On a question by D. I. Mendeleev

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In the present work, we will consider the set of all polynomials [lit. entire functions]

$$f(z) = p_0 z^n + p_1 z^{n-1} + p_2 z^{n-2} + \cdots + p_{n-1} z + p_n$$

whose degree does not exceed a given whole number n, whose absolute values do not exceed another given value L for all values of the variable z lying between the two given values a and $b > a$. Thus,

$$-L < f(z) < L \quad \text{for} \quad a < z < b.$$

The question is which bound the absolute value of the derivative

$$f'(x) = np_0 x^{n-1} + (n-1)p_1 x^{n-2} + \cdots + 2p_{n-2}x + p_{n-1}$$

of $f(x)$ with respect to x does not exceed.

This question was posed by $D.I.Mendeleev$, for $n = 2$, in his paper "The analysis of water solutions by specific weight" (§86).

The answer depends on how much the number x is pinned down.

We distinguish two cases:

1) x is a given number,
2) x is an arbitrary number between a and b.

Correspondingly, we consider two problems.

Problem No. 1

To find, for particular x, the biggest absolute value of $f'(x)$.

Solution.

We denote by y that function $f(z)$ of the ones considered by us for which $f'(x)$ in absolute value is biggest.

By the statement of the question,

$$-L \leq y \leq +L$$

for all values z that lie between a and b.

From all these values z, we single out those at which y equals $\pm L$.

Let us write them in a sequence

$$\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_s.$$

Denoting by

$$y(\alpha_i)$$

the value of y at $z = \alpha_i$, equal to $\pm L$, we notice that the sequence of $s - 1$ ratios

$$\frac{y(\alpha_2)}{y(\alpha_1)}, \frac{y(\alpha_3)}{y(\alpha_2)}, \ldots, \frac{y(\alpha_s)}{y(\alpha_{s-1})}$$

must contain at least $n - 1$ numbers equal to -1.

Indeed, in the contrary case it would not be difficult to find, among the polynomials of degree $n-2$, an infinite set of those whose ratios over y at

$$z = \alpha_1, \alpha_2, \ldots, \alpha_s$$

are negative numbers.

If then, having multiplied one of these

$$\varphi(z)$$

by $(z-x)^2$ and by a sufficiently small positive number ϵ, we add the product

$$\epsilon(z-x)^2\varphi(z)$$

to y, then we obtain the polynomial

$$Y = y + \epsilon(z-x)^2\varphi(z)$$

of degree n in z and such that, for $a < z < b$, $|Y| < L$ and, for $z = x$,

$$\frac{\mathrm{d}Y}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}z}.$$

Finally, if we multiply Y by the ratio of the number L over the biggest absolute value of Y on $a < z < b$, then the new function obtained this way will belong to the functions $f(z)$ considered by us and, at $z = x$, its derivative is bigger in absolute value than $\frac{\mathrm{d}y}{\mathrm{d}z}$.

Hence, s is no smaller than n, and the sequence of ratios

$$\frac{y(\alpha_2)}{y(\alpha_1)}, \frac{y(\alpha_3)}{y(\alpha_2)}, \ldots, \frac{y(\alpha_s)}{y(\alpha_{s-1})} \tag{1}$$

contains no less than $n-1$ numbers equal to -1.

If -1 occurs n times in the sequence (1), then it is known that y is reduced to

$$\pm L \cos n \arccos \frac{2x-a-b}{b-a} = \pm f_0(z).$$

In that case,

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\pm nL}{\sqrt{(z-a)(b-z)}} \sin n \arccos \frac{2x-a-b}{b-a} = \pm f_0'(z).$$

We study the condition under which the biggest absolute value of $f'(x)$ equals the absolute value of $f_0'(x)$.

Since we concern ourselves with absolute values, then among all functions $f(z)$ we can consider only those for which $f'(x)$ has the same sign as $f_0'(x)$.

We take, for brevity,

$$\frac{a-b}{2}\cos\frac{i\pi}{n} + \frac{b+a}{2} = \xi_i, \quad i = 0, 1, 2, \ldots, n,$$

and

$$f(z) - f_0(z) = \varphi(z).$$

Considering the value of $f(z)$ and $f_0(z)$ at

$$z = \xi_0, \xi_1, \xi_2, \ldots, \xi_n,$$

we find

$$\begin{array}{ccccccc}
f_0'(\xi_n) & = & +L & \text{and therefore} & \varphi(\xi_n) & \leq & 0 \\
f_0'(\xi_{n-1}) & = & -L & \text{and therefore} & \varphi(\xi_{n-1}) & \geq & 0 \\
f_0'(\xi_{n-2}) & = & +L & \text{and therefore} & \varphi(\xi_{n-2}) & \leq & 0 \\
\ldots & . & \ldots & \ldots & . & . & \\
f_0'(\xi_0) & = & (-1)^n L & \text{and therefore} & (-1)^n\varphi(\xi_0) & \leq & 0.
\end{array}$$

Hence, the equation

$$\varphi(z) = 0$$

must have one root between ξ_0 and ξ_1, between ξ_1 and ξ_2, ..., between ξ_{n-1} and ξ_n.

