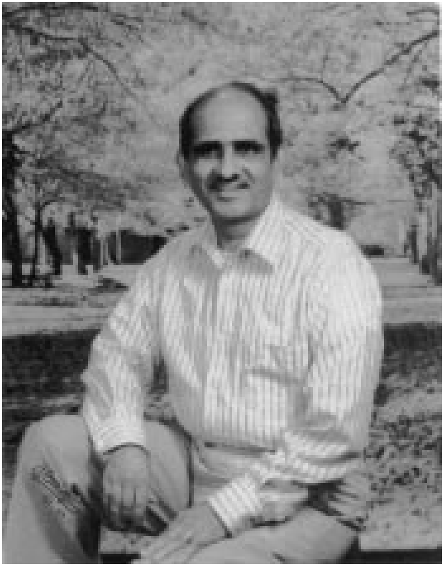

ARUN KUMAR VARMA
(1934–1994)

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In Memoriam

Arun Kumar Varma

(1934-1994)

Arun Kumar Varma was born on October 20, 1934, in the Lucknow area of India. He received his B.Sc. degree from Benares Hindu University in 1955 and his M.Sc. in mathematics from Lucknow University in 1958. These were hard times for a young mathematician in India. He told me that he rode his bike daily for dozens of miles just to get to a library, where he could copy mathematical papers which his teacher, Ambikeshwar Sharma, had designated for him. It is hard for the present day mathematician to believe, in this age of copy machines, computers, and sophisticated communication devices, that when I say he "copied" papers, I mean that he made a *handwritten* copy with neat precise handwriting, so that other students could read it as well.

He was one of the few fortunate people who could afford to leave India and attend a western university. He received his Ph.D. from the University of Alberta, Edmonton, in 1964 under the supervision of A. Sharma. Paul Turán was the outside expert at his defense who spoke strongly about Arun and praised his research. Arun was fascinated by Turán's results and his style of writing and talking; this encounter determined his further professional career. Paul Turán was his role model, but his research was also deeply influenced by the entire Hungarian school of approximation theory (L. Fejér, P. Erdős, G. Grünwald, E. Feldheim, G. Freud, and so forth). This all happened when Turán initiated the investigation of lacunary interpolation, and Arun was lucky to get involved in the topic at an early stage.

After obtaining his Ph.D., he first stayed in Edmonton to do postdoctoral work and then went to The University of Florida in Gainesville in 1967 as an assistant professor. He was promoted to associate professor in 1969 and professor in 1975. He was on the editorial board of the International Journal of Mathematics and Mathematical Sciences from 1989 to 1992. He directed one masters thesis (R. D. Spool), and four Ph.D. students (T. M. Mills, G. Howell, J. Burkett, and B. Underhill [still in progress]). He gave 57 talks, lectures, and invited addresses at international and national meetings and colloquia. His natural kindness and

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friendly behavior predestined him to do joint research with more than 20 mathematicians from all over the world.

At the age of 60, he was as devoted as ever to his research and full of future plans. A short but terminal illness took his life on December 8, 1994. He is survived by his wife Manjuka, sons Rajeev and Nitin, and daughter Latika.

His field of interest was classical approximation theory. Within this area, his main subjects of investigation were the following:

1. Interpolation by polynomials and splines.
2. Quadrature formulas.
3. Order of pointwise and uniform approximation of finitely differentiable functions by polynomials.
4. Bernstein and Markov type inequalities in L_p and uniform metrics.

In what follows I try to present his main achievements in the framework of the above classification. This presentation is by no means complete and reflects entirely my taste. Readers interested in further details should consult his list of publications at the end of this work.

