A brief account of my life and work

This account is not as short as I wanted to make it due to my long life while my life and my work cannot be separated. I belong to a generation which was too young for active duty in the First World War and too old for the Second. To give the story some structure I divide it into significant periods,

1. Romania 1903–1922. I was born on April 21, 1903, in Galatz, Romania, as the youngest of four children. My father, trained as an accountant, inherited the lumber business of my grandfather Isaac Segal, on my mother's side, but my father was not a good business man. The fortunes of the family improved in 1910 when we moved to Jassy, Romania, where my father eventually became Sub-Director (Vice President) of the newly founded "Banca Moldova". The building of the Banca Moldova, at present a main Post Office of Jassy, was new in 1910, and our family occupied one-half of its second floor. We four children enjoyed the cultural interests of our parents, especially of mother's who wrote and spoke French fluently. My father was an Austro-Hungarian subject and Consul in Jassy of the Dual Monarchy. This caused difficulties during the First World War after Romania declared war on the Axis powers. After the war we became Romanian citizens.

My parents were devoted Zionists and my mother, Rachel Segal Schoenberg, was much in demand as a public speaker at Zionist meetings; she was also a fine Romanian poetess. My father Jacob Schoenberg (1864–1930) was more practically oriented and was active in establishing agricultural stations for the training of young Jewish boys and girls which were to go as farmers to the settlements in Palestine. We frequently had as house guests prominent Zionist personalities such as Ussishkin, Sokoloff, Wilensky, Colonel Wedgwood and others. In 1926 my older sister Elsa married Ussishkin's son Sama and still lives in Jerusalem as his widow.

At the age of 12 I became interested in Physics, reading mostly French textbooks. However, I soon switched to Mathematics studying by myself the French textbooks of the Frères Jesuits; also the series of textbooks founded by Hadamard of which he wrote the Geometrie. I had plenty of time for these studies, because for the last two years of High School I prepared myself at home, as was possible at the time, taking an examination at the end of the year, and obtained the High School Diploma (Baccalaureate) in 1919. In my mathematical studies I was much helped by Titu Patriciu, a good mathematician who taught High School (Liceu).

In 1919 I started my studies at the University of Jassy, founded in 1860, by attending, during 1919–1920, the course in Projective and Descriptive Geometry given by Victor Costin. This was followed during 1920–22 by the year-long courses leading to the Licenţa (M.A.) in Mathematics: Analytic Geometry (Alexandru Myller), Algebra Including Galois Theory (S. Stoilov), Analysis (Simion Sanielevici), Theory of Analytic Functions (Vera Myller - Lebedeff), Mechanics (S. Sanielevici), Astronomy (Constantin Popovici). Also outside of this program I had courses in Differential Geometry (A. Myller) and Differential Equations, in particular the theory of Fuchs (S. Sanielevici). The two Myllers were students of Hilbert, while Sanielevici was a student of Picard. I dedicated to these mathematicians my little book [**S83**]. In June 1922 I received the M.A. degree in Mathematics (very good with distinction).

2. **Germany and Romania 1922–1930.** In the fall of 1922 my father sent me to

Germany. There I spent one semester in Göttingen, then three semesters in Berlin, and finally I went back to Göttingen for two semesters. One attraction of Berlin was that my sister Irma (1902–1984) was studying the piano there.

During the three years (1922–1925) spent in Germany, I attended courses given by Edmund Landau (Entire Functions, Trigonometric Series, The Big Fermat Problem, Analytic Number Theory), and Issai Schur (Algebra, Number Theory, Analytic Number Theory), and A. Ostrowski (Overconvergence of Power Series); also a Problem Seminar offered by Ostrowski and K. Granjot.

The idea for my Ph.D. thesis was suggested by Schur's course in Analytic Number Theory. He had discussed H. Weyl's theory [**Wy**] of uniform distribution of numbers mod 1 of 1916. Schur gave some new asymptotic properties of Euler's $\phi(n)$ function which suggested to me the more general theory of non-uniform distribution of numbers mod.l. This is what I initiated in my Ph.D. thesis [**1***]. Following a general theory, the main result concerned the asymptotic distribution function $F(x)$ in $[0, 1]$ of the special sequence $x_n = \phi(n)/n$, $(n = 1, 2, ...)$. I showed that $F(x)$ is different from x and is continuous and increasing in $[0, 1]$. Later Paul Erdős [**Er**] showed that $F(x)$ is a singular function, hence $F'(x) = 0$ almost everywhere. This is probably the most natural example of a singular function in the sense of Lebesgue.

