Department of Mathematics

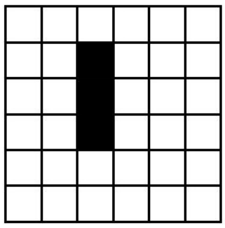
Maths 190 Assignment 4 Solutions

May 31, 2010 Due: 4pm, Monday May 31, 2010

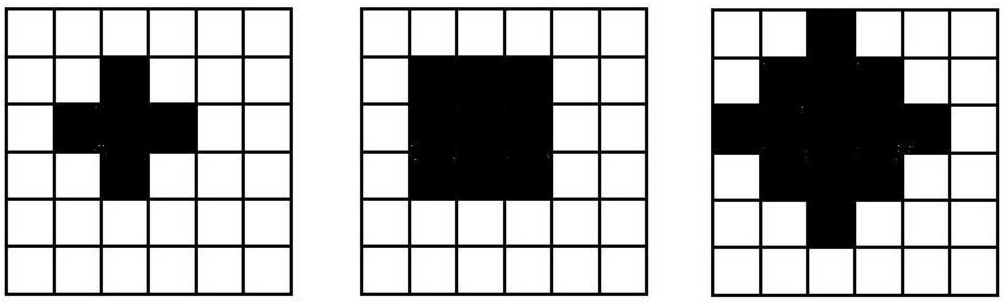
1. (6 marks) Consider the game "immortal life" which has the same birth rules as the game of life, but no death rules.

   (a) Given the following initial population, what is the population in the next 3 time steps?

   Draw your answer on the solution sheet.



SOLUTION: (2 marks each)



2. (4 marks) Let $c > 0$ and $0 < P_1 < 1$ be real numbers. Define a sequence of values generated by the formula

$$P_{n+1} = P_n + cP_n(1 - P_n).$$

   (a) (3 marks) Using Excel or any other computer programme, explore what happens with this sequence for the three values $c = 2$, $c = 2.5$ and $c = 3$ and various starting points (more precisely, make small changes to P_1 and see if they lead to large or small changes in the long term behaviour of the sequence).

   Very briefly describe what you have seen.

   (b) (1 marks) For which values c does the sequence exhibit chaotic behaviour?

SOLUTION: When $c = 2$ or 2.5 then the sequence is quite stable, in the sense that if one makes a small change to P_1 then the later values P_n are close to the original values. For example (this is $c = 2.5$):

n	P_n	P_n
1	0.69	0.7
2	1.224	1.225
3	0.536	0.535
4	1.158	1.157
5	0.700	0.701
6	1.225	1.224
7	0.535	0.535
8	1.1577	1.157

When $c = 3$ the sequence is chaotic, as the following table shows (a small change in the starting value for P_1 leads to very large changes in P_8 and P_9.

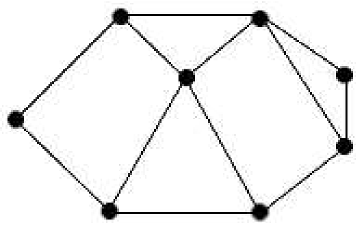
n	P_n	P_n
1	0.69	0.7
2	1.331	1.33
3	0.0067	0.013
4	0.026	0.053
5	0.102	0.202
6	0.376	0.687
7	1.080	1.332
8	0.819	0.005
9	1.263	0.019

3. (5 marks)

   (a) Explain why the below graph does not have an Euler circuit.

   (b) What is the minimum number of edges to be added to the graph before it has an Euler circuit.

   (c) On the picture on the attached page, draw these additional edges.

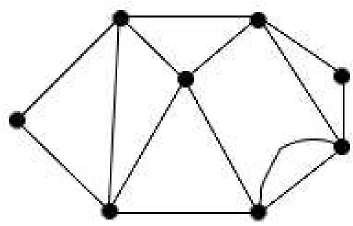


SOLUTION:

(a) (2 marks) To have an Euler circuit it is necessary that all vertices have even degree, but this graph has 4 vertices of odd degree.

(b) (2 marks) Hence, 2 edges must be added (connecting vertices of odd degree).

