Madison, Wisconsin
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Dear Phil:

Lately I have been busy with euclidean polyhedra from the point of view of distance geometry, but on and off I return to spline functions. The other day I noticed that the triangle formulae in the complex plane generalize nicely to higher order derivatives. The source is again the Hermite-Genocchi formula

$$(1) \qquad f(x_0, x_1, \ldots, x_n) = \int_{\tau_n}\!\cdots\!\int f^{(n)}(x_0 t_0 + x_1 t_1 + \cdots + x_n t_n)\, dt_1 \cdots dt_n$$

where $t_0 = 1 - t_1 - \cdots - t_n$ and where the integration is to be carried out over the simplex

$$\tau_n : \quad t_1 \geqq 0, \ldots, t_n \geqq 0, \quad \sum_1^n t_\nu \leqq 1.$$

A projection onto the real axis allows us to show that the kernel $M(x; x_0, x_1, \ldots, x_n)$ in the fundamental formula

$$(2) \qquad f(x_0, x_1, \ldots, x_n) = \frac{1}{n!}\int_{x_0}^{x_n} f^{(n)}(x)\, M(x; x_0, \ldots, x_n)\, dx, \qquad (x_0 < \cdots < x_n),$$

may be interpreted as follows: Interpret the x-axis as one of the coordinate axis of n-dimensional space. Erect at $x_0, x_1, \ldots, x_n$ a hyperplane orthogonal to the x-axis at these points and select at will in each of these hyperplanes a point, with the proviso that the simplex having these points as vertices, should have n-dimensional volume unity. If we project the volume of this simplex on the x-axis we obtain the linear density function $M(x; x_0, \ldots, x_n)$ appearing in (2).

I now pass to the complex domain. Let $z_0, z_1, \ldots, z_n$

(not all on a line)

be distinct points of the complex plane and let $f(z)$ be regular in the convex hull Π of these points. Again we erect at $z_0, z_1, \ldots, z_n$ the orthogonal complements of the plane (these are of dimension $n-2$) and select in each of them a point so that the simplex has [illegible] dim. volume unity. We now project the volume of this simplex onto the plane and denote the surface density function by

$$M(x, y;\, z_0, \ldots, z_n).$$

Then the following formula holds

$$f(z_0, z_1, \ldots, z_n) = \frac{1}{n!} \iint_{\Pi} f^{(n)}(z)\, M(x, y;\, z_0, \ldots, z_n)\, dx\, dy$$

For $n = 2$ this gives the old triangle formula

$$f(z_0, z_1, z_2) = \frac{1}{2A} \iint_{T} f''(z)\, dx\, dy$$

(P. J. Davis, Triangle formulas in the complex plane, Math. of Comp. 1964, p. 569–57?)

where A is the area of the triangle T of vertices z_0, z_1, z_2.

Joining all pairs of points z_j, z_k $(j < k)$ by segments, the polygon Π is dissected into disjoint polygons in each of which $M(x, y)$ is a polynomial in x and y of joint degree $n-2$, while $M(x, y) = 0$ outside Π. Moreover the function $M(x, y)$ has continuous partial derivatives of all orders $\leq n-3$. (*) Moreover, these properties always determine $M(x, y)$ <u>uniquely</u> up to a constant factor. Another property is this: $M(x, y) > [0]$ inside Π and

$$\log M(x, y) \qquad ((x, y) \text{ inside } \Pi)$$

is a <u>concave</u> function. In particular $M(x, y)$ has exactly one maximum point.

(*) This assumes that the points $z_0, \ldots, z_n$ are in "general position", i.e. no three are collinear.

Thus for $n = 3$ the surface $z = M(x,y)$ is a pyramid.

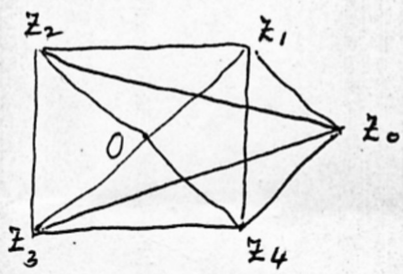
In case all the points $z_0, z_1, \dots, z_n$ are on the boundary of the polygon Π then $M(x,y)$ can be represented inside Π by means of the truncated power function x_+^{n-2}.

