Einstein Metrics

and

Global Conformal Geometry

I

Claude LeBrun
SUNY Stony Brook

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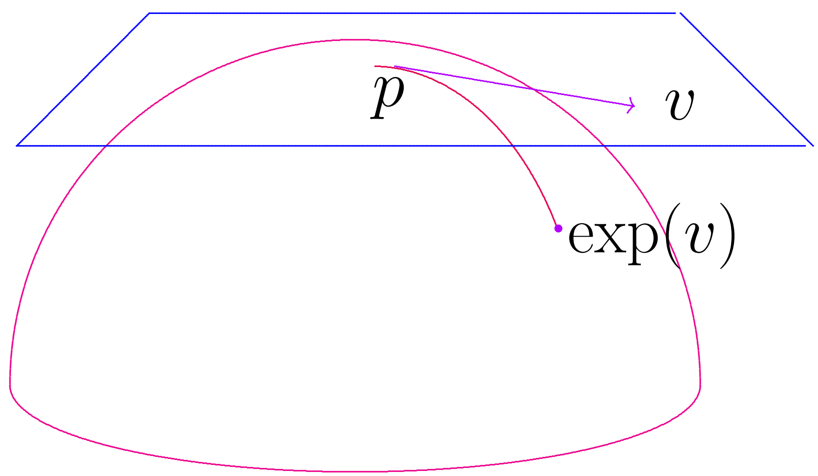

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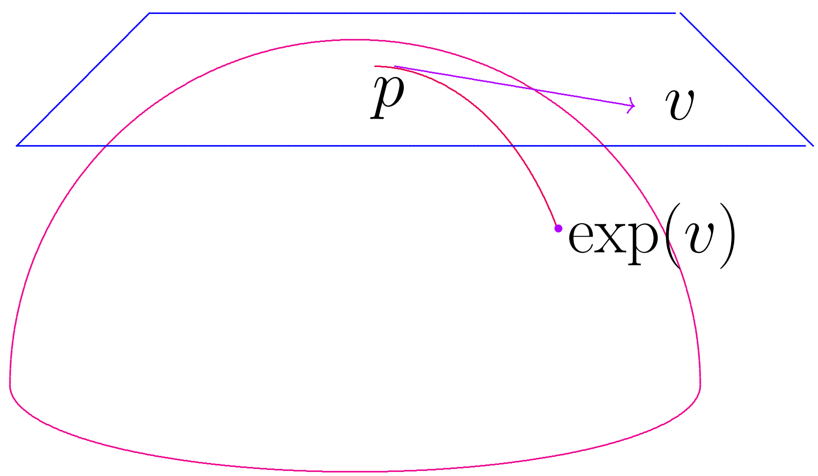

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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(Use Jacobi's equation for geodesic deviation.)

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$$d\mu_g = \sqrt{\det[g_{jk}]}\, dx^1 \wedge \cdots \wedge dx^n$$

