Department of Mathematics

Maths 260 Assignment 3 Solutions

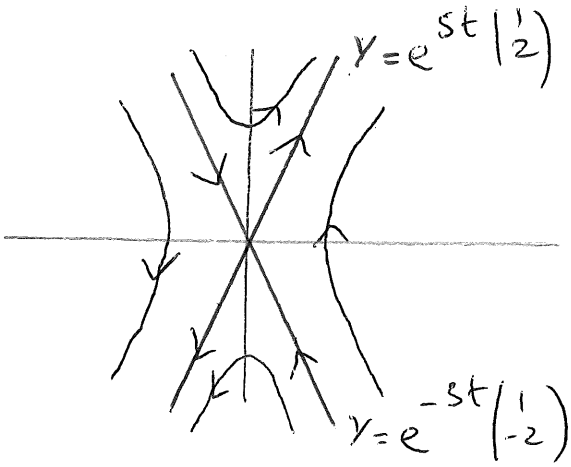
October 7, 2010 Due: 4pm, Tuesday 28th September, 2010

1. Question 1 solutions (24 marks total)

(a) i. Eigenvalues and corresponding eigenvectors,

$\lambda_1 = 5$, $\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_2 = -3$, $\mathbf{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

One positive and one negative eigenvalue, hence a saddle point at the $(0,0)$



ii. Eigenvalues are,

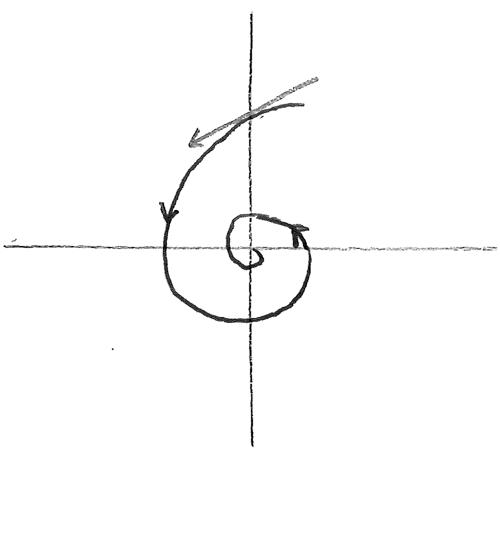
$\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$

Negative real part so a spiral sink at $(0,0)$

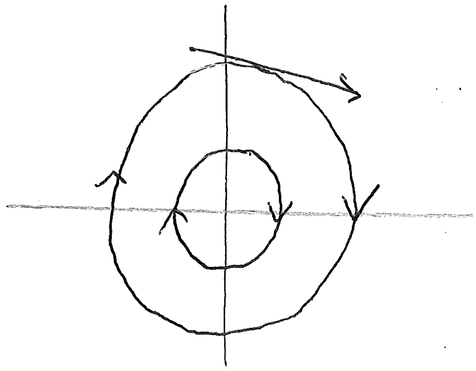
Try a point at $(0,1)$ to find direction of spiral,

$\mathbf{Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $\mathbf{AY} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

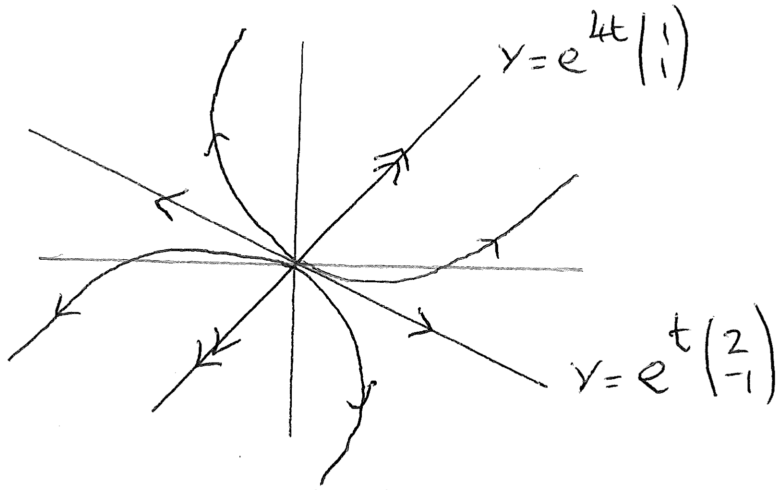
At $(0,1)$ $\dot{x} = -4$ hence spiral is anti-clockwise.