The Prussian Ministry did not recognize my High School Diploma and I could not obtain my Ph.D. degree in Germany. In 1925 I returned to Jassy becoming Assistant to Victor Costin. The following remark shows the overwhelming importance for us at the time of Western culture: During 1925–1930 on the way from Romania to Germany I passed at least half a dozen times the city of Krakow; not once did I stop there to look at this interesting ancient city. On the basis of [**1***] I obtained my Ph.D. degree at the University of Jassy in June 1926 under the direction of S. Sanielevici.

Schur liked my paper and in 1928 he said to me: "Herr Doktor, in dieser Sache sind Sie allen voraus; Sie müssen diesen Vorsprung ausnützen." (Herr Doktor, in this matter you are ahead of everybody, you must utilize this advantage). H. Davenport continued my work and there were further papers by S. Bochner and B. Jessen and also by B. Jessen and A. Wintner; for references see [**18***]. I finally wrote [**18***] using the Fourier Transform.

During 1926–1927 I served one year of military service required of high school graduates: One-half year at the Field Artillery School in Timiçoara (the old Hungarian Temeshvar) from which I graduated as a Corporal. The second half of that year I served in a (horse drawn) Field Artillery Regiment near Chişinau (the old Russian Kishinev) in Bessarabia. The year 1926–1927 was as peaceful as Europe experienced between the wars. In recollection and by old photographs my military service has some elements of musical comedy. As a lifelong vegetarian I could not eat at the officers' mess, but with my doctorate in Mathematics I was well treated.

The Hebrew University of Jerusalem was founded in 1925 and Edmund Landau, who was a great Zionist and had learnt Hebrew, was one of its founders. He arranged my visit to Jerusalem for the Spring Semester of 1928. There, at the Einstein Institute, I gave a course on "Higher Algebra" in which Benjamin Amira helped me with the language (Hebrew). From Heinrich Weber's book [**Wb**], pages 354–357, I learned about Felix Klein's note of 1892 on finding upper bounds for the number of real roots of an algebraic equation in a finite interval. Klein shows, for second degree equations only, that Jacobi's methods of using Descartes' rule of signs gives a bound which is less or equal to the bound furnished by the Budan-Fourier theorem. This result I generalized, to arbitrary degree in [**4***] and [**15***]. See also S. Karlin's book [**K**], pp. 316–318. This was my introduction to the important notions of Total Positivity and Variation Diminishing linear transformation which were to dominate many of my papers in later years.

In his doctoral dissertation [**M**] of 1933, written under the direction of A. Ostrowski, Th. Motzkin found the necessary and sufficient conditions for a linear transformation to be variation diminishing. My own derivation of these conditions is included in my work on the zeros of polynomials which I wrote up during the late thirties and later, in several papers. Besides [**4***] and [**15***], already mentioned, they are [**19**], [**42**] (jointly with Anne Whitney), [**52**] and [**53***]. Their main results are beautifully presented in Nicola Obreschkoff's excellent book [**O**].

In Jerusalem I met the older daughter Charlotte (Dolli) of Marianne and Edmund Landau and we were married in 1930 in Berlin.

3. **Chicago, Cambridge, Princeton, Waterville and Philadelphia 1930–1943.** For the year 1930–1931 I was granted a Rockefeller Fellowship and we traveled to Chicago where I was to study the Calculus of Variations under the direction of Gilbert Bliss. Professor Bliss also granted me an Assistantship for 1931–1932. In Chicago I wrote the seven papers from [**5**] to [**11**], some of them jointly with Professor Bliss. With the hindsight of over half a century it is clear that the significant work in this field was being done by Marston Morse in the Calculus of Variations in the large.

Our daughter Elizabeth was born in Chicago on September 17, 1931.