In other words, the function $\varphi(z)$ must decompose into real factors of first degree in z,

$$\varphi(z) = q(z-\eta_1)(z-\eta_2)\cdots(z-\eta_n),$$

where

$$a = \xi_0 \leq \eta_1 \leq \xi_1 \leq \eta_2 \leq \xi_2 \leq \cdots \leq \xi_{n-1} \leq \eta_n \leq \xi_n = b.$$

Concerning the coefficient q, it must be negative.

With this, we have

$$f'(x) = f_0'(x) + \left(\frac{1}{x-\eta_1} + \frac{1}{x-\eta_2} + \cdots + \frac{1}{x-\eta_n}\right)\varphi(x)$$

and

$$f_0'(x) = \frac{2^{2n-1}nL}{(b-a)^n}(x-\xi_1)(x-\xi_2)\cdots(x-\xi_{n-1})$$

since $f_0'(z)$ vanishes at

$$z = \xi_1, \xi_2, \ldots, \xi_{n-1}$$

and the leading term of the polynomial $f_0(z)$ is

$$\frac{2^{2n-1}nL}{(b-a)^n}.$$

We start with the case when x lies outside the bounds a and b.

Then each of the expressions

$$\frac{\varphi(x)}{x-\eta_1}, \frac{\varphi(x)}{x-\eta_2}, \ldots, \frac{\varphi(x)}{x-\eta_n}$$

has its sign opposite the sign of $f_0'(x)$ and so

$$|f'(x)| < |f_0'(x)|.$$

Thus, if x lies outside the bounds a and b, then the biggest value of $|f'(x)|$ equals $|f_0'(x)|$.

Now we assume that x lies between ξ_{i-1} and ξ_i.

Then

$$\frac{\varphi(x)}{x-\eta_i} = q(x-\eta_1)(x-\eta_2)\cdots(x-\eta_{i-1})(x-\eta_{i+1})\cdots(x-\eta_n)$$

has its sign the opposite of the sign of $f_0'(z)$.

This leaves us to consider the sign of the sum

$$\frac{x-\eta_i}{x-\eta_1} + \frac{x-\eta_i}{x-\eta_2} + \cdots + \frac{x-\eta_i}{x-\eta_{i-1}} + \frac{x-\eta_i}{x-\eta_i} + \cdots + \frac{x-\eta_i}{x-\eta_n} = \Sigma,$$

which, for brevity, we denote by the single letter Σ.

We now denote by $f(z)$ some arbitrary polynomial of nth degree in z that satisfies the conditions

$$-L < f(z) < +L \text{ for } a < z < b$$

and

$$\frac{f'(x)}{f_0'(x)} > 0.$$

Then the numbers

$$\eta_1, \eta_2, \ldots, \eta_n$$

can take on arbitrary values, subject only to the inequalities

$$\xi_0 \leq \eta_1 \leq \xi_1 \leq \eta_2 \leq \xi_2 \leq \cdots \leq \xi_{n-1} \leq \eta_n \leq \xi_n$$

and the coefficient q is sufficiently small in absolute value.

Taking into account this remark, it is easy to see that the smallest (extreme) value of the sum Σ is equal to the smaller of the numbers

$$\frac{x-\xi_{i-1}}{x-\xi_0} + \frac{x-\xi_{i-1}}{x-\xi_1} + \ldots + \frac{x-\xi_{i-1}}{x-\xi_{n-1}} = (x-\xi_{i-1}) \left\{ \frac{f_0''(x)}{f_0'(x)} + \frac{1}{x-a} \right\}$$

and

$$\frac{x-\xi_i}{x-\xi_1} + \frac{x-\xi_i}{x-\xi_2} + \ldots + \frac{x-\xi_i}{x-\xi_n} = (x-\xi_i) \left\{ \frac{f_0''(x)}{f_0'(x)} + \frac{1}{x-b} \right\}.$$

If the smallest value of Σ is positive, then also all values of Σ are positive and the sign of the expression

$$\left(\frac{1}{x-\eta_1} + \frac{1}{x-\eta_2} + \cdots + \frac{1}{x-\eta_n} \right) \varphi(x)$$

is the opposite of the sign of $f_0'(x)$; in addition, surely,

$$|f'(x)| < |f_0'(x)|.$$

But if the smallest value of Σ is negative, then the undetermined numbers

$$\eta_1, \eta_2, \ldots, \eta_n,$$

can be so chosen that $|f'(x)|$ exceeds $|f_0'(x)|$

From this we conclude that the biggest value of $|f'(x)|$ equals the biggest value of $|f_0'(x)|$ if and only if x lies outside the bound a and b, or else

$$a < x < b, \quad \frac{f_0''(x)}{f_0'(x)} + \frac{1}{x-a} > 0, \quad \text{and} \quad \frac{f_0''(x)}{f_0'(x)} + \frac{1}{x-b} < 0. \tag{2}$$

Instead of the rational expressions

$$\frac{f_0''(x)}{f_0'(x)} + \frac{1}{x-a} \quad \text{and} \quad \frac{f_0''(x)}{f_0'(x)} + \frac{1}{x-b}$$

it is possible to consider

$$(x-a)f_0''(x) + f_0'(x) \quad \text{and} \quad (x-b)f_0''(x) + f_0'(x),$$

since, firstly, in obedience to the inequality (2), the expressions

$$(x-a)f_0''(x) + f_0'(x) \quad \text{and} \quad (x-b)f_0''(x) + f_0'(x) \tag{3}$$

have the same sign and, secondly, our inequalities (2) hold in case the signs of the expressions (3) are the same and $a < x < b$.