iii. Eigenvalues are, $\lambda_1 = \sqrt{5}i$, $\lambda_2 = -\sqrt{5}i$ No real part, a spiral centre.
Try a point at $(0,1)$ to find direction of spiral centre,
$\mathbf{Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $\mathbf{AY} = \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$
At $(0,1)$ $\dot{x} = 3$ hence spiral centre is clockwise.



iv. Eigenvalues and corresponding eigenvector are,
$\lambda_1 = 1$, $\mathbf{v_1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\lambda_2 = 4$, $\mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Two positive eigenvalues hence a source at $(0,0)$.



(b) i. General solution, $\mathbf{Y}(t) = c_1 e^t \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ii. Solving IVP $\mathbf{Y}(0) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
Resulting in, $2 = 2c_1 + c_2$ and $3 = c_2 - c_1$
Solving gives, $c_1 = -1/3$, $c_2 = 8/3$
$\mathbf{Y}(t) = e^t \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix} + e^{4t}\begin{pmatrix} 8/3 \\ 8/3 \end{pmatrix}$

iii. As $t \to \infty$, $x(t) \to \infty$ and $y(t) \to \infty$.

2. Question 2 (9 marks total)

(a) Using $\mathbf{A}.\mathbf{v} = \lambda\mathbf{v}$

$$\mathbf{A}.\mathbf{v} = \begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+2i \\ 1 \end{pmatrix} = \begin{pmatrix} -2+6i \\ 2+2i \end{pmatrix} = (2+2i)\begin{pmatrix} \frac{-2+6i}{2+2i} \\ 1 \end{pmatrix}$$

and

$$\frac{-2+6i}{2+2i} = \frac{(-2+6i)(2-2i)}{(2+2i)(2-2i)} = \frac{8+16i}{8} = 1+2i$$

Therefore

$$\mathbf{A}.\mathbf{v} = (2+2i)\begin{pmatrix} 1+2i \\ 1 \end{pmatrix} = \lambda\mathbf{v}$$

So the eigenvalue, λ, is $2+2i$.

(b) Solution

$$\mathbf{Y}(t) = e^{(2+2i)t}\begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$$

$$\begin{aligned} e^{(2+2i)t}\begin{pmatrix} 1+2i \\ 1 \end{pmatrix} &= e^{2t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1+2i \\ 1 \end{pmatrix} \\ &= e^{2t}\begin{pmatrix} \cos 2t + i\sin 2t + 2i\cos 2t - 2\sin 2t \\ \cos 2t + i\sin 2t \end{pmatrix} \\ &= e^{2t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} + ie^{2t}\begin{pmatrix} \sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix} \end{aligned}$$

(c) General solution,

$$\mathbf{Y}(t) = c_1 e^{2t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix}$$

(d) Solving the IVP,

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = c_1 e^0\begin{pmatrix} \cos 0 - 2\sin 0 \\ \cos 0 \end{pmatrix} + c_2 e^0\begin{pmatrix} \sin 0 + 2\cos 0 \\ \sin 0 \end{pmatrix}$$

