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The Work of P. Turán on Interpolation and Approximation

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The main work of P. Turán lies in Number Theory and Analysis; in our field he has also made many important contributions. The hallmark of a good mathematician is not only the power of his methods, but also the novelty and originality of his problems, inspiring others to continue his work. This applies remarkably to Trán. I shall restrict myself almost entirely to three main themes treated by him.

Let $-1 \leqslant x_1^{(n)} < \cdots < x_{n+1}^{(n)} \leqslant 1$, $n = 1, 2,...,$ be a matrix of knots in $[-1, 1]$, and let $L_n(f, x)$ be the corresponding Lagrange interpolation polynomial of a function f. In particular, let $x_k^{(n)}$ be the zeros of the nth orthonormal polynomial with (measurable) weight $p(x)$ on $[-1, +1]$, where $p(x) > 0$ a.e., and $\int_{-1}^{+1} p\, dx < +\infty$. In the papers of Turán with Erdös [1] and Grünwald [2], the convergence of $L_n(f)$ to f is discussed. One has for Riemann integrable f, $\int_{-1}^{+1} p\, |f(x) - L_n(f, x)|^2\, dx \to 0$. This is best possible, for convergence with an exponent >2 does not hold in general. Here is an essential difference between the general theory and that of Jacobi polynomials. One has $L_n(f, x) \to f(x)$ uniformly on $[-1, +1]$ if $p(x) \geqslant m(1 - x^2)^{-1/2} > 0$ and if $f \in \operatorname{Lip} \alpha$, $\alpha > \frac{1}{2}$.

In the papers [3, 5] Turán and Erdös study the distribution of zeros of orthogonal polynomials P_n on $[-1, 1]$ in relation to the properties of the weight function p. The results which they obtain should be compared with the classical theorems of Markov, Stieltjes and others concerning the Legendre and Jacobi polynomials. Let $x_k^{(n)} = \cos \theta_k^{(n)}$, $0 \leqslant \theta_0^{(n)} < \theta_1^{(n)} < \cdots < \theta_n^{(n)} \leqslant \pi$, be the zeros of P_n. Often it is possible to prove that

$$\frac{c_1}{n} \leqslant \theta_{k+1}^{(n)} - \theta_k^{(n)} \leqslant \frac{c_2}{n}. \tag{1}$$

The method of investigation is, in general, as follows. From assumed propoerties of the weight p one derives properties of the fundamental polynomials $l_k^{(n)}(x)$, and uses them to estimate $\sum |l_k^{(n)}(x)|$; this leads to results like (1); they, in turn, are applied to the study of the behavior of the P_n. One of the theorems, for example, derives from the assumption $|l_k^{(n)}(x)|^{1/n} \leqslant$

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$1 + \epsilon$, $-1 \leqslant x \leqslant 1$, $k = 1,..., n$, $n \geqslant n(\epsilon)$, the relation, for all complex z,

$$\lim_{n\to\infty} [P_n(z)]^{1/n} = \frac{z + (z^2 - 1)^{1/2}}{2}. \tag{2}$$

Several papers of Turán [15–21, 36] deal with lacunary or Birkhoff interpolation. The problem is to find a polynomial P of degree $\leqslant 2n - 1$ satisfying $P(x_k) = y_k$ and $P''(x_k) = y_k''$, $k = 1,..., n + 1$, where $-1 \leqslant x_1 < \cdots < x_{n+1} \leqslant 1$, and y_k, y_k'' are given numbers. This problem is not always solvable. Turán and his collaborators (Balázs, Egerváry, Surányi) recognized that the situation is different if n is even and the knots $x_k = x_k^{(n)}$ are the zeros of $(1 - x^2) L_n'(x)$, where L_n is the nth Legendre polynomial. The interpolating polynomial exists, is unique, and has the form

$$P_{2n+1}(x) = \sum_1^{n+1} y_k r_k(x) + \sum_1^{n+1} y_k'' \rho_k(x). \tag{3}$$