Having considered in this fashion the case

$$y = f_0(z),$$

we turn to the others.

If y is not $= f_0(z)$, then, by the above, the sequence of ratios

$$\frac{y(\alpha_2)}{y(\alpha_1)}, \frac{y(\alpha_3)}{y(\alpha_2)}, \ldots, \frac{y(\alpha_s)}{y(\alpha_{s-1})}$$

contains $n-1$ numbers equaling -1. Also, $s=n$ and, of the two differences

$$\alpha_1 - a, \quad b - \alpha_n$$

at least one must be zero.

We take one of the functions $f(z)$ that satisfy our conditions.

The equation

$$f(z) - y = 0$$

of nth or lower degree in z has a root

$$\text{between } \alpha_1 \text{ and } \alpha_2, \text{between } \alpha_2 \text{ and } \alpha_3, \ldots, \text{between } \alpha_{n-1} \text{ and } \alpha_n.$$

In other words, the difference $f(z) - y$ must decompose into real factors of first degree in z:

$$f(z) - y = \psi(z) = (qz - r)(z - \eta_1)(z - \eta_2)\cdots(z - \eta_{n-1})$$

where

$$\alpha_1 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \alpha_{n-1} \leq \eta_{n-1} \leq \alpha_n, \quad \frac{r}{q} \geq \alpha_n \text{ or } \leq \alpha_1.$$

Also, we have

$$f'(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)_{z=x} + \left(\frac{1}{x-\eta_1} + \frac{1}{x-\eta_2} + \cdots + \frac{1}{x-\eta_{n-1}} + \frac{1}{x-\eta_n}\right)\psi(x),$$

with $\eta_n = \frac{r}{q}$.

It is not hard to see also that the sign of the difference

$$qz - r$$

is the opposite of the sign of $y(\alpha_n)$ for all values of z that lie between α_1 and α_n.

Under the conditions pointed out above, the numbers

$$\eta_1, \eta_2, \ldots, \eta_n,$$

can be given arbitrary values, subject only to the condition that q be sufficiently small.

We assume, to begin with, that x is bigger than α_n.

Then, for $\eta_n > x$, we get the inequalities

$$0 < \frac{1}{x-\eta_1} + \frac{1}{x-\eta_2} + \cdots + \frac{1}{x-\eta_{n-1}} < \frac{1}{x-\alpha_2} + \cdots + \frac{1}{x-\alpha_n}$$

$$0 > \frac{1}{x-\eta_n} > -\infty$$

and the undeterminedness of the numbers

$$\eta_1, \eta_2, \ldots, \eta_n$$

can be made use of in such a way that the quantity

$$\left(\frac{1}{x-\eta_1} + \frac{1}{x-\eta_2} + \cdots + \frac{1}{x-\eta_n}\right)\psi(x)$$

will have an arbitrary sign.

It follows that the case

$$x > \alpha_n$$

is impossible.

One shows similarly that the case $x < \alpha_1$ is impossible.

So assume that x lies between α_i and α_{i+1}.

Then, the sign of

$$\frac{\psi(x)}{x-\eta_i}$$

is the opposite of the sign of

$$(-1)^{n-i-1} y(\alpha_n)$$

and so, in order for $f'(x)$ to be, in absolute value, smaller than $\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)_{z=x}$, the sign of the sum

$$\frac{x-\eta_i}{x-\eta_1} + \frac{x-\eta_i}{x-\eta_2} + \cdots + \frac{x-\eta_i}{x-\eta_i} + \cdots + \frac{x-\eta_i}{x-\eta_n}$$

must be the same as the sign of

$$(-1)^{n-i-1} y(\alpha_n) \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)_{z=x}.$$

Now, the expression

$$(-1)^{n-i-1} y(\alpha_n) \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)_{z=x}$$

is a positive number, hence the sign of

$$(-1)^{n-i-1} y(\alpha_n)$$

is the same as that of $y(\alpha_{i+1})$ and of $\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)_{z=x}$.

On the other hand, it is not hard to see that the smallest value of the sum

$$\frac{x-\eta_i}{x-\eta_1} + \frac{x-\eta_i}{x-\eta_2} + \cdots + \frac{x-\eta_i}{x-\eta_{i-1}} + \frac{x-\eta_i}{x-\eta_i} + \cdots + \frac{x-\eta_i}{x-\eta_{n-1}} + \frac{x-\eta_i}{x-\eta_n}$$

equals the smaller of the numbers

$$(x-\alpha_i)\left\{\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \cdots + \frac{1}{x-\alpha_i} + \cdots + \frac{1}{x-\alpha_{n-1}} + \frac{1}{x-\alpha_n}\right\},$$

$$(x-\alpha_{i+1})\left\{\frac{1}{x-\alpha_2} + \frac{1}{x-\alpha_3} + \cdots + \frac{1}{x-\alpha_{i+1}} + \cdots + \frac{1}{x-\alpha_n} + \frac{1}{x-\alpha_1}\right\}$$

and therefore cannot be greater nor less than zero.

For that reason, we arrive at the following condition

$$\frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \cdots + \frac{1}{x-\alpha_{n-1}} + \frac{1}{x-\alpha_n} = 0. \tag{4}$$

Our considerations also show that, excluding the case that simultaneously

$$n = 2,\ \alpha_1 = a,\ \alpha_n = b,\ x = \frac{a+b}{2},$$

the derivative $f'(x)$ takes its biggest absolute value only for two functions $f(z)$ and these two only differ by a sign.