therefore

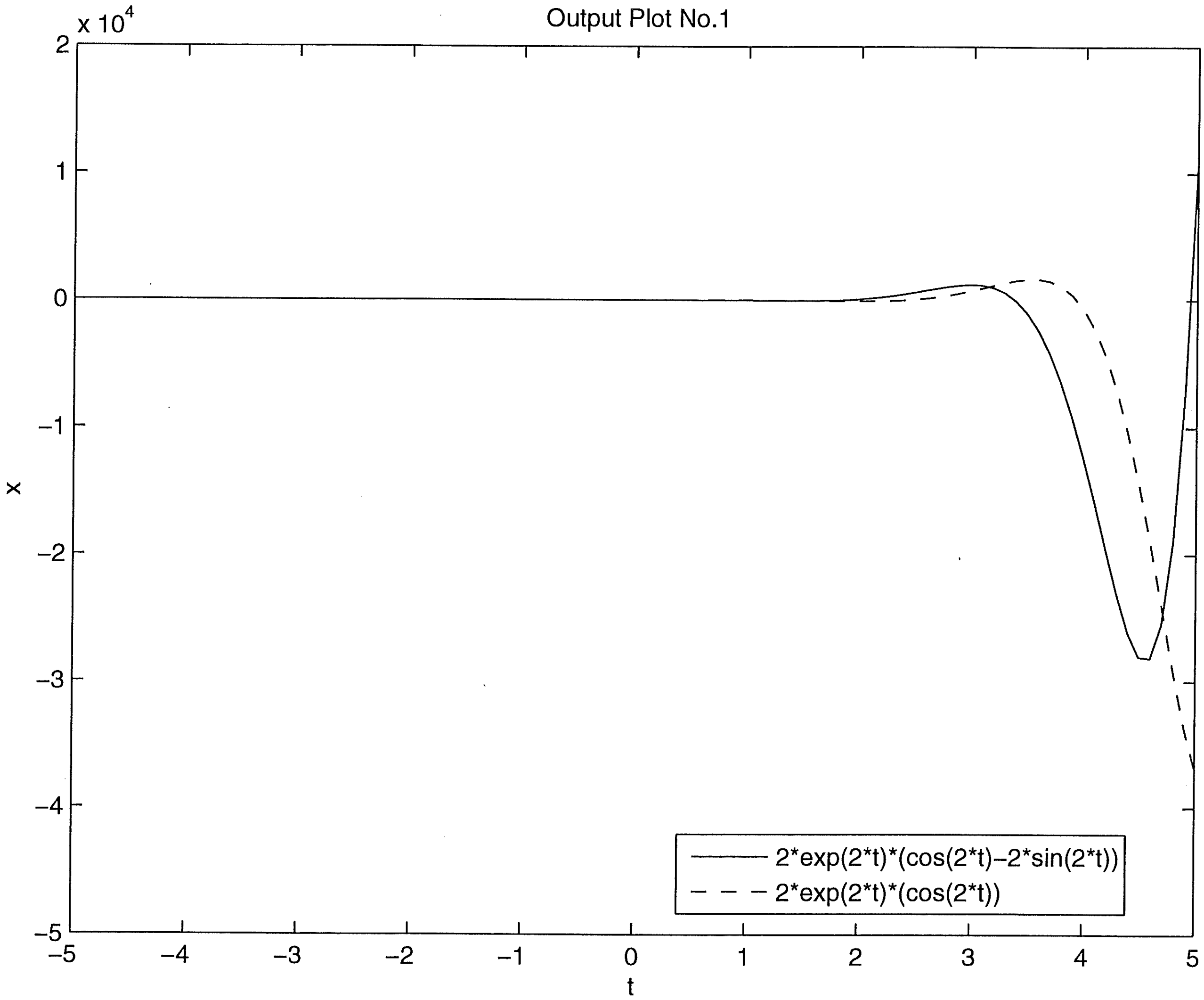
$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ 0 \end{pmatrix}$$

$c_1 = 2$ and $c_2 = 0$.
Solution to the IVP is

$$\mathbf{Y}(t) = 2e^{2t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix}$$

(e) See plot attached

(f) As $t \to \infty$ both x and y oscillate between $+\infty$ and $-\infty$.



Output Plot No.1
x 10^4
2
1
0
−1
−2
−3
−4
−5
x
−5
−4
−3
−2
−1
0
1
2
3
4
5
t
2*exp(2*t)*(cos(2*t)−2*sin(2*t))
2*exp(2*t)*(cos(2*t))

3. Question 3 (11 marks total)

(a) $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{pmatrix} \mathbf{Y}$

(b) Eigenvalues and corresponding eigenvectors are

$\lambda_1 = 1$, $\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 2 + i$, $\mathbf{v_2} = \begin{pmatrix} 0 \\ -1 - i \\ 2 \end{pmatrix}$

$\lambda_3 = 2 - i$, $\mathbf{v_3} = \begin{pmatrix} 0 \\ -1 + i \\ 2 \end{pmatrix}$

$$\mathbf{Y_1}(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{Y_2} = e^{(2+i)t} \begin{pmatrix} 0 \\ -1 - i \\ 2 \end{pmatrix} \text{ and } \mathbf{Y_3} = e^{(2-i)t} \begin{pmatrix} 0 \\ -1 + i \\ 2 \end{pmatrix}$$