Explicit formulas for the fundamental polynomials r_k, ρ_k are obtained and the sums $\sum |r_k(x)|$, $\sum |\rho_k(x)|$ are estimated. This allows to derive convergence theorems, and even interesting estimates of the derivatives $P_{2n+1}'(x)$. The proofs of the convergence theorems work if the function $f(x)$ in question is approximable by polynomials Q_n of degree $\leqslant n$ with error

$$|f(x) - Q_n(x)| = o\left(\frac{1 - x^2}{n} + \frac{1}{n^2}\right). \tag{4}$$

One can use here results of Dzyadyk and Freud which guarantee (4) for a certain class of functions. One convergence theorem states that $P_{2n-1}(x) \to f(x)$ uniformly on $[-1, +1]$ if one takes $y_k = y_{kn} = f(x_k^{(n)})$, $y_k'' = y_{kn}'' = 0$. One can also take y_{kn}'' different from zero, provided they are not too large. This reminds us of the classical Hermite–Fejér interpolation at the zeros of Chebyshev polynomials.

Interpolation of type (3) is called (0, 2)-interpolation. Since the work of Turán, many papers have appeared (by A. Sharma, O. Kis, P. O. H. Vértesi, A. K. Varma and others) which study (0, 1, 3)-, (0, 1, 2, 4)-, and other types of lacunary interpolation, by polynomials, or by trigonometric polynomials. All these authors use very special knots, for example the roots of unity in the complex plane. A good exposition of known results can be found in the review article of Sharma [5*].

It is a pity that there is no modern monograph summing up the achievements in interpolation theory; in contrast, there are several excellent texts on general approximation theory.

Another important set of papers of Turán, this time jointly with Szüsz [27–31], concerns rational approximation. The degree of rational approximation $R_n(f)$, of a function $f \in C[-1, 1]$, is the minimum of $\max_{-1 \leq x \leq 1} |f(x) - r_n(x)|$ over all rational functions r_n of degree $\leq n$. The early results obtained for rational approximation seemed to indicate that, for sufficiently large natural classes of functions (such as balls in Lipschitz and other spaces), rational approximation is not essentially better than polynomial approximation. Not everybody believed this myth, but Turán and Szüsz were the first to disprove it.

They use a theorem of D. J. Newman of 1964, according to which the degree $R_n(g)$ of rational approximation of the function $g(x) = |x|$ on $[-1, +1]$ is of order $e^{-c\sqrt{n}}$. They prove: If $f^{(k-1)}$ is absolutely continuous, and $f^{(k)}$ is of bounded variation, then $R_n(f) \leq Cn^{-k-1} \log^{2k+2} n$. (Later Popov [4*] improved this bound to Cn^{-k-1}.) If f is piecewise analytic on $[-1, 1]$, then $R_n(f) \leq e^{-c(f)\sqrt{n}}$. Here, also, Turán's work inspired important further investigations (Szabados, Freud; the latter proved [1*] that $R_n(f) \leq Cn^{-1} \log^2 n$ if f is of bounded variation and belongs to Lip α for some $\alpha > 0$).

We shall add two further examples of Turán's work, and compare them with later investigations. Let $P_n(x)$ be a real polynomial, $P_n(\pm 1) = 0$, $P_n(x) > 0$ for $-1 < x < 1$. Turán [6] found the exact lower bound of the distances of the points of absolute maximum of P_n in $[-1, +1]$ from the endpoints. For example, if n is even, this lower bound is $1 - \cos(\pi/n)$. A theorem of the same type, but for more general functions, proved to be essential in the investigations of Birkhoff interpolation by the present author (see, for example, [3*]). Turán [8] was the first to establish Gauss quadrature formulas for Hermite interpolation with odd multiplicities. Karlin and Pinkus [2*] extended this to Chebyshev systems.

"On some open problems of approximation theory" of Turán [38], which appears in English in this issue, is well written and inspiring. The author describes his work and derives from it 89 problems. Some of them have meanwhile been solved. (See notes of P. Nevai, J. Szabados, V. Sös, and the present author in this issue.)

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