1. The renewed interest in Birkhoff or lacunary interpolation (that is, when higher order derivatives of non-consecutive order are prescribed) was sparked by work of Paul Turán in the early 1960s. Motivated by the numerical solution of Sturm–Liouville-type differential equations, Turán was led to the problem of (0, 2) interpolation, i.e., when function values and second derivatives are prescribed. This was later generalized to

$$(0, m_1, m_2, ..., m_q) \tag{1}$$

interpolation (with the obvious meaning). There are a number of interesting new problems here which do not arise in connection with Lagrange or Hermite interpolation. Even the existence is not always guaranteed. This is a twofold question: Given a problem (1), we may look for lacunary interpolation polynomials for *all* possible systems of nodes or only for *some* systems. Once the problem of existence is settled, the next question is that of the *representation* of the so-called fundamental polynomials. This is, in general, a very difficult problem, and usually the formulas obtained are so involved that it is impossible to proceed to the next question: Does lacunary interpolation *converge* for some classes of functions?

Turán himself considered the roots of the integral of the Legendre polynomials

$$\pi_n(x) = -n(n-1) \int_{-1}^{x} P_n(t)\, dt \tag{2}$$

as nodes of interpolation. This was a natural choice because of the simple structure of the differential equation for π_n in (2). Inspired by Turán's work, Sharma and Varma [3] considered *trigonometric* $(0, m)$ interpolation on the equidistant nodes $x_k = 2k\pi/n$, $k = 0, \pm 1, \ldots$. This is the natural analogue of the nodes (2) in the trigonometric case. Sharma told me that they solved this first in the case $m = 3$ (the case $m = 1$ leads to the so-called Jackson polynomials, and $m = 2$ was treated earlier by O. Kis). He showed the result to Zygmund, who was not much impressed and who told him "Nice, but if I praise your work, in a short time you will come back with the solution of case $m = 4$. Why don't you generalize?"

And this is how it happened: they came up with a simple formula for the fundamental polynomials of $(0, m)$ interpolation

$$F_k(x) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{n-1} \frac{(n-j)^m \cos jx}{(n-j)^m - (-j)^m}\right] \qquad (k = 0, \pm 1, \ldots) \tag{3}$$

satisfying the conditions $F_k(x_j) = \delta_{kj}$ $(k, j = 0, \pm 1, \ldots)$. (A similar formula holds for the fundamental functions of second kind.)

The beauty of this formula is in its simplicity. It is completely suitable for further investigation of convergence, as well as several generalizations. The method of finding and estimating these fundamental polynomials is ingenious and has been used in many other situations.

Of course, Varma was intrigued by the algebraic situation. But, unfortunately, there the beauty and simplicity of the problem is lost. The formulas are too involved and hard to find. Nevertheless, it is an interesting problem. Sharma and Varma [2] solved the (0, 2) interpolation problem, and Varma [1] himself solved the (0, 1, 3) interpolation problem on the Chebyshev nodes plus the endpoints. It turned out that while Chebyshev polynomials in most situations are the model case, in lacunary interpolation this role is played by the roots of (2). Varma considered many other situations (cf. [4, 11–17, 19, 21–23, 25, 30, 49, 82, 88, 92]). For example, we proved [81] that the (0, 3) interpolation problem based on the roots of (2) is convergent for all continuous functions (this was the first case when algebraic Birkhoff interpolation proved to be convergent for all continuous functions.) Results were also obtained for the roots of unity [53, 58, 65, 66, 71]. Varma also succeeded in giving error estimates for approximation by quintic C^2 splines which interpolate Birkhoff data [34, 35, 39], as well as for cubic [60] and quartic splines [68, 90].

Varma was also interested in Hermite–Fejér interpolation. His most interesting result [41] is the pointwise estimate

$$|f(x) - H_n(f, x)| = O\left(\frac{1}{n}\sum_{k=1}^{n} \omega\left(f, \frac{\sqrt{1-x^2}}{n}\right)\right) \qquad (|x| \leqslant 1), \tag{4}$$

where $H_n(f, x)$ denotes the Hermite–Fejér interpolating polynomial based on the roots of (2). (A weaker form of (4), namely with 1 instead of $\sqrt{1-x^2}$, was proved earlier for the Chebyshev nodes by R. Bojanic.) Further related results can be found in [42, 43, 67, 73, 75].