But if

$$n = 2 \quad \text{and} \quad x = \frac{a+b}{2},$$

then the biggest absolute value of $f'(x)$ equals $\frac{2L}{b-a}$ and there exists an infinite set of different functions $f(z)$: namely, all functions of the form

$$L\left\{\frac{2z-a-b}{b-a}+q(z-a)(z-b)\right\}$$

with

$$-\frac{2}{(b-a)^2}<q<\frac{2}{(b-a)^2}.$$

We recall that of the two differences

$$\alpha_1-a,\ b-\alpha_n$$

at least one must be zero, and correspondingly distinguish three cases:

$$1)\ \alpha_1=a,\ \alpha_n<b;\quad 2)\ \alpha_1>a,\ \alpha_n=b;\quad 3)\ \alpha_1=a,\ \alpha_n=b.$$

If

$$\alpha_1=a\ \text{ and }\ \alpha_n<b,$$

then we can adjoin to the numbers

$$\alpha_1,\alpha_2,\ldots,\alpha_n$$

also some number

$$\alpha_{n+1}$$

which is bigger than b and satisfies the condition

$$y(\alpha_{n+1})=-y(\alpha_n),$$

since, as z increases continuously from α_n to $+\infty$, the ratio

$$\frac{-y}{y(\alpha_n)}$$

also changes continuously from -1 to $+\infty$.

Therefore we have

$$y=\pm L\cos n\arccos\frac{2z-a-\alpha_{n+1}}{\alpha_{n+1}-a}=\pm f_1(z).$$

The unknown α_{n+1}, in accordance with equation (4), must satisfy the equation

$$\sum_{i=1,2,\ldots,n}\frac{1}{x-\frac{a+\alpha_{n+1}}{2}-\frac{\alpha_{n+1}-a}{2}\cos\frac{i\pi}{n}}=0,\quad\text{i.e.,}\quad\frac{f_1''(x)}{f_1'(x)}+\frac{1}{x-a}=0$$

and also the inequalities

$$\alpha_{n+1}>b>\frac{a+\alpha_{n+1}}{2}+\frac{\alpha_{n+1}-a}{2}\cos\frac{\pi}{n},$$

whence

$$\frac{b-a\sin^2\frac{\pi}{2n}}{\cos^2\frac{\pi}{2n}}>\alpha_{n+1}>b.$$

Therefore, for the case

$$\alpha_1=a,\ \alpha_n<b$$

really to occur, one of the values α_{n+1} satisfying the equation

$$(x-a)f_1''(x)+f_1'(x)=0 \tag{5}$$

must lie between

$$\frac{b - a\sin^2\frac{\pi}{2n}}{\cos^2\frac{\pi}{2n}} \quad \text{and} \quad b.$$

And only one, since in the contrary case the sought-for biggest value of $f'(x)$ would be taken on by several different functions $f(z)$, but the preceding considerations show this to be impossible.

Considering then the sum

$$\sum_{i=1,2,\ldots,n} \frac{1}{x - \frac{a+\alpha_{n+1}}{2} - \frac{\alpha_{n+1}-a}{2}\cos\frac{i\pi}{n}}$$

as a function of α_{n+1}, we note this function increases continuously with α_{n+1} with the exception of those values α_{n+1} at which it is infinite.

Therefore, equation (5) cannot have multiple roots.

From this, it is not hard to conclude that the case

$$\alpha_1 = a,\ \alpha_n < b$$

occurs exactly when during the passage of α_{n+1} from b to $\frac{b-a\sin^2\frac{\pi}{2n}}{\cos^2\frac{\pi}{2n}}$ the expression

$$(x-a)f_1''(x) + f_1'(x)$$

changes its sign.

We also note that, for $\alpha_{n+1} = b$, the expression

$$(x-a)f_1''(x) + f_1'(x)$$

reduces to

$$(x-a)f_0''(x) + f_0'(x).$$

In exactly the same way, with the introduction of the variable α_0 and taking

$$L\cos n \arccos\frac{2z - \alpha_0 - b}{b - \alpha_0} = f_2(z),$$

we see that the case

$$\alpha_1 > a,\ \alpha_n = b$$

occurs exactly when during the passage of α_0 from $\frac{a-b\sin^2\frac{\pi}{2n}}{\cos^2\frac{\pi}{2n}}$ to a the expression

$$(x-b)f_2''(x) + f_2'(x)$$

changes its sign.

Then

$$y = \pm f_2(z),$$

where α_0 must satisfy the equation

$$(x-b)f_2''(x) + f_2'(x) = 0$$

and the inequalities

$$\alpha_0 < a < \frac{\alpha_0 + b}{2} + \frac{b - \alpha_0}{2}\cos\frac{(n-1)\pi}{n}.$$

We now address the case

$$\alpha_1 = a,\ \alpha_n = b,$$

which occurs exactly when neither of the preceding cases occurs.

If

$$\alpha_1 = a,\ \alpha_n = b,$$

then the equation

$$\frac{\mathrm{d}y}{\mathrm{d}z} = 0$$

of $(n-1)$st degree in z has the $n-2$ roots

$$\alpha_2, \alpha_3, \ldots, \alpha_{n-1}$$

between a and b and one root outside these bounds.