Writing $\mathbf{Y_2}$ in terms of real valued functions.

$$\begin{aligned} e^{(2+i)t} \begin{pmatrix} 0 \\ -1 - i \\ 2 \end{pmatrix} &= e^{2t}(\cos t + i \sin t) \begin{pmatrix} 0 \\ -1 - i \\ 2 \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} 0 \\ -\cos t + \sin t \\ 2\cos t \end{pmatrix} + ie^{2t} \begin{pmatrix} 0 \\ -\cos t - \sin t \\ 2\sin t \end{pmatrix} \end{aligned}$$

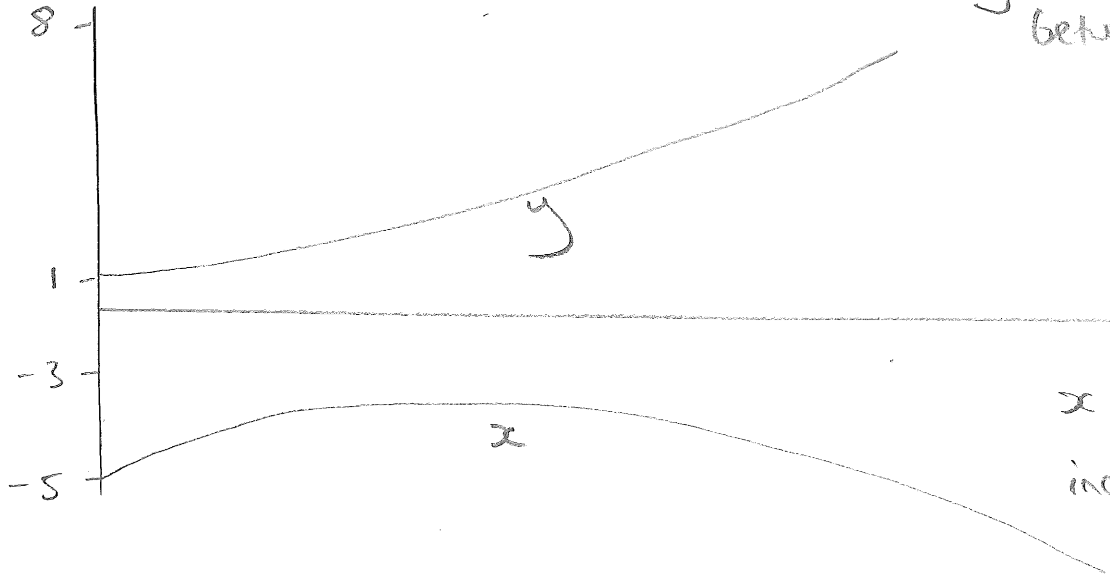
You could do the same thing here with $\mathbf{Y_3}$ with the same result.

Therefore $\mathbf{Y}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ -\cos t + \sin t \\ 2\cos t \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ -\cos t - \sin t \\ 2\sin t \end{pmatrix}$

(c) All eigenvalues have positive real part, hence a source.

Question 4 solutions

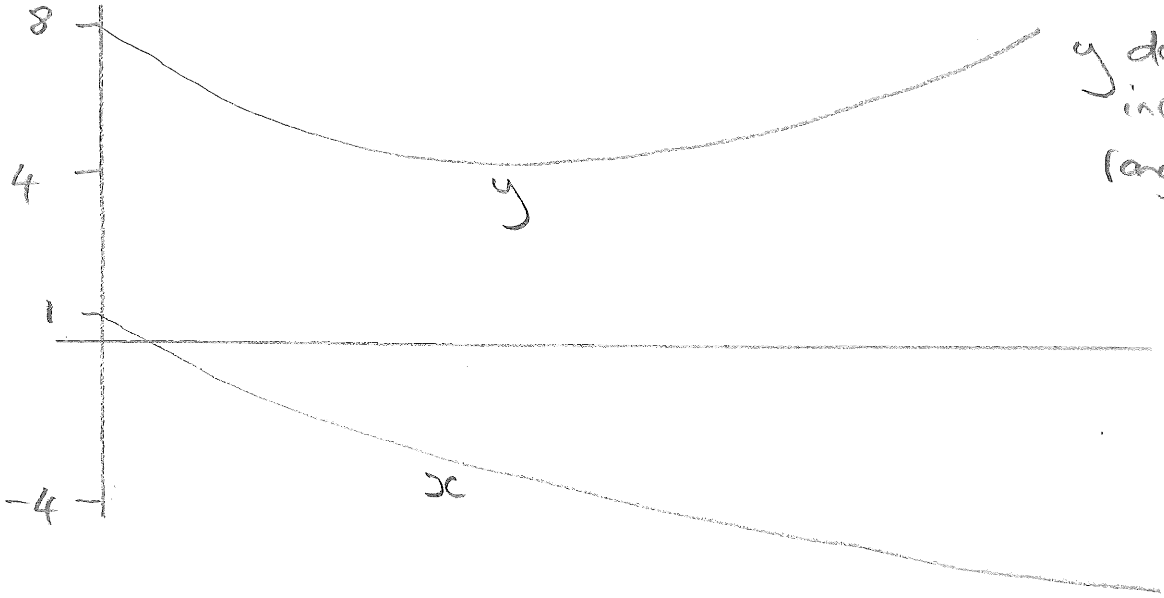
i)