In connection with (0, 1, 2) Hermite interpolation for arbitrary systems of nodes we proved [74], among other results, that the norm of the fundamental function of first kind is always $\geqslant c \log n$. This was the starting point of a much more general statement.

The investigation of mean convergence of Lagrange interpolation was initiated by Erdős, Feldheim, and Turán in the 1940s, and it is now a widely investigated area. Varma [50] succeeded in proving an L_p convergence result of Erdős–Feldheim type for extended Chebyshev nodes of the second kind (originally proved for the Chebyshev nodes). See also [70, 72, 77] for related results.

2. The theory of quadrature formulas has a direct connection with the theory of interpolation. Once we have an interpolation formula (that is, we have an explicit representation for the fundamental polynomials), there is hope that by integration we can find the Cotes numbers. Another interesting problem is that of the exactness of the resulting formula.

In [48] Varma determined the Cotes numbers of quadrature of (0, 2) interpolation based on the roots of (2), thus solving several problems posed by Turán. He also showed that the Cotes numbers of first and second kind are positive and have the simple representation

$$Q_n(f) = \frac{3(f(1)+f(-1))}{n(2n-1)} + \frac{2(2n-3)}{n(n-2)(2n-1)} \sum_{i=2}^{n-1} \frac{f(x_i)}{P_{n-1}^2(x_i)} + \frac{1}{n(n-1)(n-2)(2n-1)} \sum_{i=2}^{n-1} \frac{(1-x_i^2)\, f''(x_i)}{P_{n-1}^2(x_i)}, \tag{5}$$

where x_i's are the roots of (2). Using Turán's terminology, he found a so-called *good* matrix. For the Chebyshev roots and (0, 1, 3) interpolation he also found the formula [56], but here the coefficients are no longer positive.

Finding the Cotes numbers is a very complicated process of integrating the fundamental functions. In 1986 Varma found an ingenious shortcut [59] to get (5) *without using the fundamental functions of interpolation.* This is perhaps his most significant achievement in this area. In [69] and [97] this method was applied in the (0, 3) and (0, 4) situation.

Another interesting problem of Turán was that of finding quadrature formulas for weighted integrals based on values of consecutive derivatives of the function at the n nodes up to order $2k-2$, exact for polynomials of degree at most $2kn-1$. In the case of the Chebyshev weight, generalizing an earlier result of Micchelli and Rivlin, Varma [54] solved this problem for $k=3$. It should be mentioned that in this case the Cotes numbers are not all positive.

3. Since Faber it has been known that Lagrange interpolation is not suitable for proving Weierstrass' theorem and, therefore, certainly not for Jackson theorems on the order of best polynomial approximation. It is, however, possible to manipulate interpolation polynomials to achieve this goal. The first such result is due to Freud, and Varma made some nice contributions in this area. Answering a question raised by Steckin, starting from the Hermite–Fejér interpolating polynomials based on the Chebyshev roots, in [28] Varma constructed a linear procedure of degree at most $2n-1$ which interpolates at the n Chebyshev roots and uniformly approximates the function to the order $O(\omega_r(f, 1/n))$, thus realizing a generalized Jackson theorem. By using a Bernstein-type averaging process applied earlier by Grünwald, Varma and T. M. Mills [29] proved that a linear process based on the function values at the Chebyshev roots exists which realizes a Timan-type estimate $O(\omega(f, \sqrt{1-x^2}/n + 1/n^2))$ (see also [32, 36, 36a]). Later he and T. M. Mills [37] showed that if instead of the Chebyshev nodes one takes the Chebyshev nodes of the second kind plus the endpoints ± 1, then even $O(\omega(f, \sqrt{1-x^2}/n))$ is possible by using a properly chosen Bernstein-type averaging process. In a recent work [83] Varma and Xiang Ming Yu they were able to replace ω by ω_2, the modulus of smoothness, thus answering a question raised by R. DeVore.