We denote this last root by the letter β and assume for definiteness that $\beta > b$.

In this case, $|y|$, as z moves from b to β, grows but, as z grows even larger, first diminishes to zero and then grows without bound.

Also, the equation

$$y^2 - L^2 = 0$$

of $2n$th degree in z certainly has the $n-2$ double roots

$$\alpha_2, \alpha_3, \ldots, \alpha_{n-1}$$

and the two simple roots

$$a, b$$

and also two roots which we will denote by the letters

$$\gamma \text{ and } \delta.$$

These last two roots are bigger than β.

It follows that

$$y^2 - L^2 = p_0^2(z-\alpha_2)^2(z-\alpha_3)^2 \cdots (z-\alpha_{n-1})^2(z-a)(z-b)(z-\gamma)(z-\delta)$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}z} = np_0(z-\alpha_2)(z-\alpha_3)\cdots(z-\alpha_{n-1})(z-\beta),$$

from which we derive the first order differential equation

$$y^2 - L^2 = \frac{(z-a)(z-b)(z-\gamma)(z-\delta)}{n^2(z-\beta)^2}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2. \tag{6}$$

E. I. Zolotarev, in his work “The application of elliptic functions to questions concerning functions that deviate least and most from zero”, expressed the solution of that last equation in terms of elliptic functions.

Without relying on E. I. Zolotarev’s formulas, we show how it is possible to reduce our problem to three algebraic equations.

For this, we obtain from the equation (6) by differentiation

$$\begin{aligned} n^2(z-\beta)^3 y &= (z-a)(z-b)(z-\gamma)(z-\delta)y'' \\ &+ \frac{1}{2}(z-a)(z-b)(z-\gamma)(z-\delta)(z-\beta)\left\{\frac{1}{z-a} + \cdots + \frac{1}{z-\delta} - \frac{2}{z-\beta}\right\} y'. \end{aligned} \tag{7}$$

Taking now

$$y = p_0(z-\beta)^n + p_1'(z-\beta)^{n-1} + \cdots + p_{n-2}'(z-\beta)^2 + p_n'$$

and rewriting equation (7) in powers of $z-\beta$ and comparing coefficients, we arrive at a system of $n+1$ equations in the $n+2$ unknowns

$$\frac{p_1'}{p_0}, \frac{p_2'}{p_0}, \ldots, \frac{p_{n-2}'}{p_0}, \frac{p_n'}{p_0}, \beta, \gamma, \delta.$$

It is not hard to eliminate the unknowns

$$\frac{p'_1}{p_0}, \frac{p'_2}{p_0}, \ldots, \frac{p'_{n-2}}{p_0}, \frac{p'_n}{p_0}$$

which depend linearly on the remaining three

$$\beta, \gamma, \delta.$$

By eliminating

$$\frac{p'_1}{p_0}, \frac{p'_2}{p_0}, \ldots, \frac{p'_{n-2}}{p_0}, \frac{p'_n}{p_0}$$

we arrive at two algebraic equations for the unknowns

$$\beta, \gamma, \delta.$$

Now, condition (4) gives the third equation

$$\left(\frac{y''}{y'}\right)_{z=x} + \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-\beta} = 0. \tag{8}$$

As concerns the coefficient p_0, it is determined from the condition

$$y(a) = \pm L.$$

We arrive at the same results also in case β is less than a except that, for $\beta < a$, γ and δ must be less than β.

For what is to follow it is important to notice that in each case the expression

$$\frac{(z-\gamma)(z-\delta)}{(z-\beta)^2}$$

is greater than unity for all values of z lying between a and b.

We now show that equation (6) may be transformed into two first-order linear differential equations with two unknown polynomials.

Here, for definiteness, we'll take

$$y(a) = L; \quad a < b < \beta < \gamma < \delta.$$

Let n be even.

Then, denoting the products

$$(z-\alpha_2)(z-\alpha_4)\cdots(z-\alpha_{n-2}) \text{ and } (z-\alpha_3)(z-\alpha_5)\cdots(z-\alpha_{n-1})$$

respectively by

$$U \text{ and } V,$$

we obtain

$$\begin{aligned} y - L &= p_0(z-a)(z-\delta)V^2 \\ y + L &= p_0(z-b)(z-\gamma)U^2 \\ y' &= p_0\{2(z-a)(z-\delta)V' + (2z-a-\delta)V\}V \\ &= p_0\{2(z-b)(z-\gamma)U' + (2z-b-\gamma)U\}U \\ &= np_0(z-\beta)UV \end{aligned}$$

and thereby arrive at the desired two linear first-order differential equations

$$2(z-a)(z-\delta)V' + (2z-a-\delta)V = n(z-\beta)U$$

$$2(z-b)(z-\gamma)U' + (2z-b-\gamma)U = n(z-\beta)V.$$

In similar fashion, for odd n, denoting the products

$$(z-\alpha_2)(z-\alpha_4)\cdots(z-\alpha_{n-1}) \quad \text{and} \quad (z-\alpha_3)(z-\alpha_5)\cdots(z-\alpha_{n-2})$$

respectively by

$$U \quad \text{and} \quad V,$$

we obtain

$$2(z-a)(z-b)(z-\gamma)V' + \{3z^2 + 2(a+b+\gamma)z + ab + a\gamma + b\gamma\}V = n(z-\beta)U$$

$$2(z-\delta)U' + U = n(z-\beta)V.$$

Examples.

