# Maths 260 Lecture 10

- ▶ **Topic for today:** Bifurcations
- ▶ **Reading for this lecture:** BDH Section 1.7
- ▶ **Suggested exercises:** BDH Section 1.7, #1,3,9
- ▶ **Reading for next lecture:** BDH Section 1.7, again
- ▶ **Today's handouts:** Tutorial 4 questions

# Bifurcations

Many DE models contain parameters, i.e., quantities that do not depend on the independent and dependent variables but may take on different values.

We are interested in how the behaviour of solutions changes as parameters are changed, particularly the long term behaviour.

Questions we sometimes ask are:

▶ What are the solutions like over a range of parameter values?

▶ How good is our model if we only know the parameter values roughly?

# Bifurcations

A small change in the value of a parameter usually results in a small change in solutions to the DE.

A **bifurcation** occurs when a small change in parameter gives a qualitative change in the behaviour of solutions.

We focus on autonomous equations that depend on one parameter, i.e., equations of the form

$$\frac{dy}{dt} = f_k(y)$$

This is a one-parameter family of DEs - we get one DE for each choice of the parameter $k$.

## Example 1:

Consider the DE

$$\frac{dy}{dt} = f_h(y) = y(1 - y) - h$$

This is a one-parameter family of equations, with the parameter being $h$.

Draw the phase line when $h = 0$ and when $h = 1$.

## Example 1:

We see that the qualitative behaviour of solutions changes between $h = 0$ and $h = 1$. There must be a bifurcation at some value of $h$ in the interval $(0, 1)$.

To locate the value of $h$ at which the bifurcation occurs, we find and classify equilibria as a function of $h$.

A bifurcation occurs at the $h$ value at which the number and/or type of equilibrium solutions changes.

# Bifurcation Diagrams

A bifurcation diagram is a picture, in the $k - y$ plane, of the phase lines near a bifurcation value. Here $k$ is the parameter and $y$ is the dependent variable.

A bifurcation diagram summarises the qualitative changes in solutions that occur as the parameter passes through the bifurcation value.

# Procedure for drawing a bifurcation diagram

1. Draw $k$ and $y$ axes and label them.

2. Plot curves showing the position of equilibria as $k$ varies.

3. Sketch representative phase lines, including at least one for each of $k < k_0$, $k = k_0$, and $k > k_0$, where $k_0$ is a bifurcation value.

4. Label any significant values of $k$ and $y$, including bifurcation values.

## Example 2:

For the family of equations

$$\frac{dy}{dt} = k - y^2$$

find the value(s) of $k$ where a bifurcation occurs. Sketch the bifurcation diagram.

## Example 3:

Draw the bifurcation diagram for the one-parameter family

$$\frac{dy}{dt} = y(1 - y) - h,$$

where $h$ is the bifurcation parameter.

## Important ideas from today:

► A bifurcation occurs when a small change in parameter gives a qualitative change in the behaviour of solutions.

► A bifurcation diagram is a picture that summarises the qualitative changes in behaviour of solutions that occur when a parameter passes through a bifurcation value.