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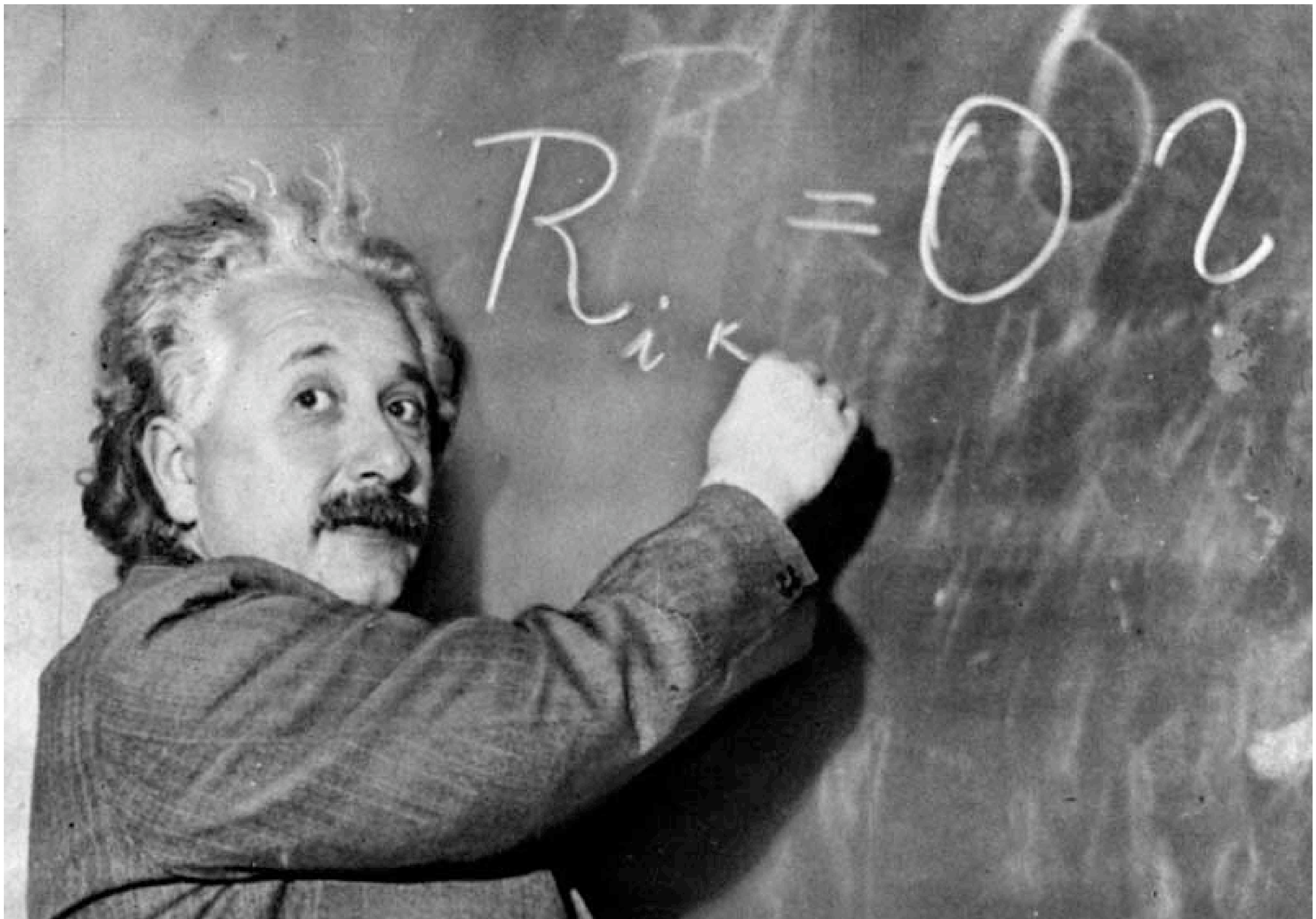

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Proposition. *If* $n \geq 3$*, a Riemannian* n*-manifold* (M^n, g) *is Einstein iff the trace-free part of its Ricci tensor vanishes:*

$$\mathring{r} := r - \frac{s}{n} g = 0.$$

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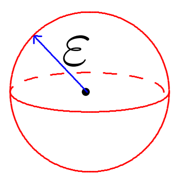

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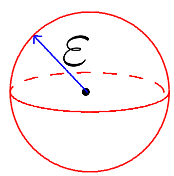

$$B_\varepsilon(p) = \{q \in M \mid \exists \text{ path from } p \text{ to } q \text{ of length } < \varepsilon\}$$

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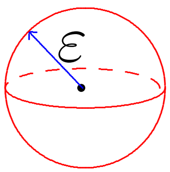

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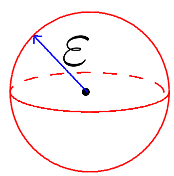

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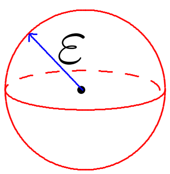

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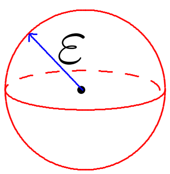

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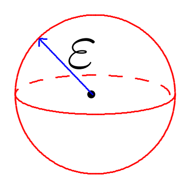

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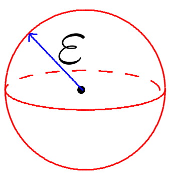

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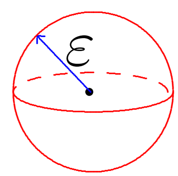

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On a 3-manifold,

$$\frac{s}{2} - r(v, v) = K(v^{\perp})$$

for any unit vector v, so Einstein $\Rightarrow$ constant sectional curvature $\lambda/2$.