I $n = 2$.

In this case,

$$f_0(z) = \frac{L}{(b-a)^2}\left\{8(z-a)(z-b) + (b-a)^2\right\},$$

$$f_0'(z) = \frac{8L}{(b-a)^2}(2z-a-b), \quad f_0'' = \frac{16L}{(b-a)^2}$$

$$(x-a)f_0''(x) + f_0'(x) = \frac{8L}{(b-a)^2}(4x-3a-b)$$

$$(x-b)f_0''(x) + f_0'(x) = \frac{8L}{(b-a)^2}(4x-3b-a).$$

Hence, for

$$x > \frac{3b+a}{4} \quad \text{or} \quad x < \frac{3a+b}{4},$$

the greatest value of $|f'(x)|$ equals the absolute value of

$$f_0'(x) = \frac{8L}{(b-a)^2}(2x-a-b).$$

Turning now also to the functions $f_1(z)$ and $f_2(z)$, we find

$$f_1(z) = \frac{L}{(\alpha_3-a)^2}\left\{8(z-a)(z-\alpha_3) + (\alpha_3-a)^2\right\},$$

$$(x-a)f_1''(x) + f_1'(x) = \frac{8L}{(\alpha_3-a)^2}(4x-3a-\alpha_3)$$

$$f_2(z) = \frac{L}{(b-\alpha_0)^2}\left\{8(z-\alpha_0)(z-b) + (b-\alpha_0)^2\right\},$$

$$(x-b)f_2''(x) + f_2'(x) = \frac{8L}{(b-\alpha_0)^2}(4x-3b-\alpha_0).$$

$$\alpha_3 = 4x-3a, \quad \alpha_0 = 4x-3b$$

from which we conclude that the absolutely largest value of $f'(x)$

$$\text{for} \quad \tfrac{3a+b}{4} < x < \tfrac{a+b}{2} \quad \text{equals} \quad \tfrac{-8L}{(\alpha_3-a)^2}(2x-\alpha_3-a) = \tfrac{L}{x-a}$$

$$\text{for} \quad \tfrac{a+b}{2} < x < \tfrac{3b+a}{4} \quad \text{equals} \quad \tfrac{8L}{(b-\alpha_0)^2}(2x-\alpha_0-b) = \tfrac{L}{b-x}.$$

As to the function y determined by the differential equation (6), for $n = 2$ it plays no role in our question.

II $\quad n = 3$.

Taking for simplicity of the results

$$a = -1 \text{ and } b = +1,$$

we find

$$f_0(z) = L(4z^3 - 3z), \quad f_0'(z) = 3L(4z^2 - 1), \quad f_0''(z) = 24Lz$$

$$(x-a)f_0''(x) + f_0'(x) = 3L(12x^2 + 8x - 1) = 36L(x-\omega_1)(x-\omega_2)$$

$$(x-b)f_0''(x) + f_0'(x) = 3L(12x^2 - 8x - 1) = 36L(x-\omega')(x-\omega''),$$

where

$$\omega_1 = \frac{-2-\sqrt{7}}{6} < \omega' = \frac{2-\sqrt{7}}{6} < \omega_2 = \frac{-2+\sqrt{7}}{6} < \omega'' = \frac{2+\sqrt{7}}{6}.$$

Hence, for

$$x < \omega_1, \quad \omega' < x < \omega_2 \text{ or } x > \omega'',$$

the absolutely largest value of $f'(x)$ equals the absolute value of

$$f_0'(x) = 3L(4x^2 - 1).$$

Turning now to the functions $f_1(z)$ and $f_2(z)$, we find

$$f_1(z) = L\left\{4\left(\frac{2z-1-\alpha_4}{\alpha_4+1}\right)^3 - 3\frac{2z+1-\alpha_4}{\alpha_4+1}\right\}$$

$$(x-a)f_1''(x) + f_1'(x) = \frac{6L}{(\alpha_4+1)^3}\left[16(2x+1-\alpha_4)(x+1) + 4(2x+1-\alpha_4)^2 - (\alpha_4+1)^2\right]$$

$$(x-b)f_2''(x) + f_2'(x) = \frac{6L}{(1-\alpha_0)^3}\left\{16(2x-1-\alpha_0)(x-1) + 4(2x-1-\alpha_0)^2 - (1-\alpha_0)^2\right\}$$

The expression

$$16(2x+1-\alpha_4)(x+1) + 4(2x+1-\alpha_4)^2 - (\alpha_4+1)^2$$

for $\alpha_4 = 1$ turns into

$$48x^3 + 32x - 4 = 48(x-\omega_1)(x-\omega_2),$$

and for $\alpha_4 = \frac{1+\sin^2\frac{\pi}{6}}{\cos^2\frac{\pi}{6}} = \frac{5}{3}$, it turns into

$$\begin{aligned} 32(x-\frac{1}{3})(x+1) + 16(x-\frac{1}{3})^2 - \frac{64}{9} &= 48x^2 + \frac{32}{3}x - 16 = \\ &= 48(x-\epsilon_1)(x-\epsilon_2), \end{aligned}$$

where

$$\epsilon_1 = \frac{-1-\sqrt{28}}{9} \text{ and } \epsilon_2 = \frac{-1+\sqrt{28}}{9}.$$

From this we conclude that the biggest absolute value of $f'(x)$ equals the absolute value of $f_1'(x)$ in those cases when $\omega_1 < x < \epsilon_1$ or $\omega_2 < x < \epsilon_2$.