(c) (1 mark) There are many such pictures. For example



4. (5 marks) A soccer ball is made up of pentagons and hexagons. Let n be the number of pentagons in a soccer ball and m the number of hexagons.

   (a) Show that $m = 5n/3$.

   (b) Consider the graph whose edges are the joins between pentagons and hexagons and whose vertices are corners of pentagons.
   Show that $V = 5n$, $E = 15n/2$ and $F = 8n/3$.

   (c) Use the Euler characteristic to solve for n and hence determine the number of pentagons and hexagons in a soccer ball.

SOLUTION:

   (a) (1 mark) Each pentagon has 5 hexagons around it, and each hexagon arises in this way from 3 pentagons. Hence, computing $5n$ counts every hexagon 3 times. So $m = 5n/3$.

   (b) (1 mark each)

      i. Each vertex arises in exactly one way as the corner of a pentagon. Since each pentagon has 5 corners we have $V = 5n$.

      ii. There are 5 edges around each pentagon. There are also 5 edges radiating out from each pentagon, and each of these meets another pentagon (and so is counted twice). Hence $E = 5n + \frac{1}{2}5n = 15n/2$.

      iii. The number of faces is $n + m = n + 5n/3$.

   (c) (1 mark) Now

$$2 = V - E + F = 5n - 15n/2 + 8n/3 = n/6$$

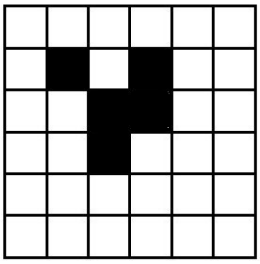
   and so $n = 12$ and $m = 20$.

5. (3 marks) Draw a graph on the surface of a torus such that the Euler formula $V-E+F = 2$ does not hold.

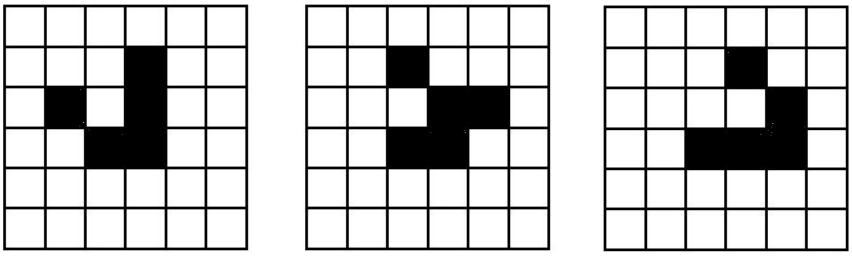
   SOLUTION: Draw a loop from one vertex to itself around the middle of the torus. We have $V = 1$, $E = 1$ and $F = 1$. Hence $V - E + F = 1$.

Tutorial Questions

Question 4 of Tutorial 9 (5 marks) For the below initial population in the Game of Life find out the behaviour of the population on an infinite grid — compute the next few generations and decide if the population will explode, go extinct, become periodic, migratory, etc.



SOLUTION: (4 marks for picture, 1 mark for describing the behaviour) The shape is the glider. It is **migratory**.



Question 4 of Tutorial 10 (5 marks) Give two different arguments to show that a Möbius band is not equivalent by distortion to a cylinder.

SOLUTION: The number of sides of a shape is preserved by distortions: stretching or bending a shape with n sides results in a shape with n sides. A cylinder has 2 sides while a Möbius band has 1 side. Hence the cylinder and the Möbius band are not equivalent by distortion.

Similarly, the number of edges of a shape is preserved by distortions: stretching or bending a shape with n edges results in a shape with n edges. A cylinder has 2 edges while a Möbius band has 1 edge. Hence the cylinder and the Möbius band are not equivalent by distortion.

Question 5 of Tutorial 11 (5 marks) Is it possible to draw a connected graph in the plane with an odd number of faces, an even number of vertices, and an even number of edges? (Don't forget to count the infinite outside of your graph as a face.) If so, draw one; if not, explain why not.

SOLUTION: If such a graph exists then it satisfies the Euler characteristic $V - E + F = 2$. Now, if V and E are even and F is odd then $V - E + F$ is odd. Since 2 is even it follows that such a graph cannot exist.