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- When $n \geq 6$, wide open. Maybe???

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where $V = \mathrm{Vol}(M, g)$ inserted to make scale-invariant.

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Difficulty: $L^2_1 \hookrightarrow L^p$ bounded, but not compact.

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Has $s =$ constant.

Unique up to scale when $s \leq 0$.

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If g has s of fixed sign, agrees with sign of $Y_{[g]}$.

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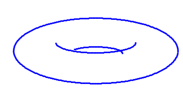
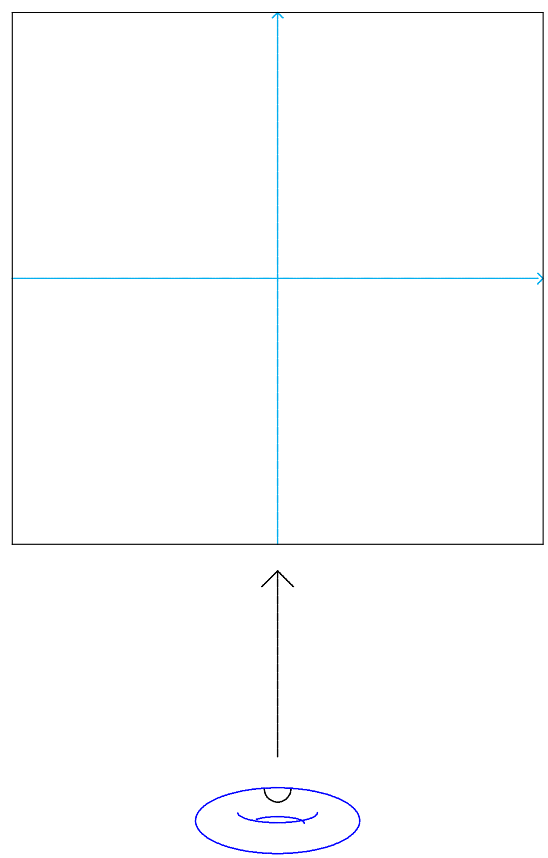
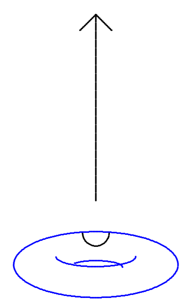
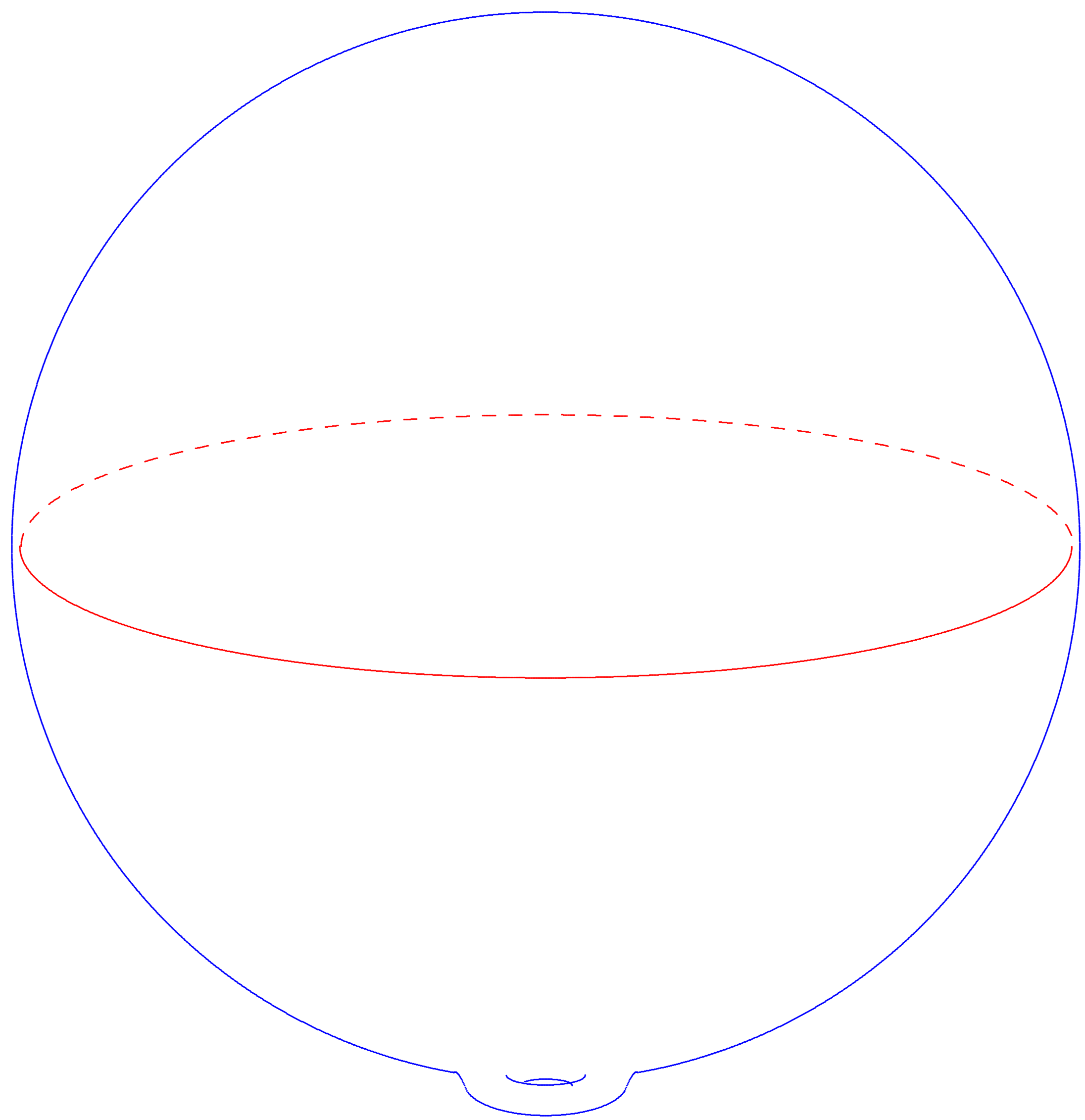