Here, the number α_4 must be determined from the equation

$$16(2x+1-\alpha_4)(x+1) + 4(2x+1-\alpha_4)^2 - (\alpha_4+1)^2 = 0.$$

In order to give the expression $f_1'(x)$ a possibly simple form, we take

$$\alpha_4 = -1 + \xi \quad \text{and} \quad x + 1 = t.$$

Then

$$f_1'(x) = -t f_1''(x)$$

$$f_1''(x) = \frac{4.3.2.2^2(2t-\xi)}{\xi^3} L = 96 \left\{ 2\left(\frac{t}{\xi}\right)^3 - \left(\frac{t}{\xi}\right)^2 \right\} \frac{L}{t^2}$$

$$48t^2 - 32t\xi + 3\xi^2 = 0, \quad \frac{t}{\xi} = \frac{4 \pm \sqrt{7}}{12}$$

$$2\left(\frac{t}{\xi}\right)^3 - \left(\frac{t}{\xi}\right)^2 = \frac{10 \pm 7\sqrt{7}}{144.6}$$

$$f'(x) = -\frac{10 \pm 7\sqrt{7}}{9} \frac{L}{x+1}.$$

Of the two signs $\pm$ for $\sqrt{7}$ one must take the one for which

$$\alpha_4 = -1 + \frac{12}{4 \pm \sqrt{7}}(x+1)$$

lies between 1 and $\frac{5}{3}$.

And the inequalities

$$\frac{5}{3} > \alpha_4 > 1$$

are equivalent to

$$\frac{-1 \pm \sqrt{28}}{9} > x > \frac{-2 \pm \sqrt{7}}{6}.$$

Comparing these last inequalities with the ones found earlier

$$\omega_1 < x < \epsilon_1 \quad \text{or} \quad \omega_2 < x < \epsilon_2,$$

we see that the absolutely largest value of $f'(x)$

$$\begin{array}{rll} \text{for} & \omega_1 < x < \epsilon_1 & \text{equals} \quad \frac{7\sqrt{7}-10}{9}\frac{L}{x+1} \\ \text{and for} & \omega_2 < x < \epsilon_2 & \text{equals} \quad \frac{7\sqrt{7}+10}{9}\frac{L}{x+1}. \end{array}$$

Similarly, taking

$$\frac{1-\sqrt{28}}{9} = \epsilon' \quad \text{and} \quad \frac{1+\sqrt{28}}{9} = \epsilon''$$

we find that the absolutely largest value of $f'(x)$

$$\begin{array}{rll} \text{for} & \epsilon'' < x < \omega'' & \text{equals} \quad \frac{7\sqrt{7}-10}{9}\frac{L}{1-x} \\ \text{but for} & \epsilon' < x < \omega' & \text{equals} \quad \frac{7\sqrt{7}+10}{9}\frac{L}{1-x}. \end{array}$$

If now x lies

$$\text{between } \epsilon_1 \text{ and } \epsilon' \quad \text{or} \quad \text{between } \epsilon_2 \text{ and } \epsilon''$$

then the biggest absolute value of $f'(x)$ coincides with that of the function y which is determined by the equations (6) and (8) for $n = 3$, $a = -1$, $b = +1$.

In our example, the differential equation (6) can be changed into the two equations

$$y - L = p_0(z^2 - 1)(z - \gamma), \quad y - L = p_0(z^2 - \alpha_2)^2(z - \delta),$$

from which we deduce that

$$\gamma = \delta + 2\alpha_2, \quad -1 = \alpha_2^2 + 2\alpha_2\delta, \quad -2L = p_0(\alpha_2^2\delta + \gamma)$$

$$\delta = -\frac{1 + \alpha_2^2}{2\alpha_2}, \quad \gamma = \frac{3\alpha_2^2 - 1}{2\alpha_2}, \quad p_0 = \frac{4\alpha_2 L}{(1 - \alpha_2^2)^2}.$$

Now, equation (8) becomes

$$\frac{1}{x - \alpha_2} + \frac{1}{x + 1} + \frac{1}{x - 1} = 0.$$

Hence

$$x - \alpha_2 = \frac{1 - x^2}{2x}, \quad \alpha_2 = \frac{3x^2 - 1}{2x}$$

$$1 - \alpha_2 = \frac{1 + 2x - 3x^2}{2x} = \frac{(1 - x)(1 + 3x)}{2x}, \quad 1 - \gamma = \frac{(1 - \alpha_2)(1 + 3\alpha_2)}{2\alpha_2}$$

$$1 + \alpha_2 = \frac{3x^2 + 2x - 1}{2x} = \frac{(1 + x)(3x - 1)}{2x}, \quad 1 + \gamma = \frac{(1 + \alpha_2)(3\alpha_2 - 1)}{3\alpha_2}$$

$$1 + 3\alpha_2 = \frac{9x^2 + 2x - 3}{2x} = \frac{9(x - \epsilon_1)(x - \epsilon_2)}{2x}$$

$$3\alpha_2 - 1 = \frac{9x^2 - 2x - 3}{2x} = \frac{9(x - \epsilon')(x - \epsilon'')}{2x}$$

$$\begin{aligned}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)_{z=x} &= \frac{4\alpha_2 L}{(1 - \alpha_2^2)^2}\left\{3x^2 - \frac{3\alpha_2^2 - 1}{\alpha_2}x - 1\right\} = \frac{4(x - \alpha_2)(3x\alpha_2 + 1)}{(1 - \alpha_2^2)^2}L \\ &= -\frac{16x^3 L}{(1 - 9x^2)(1 - x^2)}.\end{aligned}$$