For 1932–1933 we went to Cambridge MA where I attended courses at Harvard (David Widder) and M.I.T. (Dirk Struik and Jesse Douglas). In the Fall of 1933 we went to Princeton NJ where I became a Fellow of the Institute for Advanced Studies, newly founded by Abraham Flexner. I also received a small stipend for helping with the publication of the Annals of Mathematics. Under the influence of Leonard M. Blumenthal I became interested in the Distance Geometry which had been initiated by Karl Menger in Vienna in the late twenties. Stimulated by a note of M. Fréchet in the Annals of Mathematics, I wrote the series of papers from [**22**] to [**30**] on isometric imbedding of metric spaces in Hilbert space and its connection with Analysis. The dominant notions are the *Positive Definite Functions* and *Isometric Imbedding.* Some of these results are sometimes quoted in the literature only because of their purely analytic or probabilistic connections, without mention of their geometric imbedding interpretations: As examples see A. M. Yaglom [**Y**], G. Choquet [**Ch**], N. I. Akhiezer [**A**]. My main papers on the subject are [**25***], [**26***], and the joint paper [**28***] with J. von Neumann.

I recently noticed with particular pleasure that after all these years the three authors C. Berg, J. Christensen and P. Ressel discuss my work in their recent book [**BCR**]. They simplify my theory and at the same time generalize it to semigroups.

This was the worst part of the Depression and teaching positions were hard to obtain. Oswald Veblen suggested to Arnold Dresden that I might teach at Swarthmore College while Dresden went on his sabbatical leave in January 1935. We then moved to Swarthmore PA where I was Acting Assistant Professor. On Dresden's return from Cambridge

University and Princeton in January 1936, I continued to teach at Swarthmore until June 1936. At that time Marston Morse, as a Colby College graduate, had recommended me for a position at Colby College.

We spent five enjoyable years (1936–1941) in Waterville ME. I usually drove by myself to Princeton for a few summer months. At Colby I taught also Mechanics and a non-technical Mathematics course open to any student. We made lifelong friends such as Mary Hatch Marshall, for many years Professor of English at Syracuse University. In 1937 our second daughter, Beatrice, was born on October 17.

In 1933 my father-in-law lost his Professorship in Göttingen due to the Nazis and moved to Berlin; there he died on February 14, 1938. His widow Marianne Landau (1886–1963), daughter of the Nobel Laureate Paul Ehrlich, the founder of Chemotherapy, left Berlin and joined our family in Waterville.

At Colby I continued working on Distance Geometry and its connection with Analysis resulting in papers -already mentioned above and in [**Bl**] and [**BCR**], see [**Bl**] on pages 132–136. My one-page note [**24***], "On the Peano curve of Lebesgue" was written while proctoring a two-hour examination in Mechanics in 1937.

In 1941 John Robert Kline offered me a position at the University of Pennsylvania where I was to remain until 1965, with the interruptions of sabbaticals and leaves.

4. **Aberdeen MD 1943–1946.** Several mathematicians, 'among them O. Veblen, Mina Rees and Leo Zippin, were concerned with making mathematicians available for the war effort. Leo Zippin, as a Corporal, followed the development of the first electronic computer, the ENIAC, at the Moore School of the University of Pennsylvania, which was to be moved to the Ballistics Research Laboratory (the BRL) in Aberdeen MD. Leo Zippin arranged my going to the BRL for the duration.

The morning in August 1943 of my reporting for duty, Major A. A. Bennett, of Brown University, then Chief of the Computing Branch of the BRL, told me what my particular problem was to be: Trajectories of projectiles were until then computed with desk calculators by hand. Into these computations entered tables of the drag-functions of air resistance, about 24 of them, which were obtained empirically by firings of various types of projectiles. As the step of integration used in these trajectory computations was rather large and the methods of numerical integrations fairly complicated, it did not much matter that the 4-place drag-function tables were rather rough. In performing these computations on the ENIAC, which was very fast, a much simpler integration method of very small step could be used. In these methods, the accumulation of the round-off errors was unacceptable due to the rough drag-function tables; they needed to be smoothed by being approximated by analytic functions. To do this was my problem.

I solved this problem by what later I called Cardinal Spline Interpolation and Cardinal Spline Smoothing. To insure the smoothness, to 8 decimal places, of the second derivative of the approximant of the drag-function, I used the heat-flow equation on the real axis to produce approximations regular on the entire real axis.

Only the fine resources of the BRL made this work possible. We had advanced (but not electronic) punched-card machines which were well suited for this work. A full account of it is given in my papers [**31***]and [**32***]. Equidistant splines were used before me, e.g., by C. Runge [**R**] (quadratic splines, 1901) and Quade and Collatz [**QC**] (arbitrary degree,

1938) for the approximation of the Fourier coefficients of periodic functions, but not for the approximation of unctions. From January 1945 to January 1946 I was Chief of the Punched-Card-Section of the Computing Branch.