$$g_{jk} = \delta_{jk} + O(|x|^2)$$





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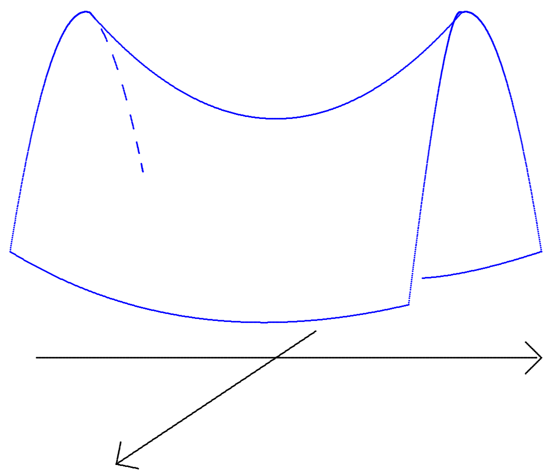

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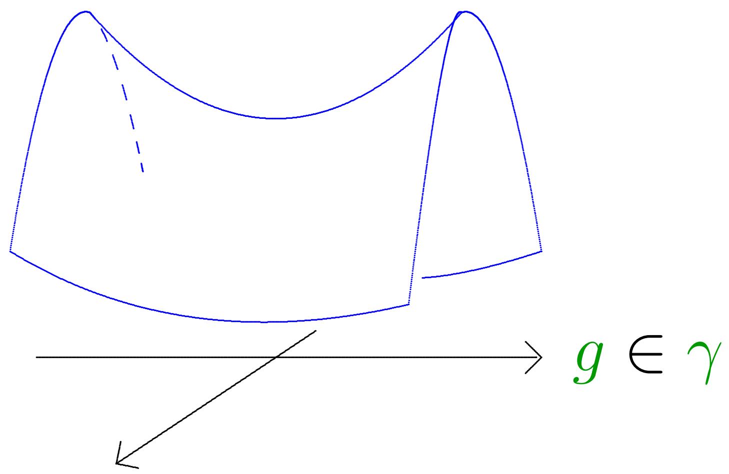

