box next to 'Number of steps ...' and select $\boxed{\text{3 (8 steps)}}$

5. Click $\boxed{\text{Proceed}}$. You should obtain the output shown below.



NUMERICAL Display

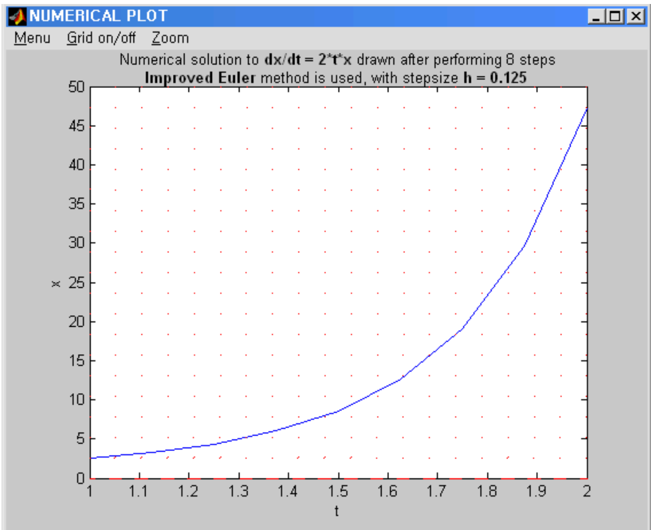
Differential equation: dx/dt = 2*t*x

Method: Improved Euler

Initial t = 1 Final t = 2 Initial x = 2.5 Solution at final t = 50.2138

No. of Steps	Solution at final t	Error	Effective order
1	2.000000000000e+001	3.021384230797e+001	
2	3.187500000000e+001	1.833884230797e+001	0.7203070964
4	4.198383092880e+001	8.230011379167e+000	1.1559362371
8	4.736406317175e+001	2.849779136215e+000	1.5300443136
16	4.937471161572e+001	8.391306922480e-001	1.7638826828
32	4.998669607974e+001	2.271462282303e-001	1.8852741743
64	5.015481582623e+001	5.902648173559e-002	1.9441870918
128	5.019880266921e+001	1.503963875842e-002	1.9725924378
256	5.021004688249e+001	3.795425482657e-003	1.9864363831
512	5.021288900522e+001	9.533027512347e-004	1.9932552616

Print Close



6. From the output, we see that the approximate solution at $t = 2$ using 8 steps is 47.36 with an approximate error of 2.8. The effective order at this step-size is given as approximately 1.5, i.e., halving the step-size reduces the error by about $2^{1.5}$

Notes:

i. To take a copy of the graph in the NUMERICAL PLOT window, choose *Copy to clipboard* from *Menu*. This takes the current plot window as an image which you can then paste into Microsoft Word, or other programs.

1. You can enable/disable plotting of the direction field in the NUMERICAL PLOT window by checking or unchecking the "Draw field lines" box. You can also specify the number of direction field line segments across the page (default is 20).

ii. You can put the grid on the current plot window by clicking the "Grid on/off" menu item.

iii. You can zoom in by first choosing *Zoom in* from the "Zoom" menu then left-clicking on the plot area, or click and drag to set an area to be zoomed into. You can zoom out by right-clicking on the **zoomed in** plot area, or by selecting *Zoom to original size* from the "Zoom" menu.

4 pplane

The function *pplane* can be used to investigate the behaviour of solutions to systems of two, first order differential equations. It will compute numerical approximations to solutions of a system of differential equation and plot these approximate solutions in the phase plane.

Exercise 4: Use *pplane* to draw the slope field for the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + y), \\ \frac{dy}{dt} &= y(2 - x - 3y). \end{aligned}$$

Plot the phase portrait for the system of equations including the solution that passes through the point $(x, y) = (1, 2)$.

Procedure

1. Open *pplane* and enter the equations in the dialogue box.

2. Choose some appropriate values for the ranges of x and y by altering the entries in the appropriate boxes near the bottom of the window. For this example, suitable ranges are $x_{min} = -3, x_{max} = 3$, $y_{min} = -3, y_{max} = 3$.

3. Click $\boxed{\text{Proceed}}$.

4. To draw the solution that passes through the point $(x, y) = (1, 2)$ there are two options:

- Mouse click Move the mouse until the arrowhead is at the point $(1, 2)$, then click the left mouse button. Use the grid lines, and watch the text in the lower left of the window, to help you locate the point you wish to click.
- Keyboard input Alternatively, you can select *Keyboard Input* from the *Solutions* menu, and enter in $(x, y) = (1, 2)$.

5. Plot some other solutions to the differential equation by clicking on other points of your choice in the slope field.

6. To erase all the solutions you have drawn so far, select *Erase all solutions* from the *Edit* menu. If you want to erase just one solution, select *Delete a graphics object* from the *Edit* menu, and click on the solution curve you wish to remove.

7. You should be able to estimate where the equilibrium points for the system are by looking for places where solutions move very slowly or places where the slope marks change direction dramatically. For example, there seems to be an equilibrium point near $(x, y) = (1.25, 0.25)$. To determine the position of this equilibrium, first choose *Find an equilibrium point* from the *Solutions* menu then click in the vicinity of $(1.25, 0.25)$; a red circle will appear at the equilibrium point and the values of x and y will be reported in a pop-up window.

8. To print the graph, choose *Print...* in *Menu*, from the plot window.

Notes:

i. The solution curve is usually plotted in two parts: first by starting at the initial point and letting time increase, then by starting at the initial point and letting time decrease. You should watch carefully to see which direction along the solution curve corresponds to increasing time, because when you have printed out your completed phase portrait you will need to draw an arrow on each solution curve to indicate the direction of increasing time.

ii. To see which numerical method has been used to plot the solution or to alter the numerical method, choose *Solver* from the *Options* menu. To see which step-size has been used or to alter the step-size, choose *Settings* from the *Options* menu. For this example, you will need to use a step-size of less than 0.05 with the Runge-Kutta method to get a nice phase portrait. Numerical methods for systems of equations will be discussed in lectures.