y increases monotonically between [1, 8]

x between [-5, -3] increases then decreases

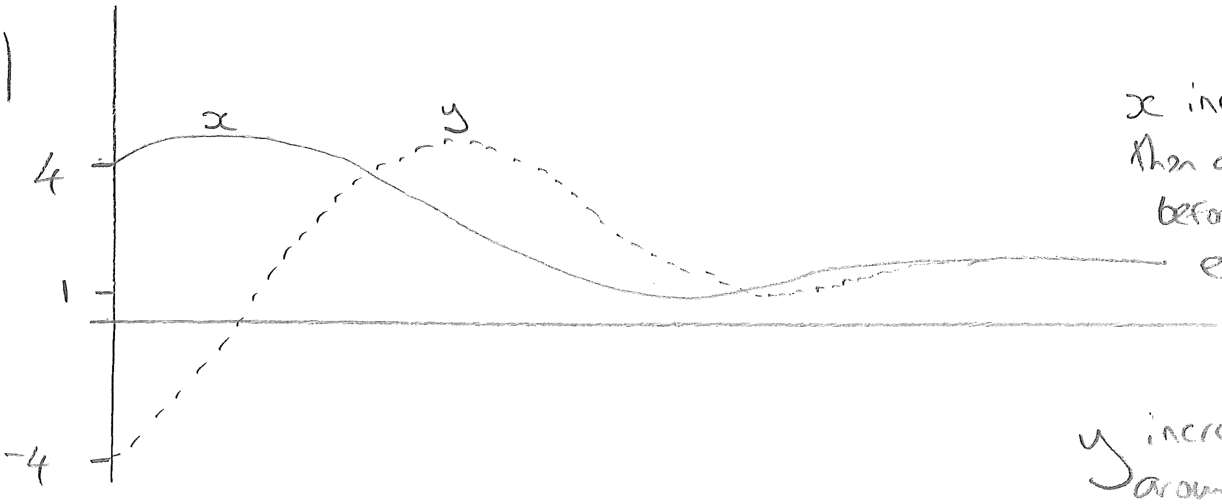
ii)



y decreases then increases in range [4, 8]

x decreases in range [-4, 1]

iii)



x increases slightly then decreases before increasing to equilibrium at 1

y increases to around $y = 4$ then decreases before increasing to equilibrium at $y = 1$

b) Spiral sink at (1, 1)
Saddle at (-2, 4)

5. Question 5 (13 marks total)

(a) Euler's method is:

$$x_{j+1} = x_j + h\dot{x}(x_j, y_j, t_j)$$
$$y_{j+1} = y_j + h\dot{y}(x_j, y_j, t_j)$$

With $h = 1$ we get

j	t_j	x_j	y_j	$\dot{x}(x_j, y_j, t_j)$	$\dot{y}(x_j, y_j, t_j)$
0	0	1	1	0.5	0
1	1	1.5	1	-1	0.5
2	2	0.5	1.5	-0.25	-0.75
3	3	0.25	0.75		

(b) Solution plotted in figure below.