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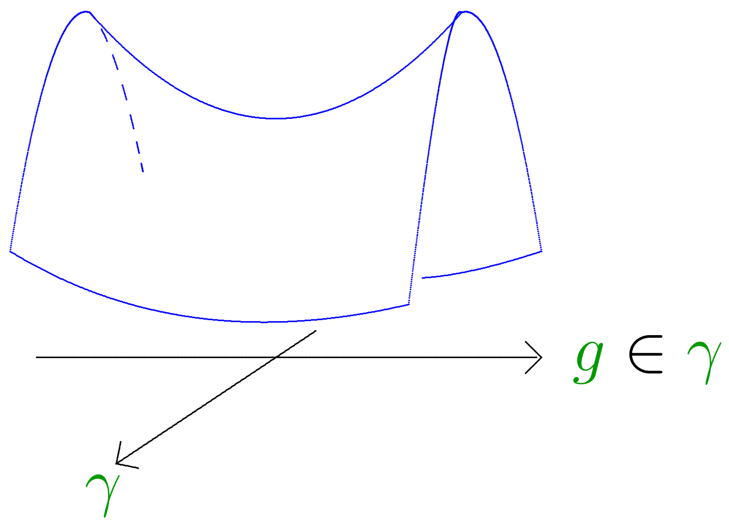

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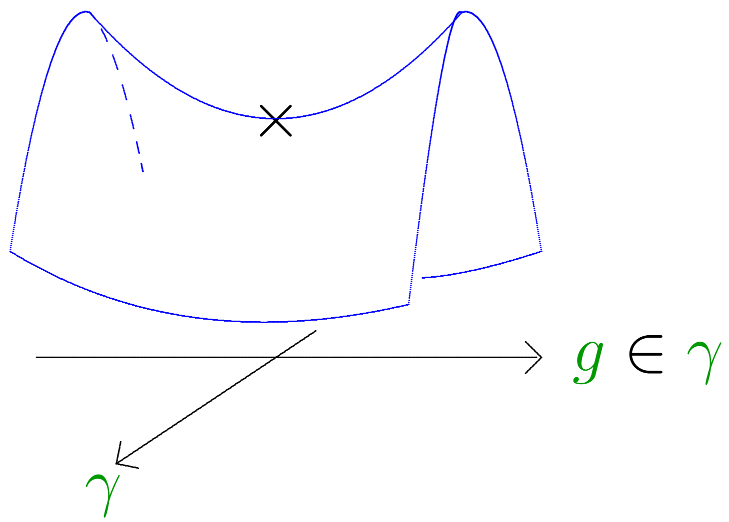

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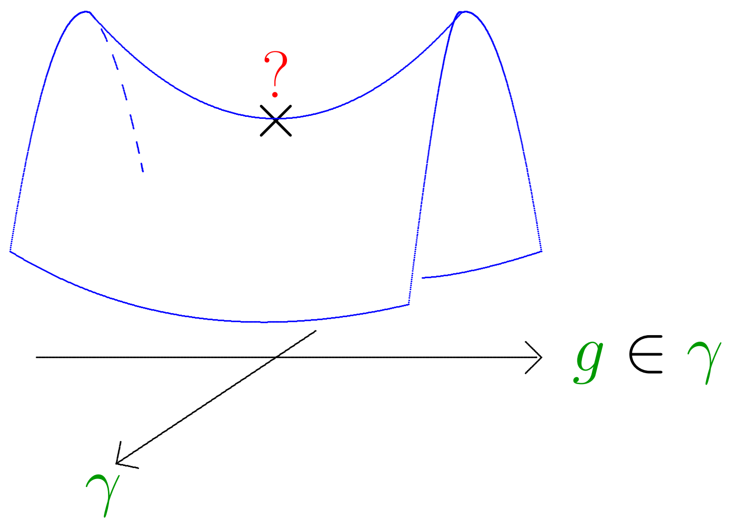