Here is an example: For $n = 4$ and

$$z_0 = 2, \quad z_1 = 1+i, \quad z_2 = -1+i, \quad z_3 = -1-i, \quad z_4 = 1-i$$

$M(x,y)$ is up to a positive factor, which I did not determine, identical <u>inside Π</u> to

$$2 + 4x - 4x^2 - 6y^2 + 6(x-1)_+^2 + (x+3y-2)_+^2 + 3(y-x)_+^2 + (x-3y-2)_+^2 + 3(-y-x)_+^2.$$



I am very much interested in these 2-dimensional frequency functions

$$M(x,y; z_0, \dots, z_n)$$

because I suspect that the limits of such frequency functions (with the z_j depending on n and as $n \to \infty$) in the usual sense of probability theory, ought to be interesting frequency functions. The similar question on the line led to the Polya frequency functions. What will one get in the plane, if anything?

I hope these remarks did not bore you.

With greetings and good wishes

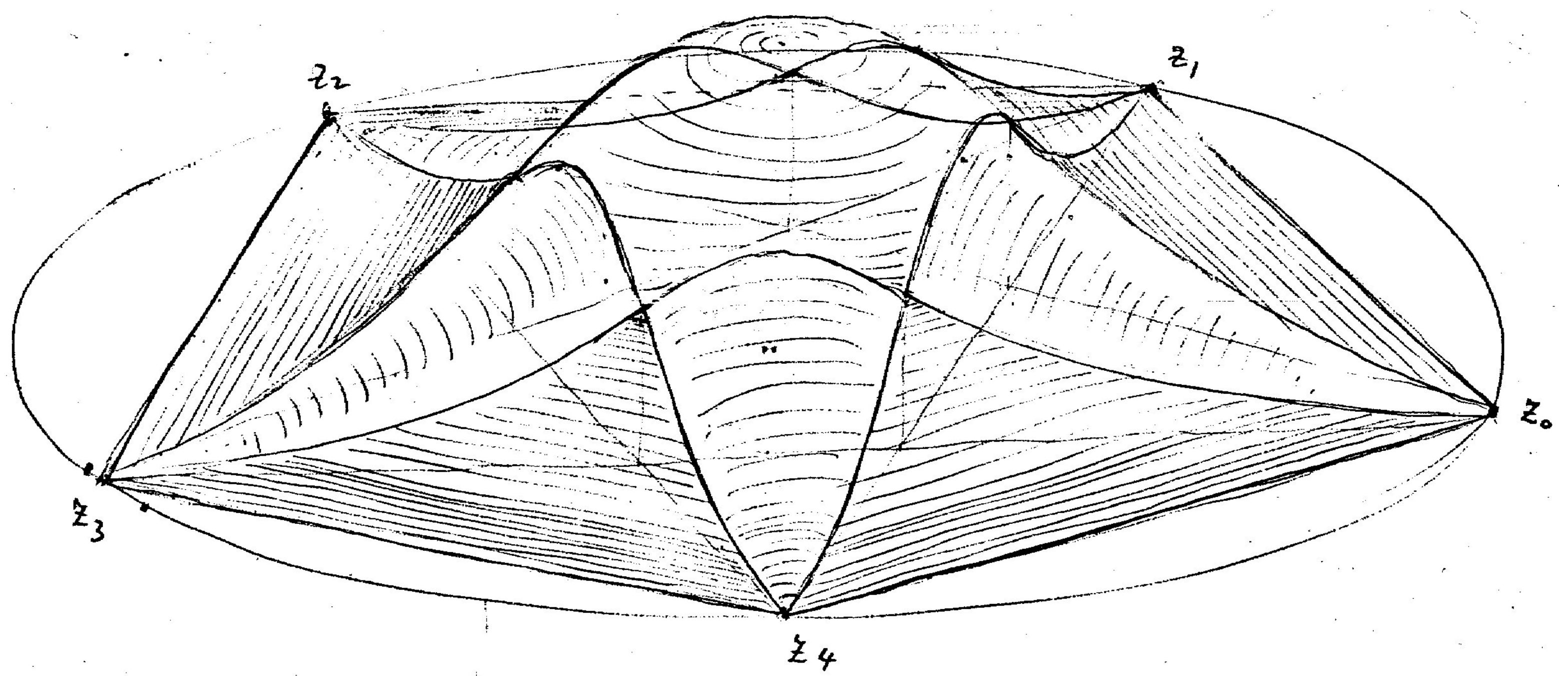
Yours

Iso

(I. J. Schoenberg)

I add a sketch of the second-degree $M(x,y)$ $(n = 4)$ for the case when $z_0, \dots, z_4$ are the vertices of a regular pentagon. All vertical sections are, of course, ordinary spline functions.

I send copies of this letter to H. B. Curry, S. Karlin and T. S. Motzkin



A sketch of the spline function $z = M(x, y; z_0, z_1, z_2, z_3, z_4)$