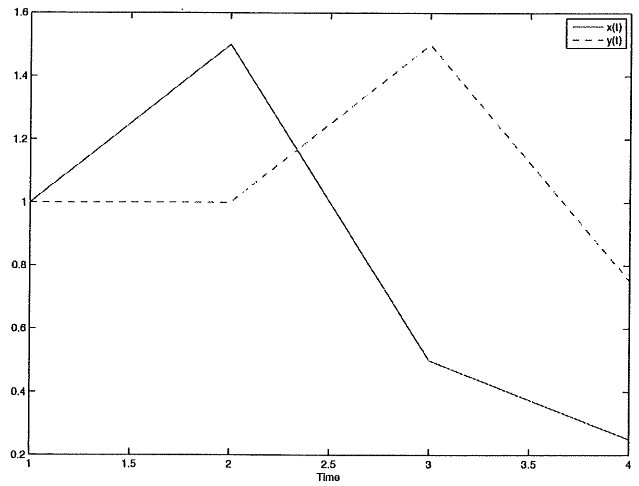


Figure 1: Plot of $x(t)$ and $y(t)$ from (a)

(c) i. To 4 decimal places $x(3) = 0.1228$ and $y(3) = 0.3866$

ii. Table of errors to 4dp is as follows,

Numbers of steps	Error in $x(3)$	Error in $y(3)$
12	0.0073	0.0063
24	0.0015	0.0018

iii. To find the order we use the following:
12 steps has $h = 0.25$
24 steps has $h = 0.125$
We denote error in x with step-size h as $E_x(h)$ and error in y as $E_y(h)$ then
$E_x(0.25) = 0.0073$, $E_y(0.25) = 0.0063$
$E_x(0.125) = 0.0015$, $E_y(0.125) = 0.0018$. The effective order for the x and y component is then given by,

$$q_x(0.125) = \frac{\log(E_x(0.25)) - \log(E_x(0.125))}{\log 2}$$
$$\approx \frac{-4.92 - (-6.5)}{0.693} = 2.27$$

$$
\begin{aligned}
q_y(0.125) &= \frac{\log(E_y(0.25)) - \log(E_y(0.125))}{\log 2} \\
&\approx \frac{-5.067 - (-6.32)}{0.693} = 1.8075
\end{aligned}
$$

We estimated the effective order using $x(3)$ as 2.27, with the order estimated in $y(3)$ as 1.8075. Therefore we predict an order 2 method was used.

iv. We think an order 2 method was used from (iii). Examples of order 2 numerical methods are the Improved Euler, the Midpoint method and the second order Runge-Kutta method.

6. Question 6 (5 marks)

(a) Looking at the differential equation in x. The general solution to

$$\frac{dx}{dt} = 2$$

is $x = 2t + c$, solving the IVP gives $c = -1$ hence $x(t) = 2t - 1$.
The IVP partially de-couples. Solving for x we find as above,

$$x = 2t - 1$$

We substitute this into the equation for y giving,

$$\frac{dy}{dt} = 2t - 1 - \frac{y}{t}$$

Writing in standard form

$$\frac{dy}{dt} + \frac{y}{t} = 2t - 1$$

We use an integrating factor to solve.

$$\mu(t) = \exp(\int 1/t \, dt) = \exp(\ln t) = t$$

We multiply by the integrating factor μ.

$$t\frac{dy}{dt} + y = 2t^2 - t$$

which can be re-written,

$$\frac{d}{dt}(ty) = 2t^2 - t$$

Integrating with respect to t,

$$ty = \int 2t^2 - t \, dt$$

We integrate the RHS,

$$ty = \frac{2}{3}t^3 - \frac{1}{2}t^2 + c$$

Thus the general solution for y is,

$$y = \frac{2}{3}t^2 - \frac{1}{2}t + \frac{c}{t}$$

Solving for the initial condition $y(1) = 1$ gives,

$$y(t) = \frac{2}{3}t^2 - \frac{1}{2}t + \frac{5}{6}\frac{1}{t}$$

We now want to find out solutions at $t = 3$. Plugging in $t = 3$ gives,

$$x(3) = 2 \times 3 - 1 = 5$$

$$y(3) = \frac{2}{3} \times 3^2 - \frac{1}{2} \times 3 + \frac{5}{6} \times \frac{1}{3} = \frac{43}{9} = 4.7778$$

(b) Eulers method is exact for linear solutions hence $x(3)$ doesn't change. Linear solutions are found when the RHS of a differential equation is constant.