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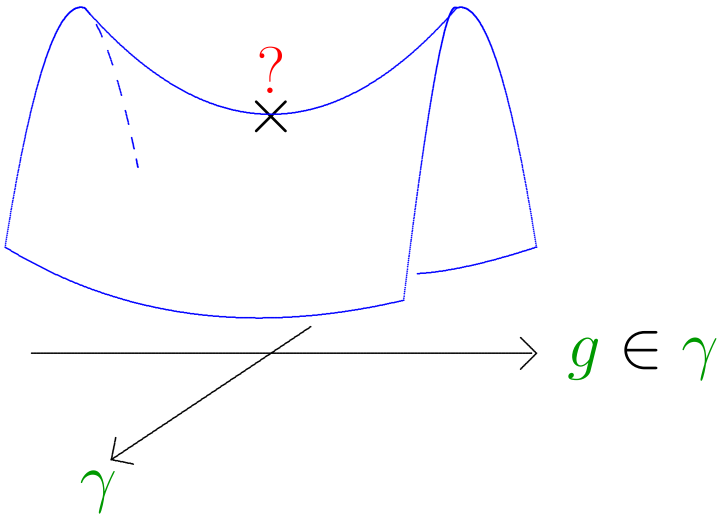

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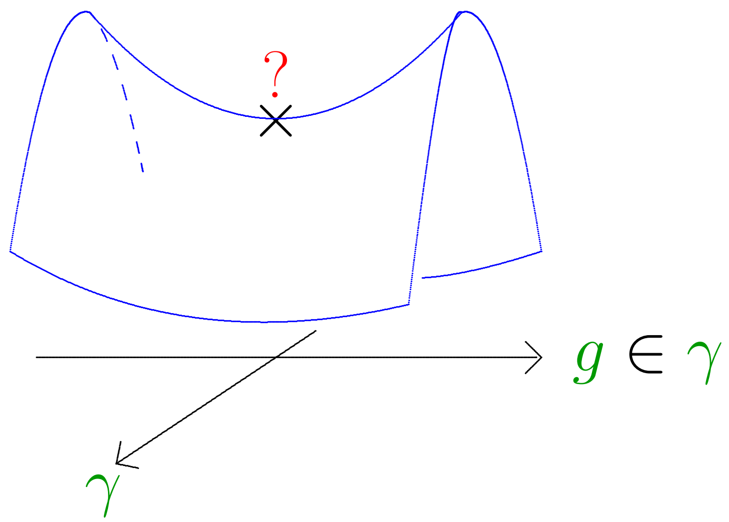

Too good to be true! But . . .

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$\mathcal{Y}(M) > 0 \iff M$ admits g with $s > 0$.

Problem. *Compute actual value of $\mathcal{Y}(M)$ for concrete, interesting manifolds.*

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Example The round metric on S^n is a supreme Einstein metric.

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Problem. *Think of your favorite examples of Einstein metrics. Are are any of them supreme?*

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S^3/Γ open, except when $\Gamma = \mathbb{Z}_2$.

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Open question for hyperbolic 4-manifolds $\mathcal{H}^4/\Gamma$!

Theorem (Petean)**.** *Let* M^n *be a simply connected* n*-manifold,* $n \geq 5$*. Then* $\mathcal{Y}(M) \geq 0$*.*

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Inspiration:

Theorem (Gromov/Lawson). *Let M^n be a simply connected n-manifold, $n \geq 5$. If M is not spin, then M carries a metric g with $s > 0$. That is,*

$$w_2(TM) \neq 0 \Longrightarrow \mathcal{Y}(M) > 0.$$

Theorem. *Let M be a compact simply connected n-manifold, $n \geq 3$. If $n \neq 4$, $\mathcal{Y}(M) \geq 0$.*

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Theorem. *There exist infinitely many compact simply connected 4-manifolds with $\mathcal{Y}(M) < 0$.*

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This is intimately tied to the fact that $\mathcal{Y}(M)$ depends strongly on the smooth structure in dimension four.