Now it is not hard to see that, for

$$\epsilon_1 < x < \epsilon' \text{ or } \epsilon_2 < x < \epsilon'',$$

the function y constructed by us satisfies all the aforementioned conditions and the absolutely largest value of $f'(x)$ equals the absolute value of

$$\frac{16x^3 L}{(1 - 9x^2)(1 - x^2)}.$$

Problem No. 2

To find the biggest absolute value of $f'(x)$ for all x lying between a and b.

Solution.

In solving the previous problem, we found all those functions $f(z)$ for which $f'(x)$ takes on its largest absolute value.

One of our results is the fact that, for

$$\frac{(x-b)f_0''(x)+f_0'(x)}{(x-a)f_0''(x)+f_0'(x)}>0,$$

the absolutely largest value of $f'(x)$ equals

$$|f_0'(x)| = |\frac{nL\sin n\,\operatorname{arc}\cos\frac{2x-a-b}{b-a}}{\sqrt{(x-a)(b-x)}}|.$$

Assuming now

$$x=\frac{a+b}{2}+\frac{b-a}{2}\cos\varphi,$$

we find that

$$f_0(x)=L\cos n\varphi,\quad f_0'(x)=\frac{2nL\sin n\varphi}{(b-a)\sin\varphi},$$

$$f_0''(x)=\frac{4nL\{\sin n\varphi\cos\varphi-n\cos n\varphi\sin\varphi\}}{(b-a)^2\sin^2\varphi},$$

$$\frac{(x-b)f_0''(x)+f_0'(x)}{(x-a)f_0''(x)+f_0'(x)}=\frac{1-\cos\varphi}{1+\cos\varphi}\frac{\sin n\varphi+n\cos n\varphi\sin\varphi}{\sin n\varphi-n\cos n\varphi\sin\varphi}.$$

If $0<\varphi<\frac{\pi}{2n}$ or $\pi>\varphi>\pi-\frac{\pi}{2n}$, then

$$|\sin n\varphi|>|n\cos n\varphi\sin\varphi|$$

and

$$\frac{(x-b)f_0''(x)+f_0'(x)}{(x-a)f_0''(x)+f_0'(x)}>0.$$

On the other hand, from the formula

$$f_0'(x)=\frac{2nL\sin n\varphi}{(b-a)\sin\varphi},$$

it is evident that, for $a\le x\le b$, the largest absolute value of $f_0'(x)$ equals

$$\frac{2n^2L}{b-a}$$

and occurs when $x=a$ and $x=b$.

Therefore, for all values of x lying

$$\text{between } a \text{ and } \frac{a+b}{2}-\frac{b-a}{2}\cos\frac{\pi}{2n} \quad\text{or}\quad \text{between } \frac{a+b}{2}+\frac{b-a}{2}\cos\frac{\pi}{2n} \text{ and } b,$$

the absolutely largest value of $f'(x)$ equals

$$\frac{2n^2L}{b-a}.$$

We assume now that x lies between

$$\frac{a+b}{2}-\frac{b-a}{2}\cos\frac{\pi}{2n}\ \text{ and }\ \frac{a+b}{2}+\frac{b-a}{2}\cos\frac{\pi}{2n}.$$

In this case,

$$(x-a)(b-x) = \left(\frac{b-a}{2}\right)^2 - \left(\frac{b+a}{2} - x\right)^2 > \left(\frac{b-a}{2}\right)^2 \sin^2\frac{\pi}{2n} > \left(\frac{b-a}{2}\right)^2 \frac{1}{n^2}.$$

The derivative $f'(x)$ takes on its absolutely largest value at one of the aforementioned functions

$$f_0(x), f_1(x), f_2(x)$$

or for the function y that satisfies the differential equation (6).

But, by the above observed

$$|f_0'(x)| < \frac{2n^2L}{b-a},$$

and, in the same way, we see that

$$|f_1'(x)| < \frac{2n^2L}{\alpha_{n+1}-a} < \frac{2n^2L}{b-a}$$

and

$$|f_1'(x)| < \frac{2n^2L}{b-\alpha_0} < \frac{2n^2L}{b-a}.$$

Also, from equation (6) and for

$$\frac{a+b}{2} - \frac{b-a}{2}\cos\frac{\pi}{2n} < x < \frac{a+b}{2} + \frac{b-a}{2}\cos\frac{\pi}{2n},$$

the inequality

$$\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2_{z=x} < \frac{n^2}{(x-a)(b-x)}L^2 < \frac{4n^4}{(b-a)^2}L^2$$

results, and therefore

$$|\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)_{z=x}| < \frac{2n^2L}{b-a}.$$

All these results show that the sought-for biggest absolute value of $f'(x)$ equals

$$\frac{2n^2L}{b-a}.$$