4. Ever since the celebrated inequalities of Bernstein and Markov for derivatives of polynomials were established, efforts have been made to sharpen and generalize these inequalities to polynomials with restricted roots or coefficients, and for other (weighted) metrics. The starting point of these generalizations is the following beautiful theorem of Erdős. If a polynomial p_n of degree n has only real roots all of which are outside the interval $(-1, 1)$ then

$$\max_{|x| \leqslant 1} |p_n'(x)| < \frac{en}{2} \max_{|x| \leqslant 1} |p_n(x)|. \tag{6}$$

Another fundamental result is the inequality of Turán, which says that if all roots of p_n are in $[-1, 1]$ then

$$\max_{|x| \leqslant 1} |p_n'(x)| > \frac{\sqrt{n}}{6} \max_{|x| \leqslant 1} |p_n(x)|. \tag{7}$$

Varma [31, 33] extended (7) to L_2 metrics by proving that for these polynomials

$$\|p_n'\|_{L_2(a,\,b)} > \sqrt{\frac{n}{2}}\,\|p_n\|_{L_2(a,\,b)}$$

holds. Here the constant is asymptotically best possible.

Further generalizing these results, Varma [38, 40, 51, 55] investigated the quantities

$$I_n := \frac{\|w_1 p_n'\|_{L_2(a,\,b)}}{\|w_2 p_n\|_{L_2(a,\,b)}} \tag{8}$$

over the classes of polynomials mentioned above, using different weight functions w_1, w_2, on finite and infinite intervals (a, b).

Another problem is the investigation of (8) for polynomials with positive coefficients. For this case Varma [44] proved that if $w_1(x) = w_2(x) = x^{\alpha/2} e^{-x/2}$, $a = 0$, $b = \infty$, then

$$I_n \leqslant \begin{cases} \dfrac{n^2}{(2n+\alpha)(2n+\alpha-1)}, & \text{if} \quad \alpha \geqslant (\sqrt{5}-1)/2, \\[2ex] \dfrac{1}{(2+\alpha)(1+\alpha)}, & \text{if} \quad 0 \leqslant \alpha \leqslant 1/2, \end{cases}$$

and this is sharp in a sense.

We also started in [45] to generalize the inequality of Erdős (6) to the case where the polynomial has only one root in $(-1, 1)$, while all the other roots are real and are located outside $(-1, 1)$. This was the starting point of the beautiful generalization to $k \leqslant n$ roots inside $(-1, 1)$ (see works of P. Borwein, T. Erdélyi, and A. Máté).

Varma's technique of investigating quantities like (8) was good mainly for L_2 metrics, but in some situations he was able to overcome the difficulties arising from the use of other L_p metrics (see, e.g., [76]).

Another problem in this area is to restrict the polynomials by some curved majorant and obtain Markov and Bernstein type inequalities. This problem was suggested by Turán, who was particularly interested in conditions like

$$|p_n(x)| \leqslant \sqrt{1-x^2} \qquad (|x| \leqslant 1)$$

(circular majorant) and

$$|p_n(x)| \leqslant 1 - x^2 \qquad (|x| \leqslant 1)$$

(parabolic majorant). Varma [80, 87] was able to get asymptotically sharp results in the case of weighted L_2 spaces.

Finally I mention one of his most recent results with B. Bojanov [94]. They proved Kolmogorov type inequalities

$$\|p_n^{(j)}\|^2 \leqslant A\,\|p_n^{(m)}\|^2 + B\,\|p_n\|^2 \qquad (A, B > 0 \text{ depend on } j, m, n)$$

for $0 < j < m \leqslant n$, where the norm is L_2 norm on $(-\infty, \infty)$ with weight e^{-x^2}. This is an analogue of Kolmogorov's inequalities for smooth functions on the real line.

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