In my work described above I was much helped by the actuarial work on so-called *Osculatory Interpolation*, and in particular by T. N. E. Greville's article [**G44**]. Greville's Selected Papers include the paper [**GV**] by Greville and Hubert Vaughan, which is a masterful exposition of the actuarial work on the subject. In [**31***] and' [**32***] I also had to solve a numerical problem on Laurent series whose solution, by Hans Rademacher and me, is given in [**33**].

5. **Philadelphia, Princeton, Madison 1946–1965.** In January 1946, the family moved back to Swarthmore, while I returned to the University of Pennsylvania where I remained, with some leaves and sabbaticals, until 1965. The beneficial effect of the work done in Aberdeen on my mathematical development soon became apparent. Work on a good applied problem becomes rewarding if one is prepared to follow purely mathematical tangents which suggest themselves.

In [**31***] and [**32***] I considered exclusively spline functions having equidistant knots. Haskel Curry recognized the possibility of defining splines with arbitrary non-equidistant knots and we wrote in 1946 essentially the first part of [**86***] which was only published in 1966. I wrote the second part of [**86***] which led to one of the four different characterizations of the so-called *Pólya frequency functions*.

In [**31***] I gave the conditions under which a convolution transformation

$$y_n = \sum_k a_{n-k} z_k \tag{1}$$

may be regarded as a smoothing formula; in the second part of [**35**] are given generating functions of totally positive sequences (a_n). A sequence (a_n) is called totally positive (tp) provided that the cyclic matrix

$$||a_{n-k}|| \quad (-\infty < n, k < \infty)$$

has only non-negative minors; see my invited address [**48***].

The continuous analogue of (1) is the convolution

$$g(x) = \int_{-\infty}^{\infty} \Lambda(x-y) f(y)\, \mathrm{d}y. \tag{2}$$

and the characterization of the variation diminishing transformation of the form (2) led to what is likely my best work: in [**43***] I solved a problem that was initiated by Pólya [**P**] in 1915. This problem was as follows: Let B2 denote the class of Laguerre-Pólya-Schur entire functions of the form

$$\phi(z) = \mathrm{e}^{-\gamma z^2} + \delta z \prod_{k=1}^{\infty} (1+\delta_k z) \mathrm{e}^{-\delta_k z}, \quad (0 < \gamma + \sum_{k=1}^{\infty} \delta_k^2 < \infty). \tag{3}$$

These were shown by Laguerre and Pólya to be the only possible limits of real polynomials $P(z)$, with $P(0) = 1$, having only real zeros. Pólya [**P**] and H. Hamburger [**H**] showed in 1915 and 1920, respectively, that the reciprocal

$$1/\phi(x)$$

is represented in its vertical strip of regularity containing the origin by the Stieltjes integral

$$\frac{1}{\phi(z)} = \int_{-\infty}^{\infty} \mathrm{e}^{-zx} \, \mathrm{d}\alpha(x), \tag{4}$$

where $\alpha(x)$ is bounded and non-decreasing. In [**43***] I showed that (4) may be replaced by

$$\frac{1}{\phi(z)} = \int_{-\infty}^{\infty} \mathrm{e}^{-zx} \Lambda(x) \, \mathrm{d}x, \tag{5}$$

where $\Lambda(x)$ is summable and *totally positive.* The last condition means that if

$$x_1 < x_2 < \cdots < x_n, \qquad y_1 < y_2 < \cdots < y_n, \quad (n = 1, 2, \ldots) \tag{6}$$

then

$$\det ||\Lambda(x_i - y_k)|| \ \geq \ 0. \tag{7}$$

I called such $\Lambda(x)$ *Pólya frequency functions.* In a letter of October 1, 1946, Pólya wrote to me: "... Since a long time I have not seen a result in which I were more interested. ..." See also the book [**HW**] by Hirschman and Widder and particularly the important book [**K**] of Samuel Karlin. There are four distinct characterizations of Pólya frequency functions described in [**43***], [**39***], [**47***] and [**86***]. These four characterizations are conveniently stated in my monograph [**114**], Lecture 1, 3. In jest N. G. de Bruijn suggested that these frequency functions should be called Pólyamials!

The discrete analogues of Pólya frequency functions are the totally positive sequences (a_n) with $\sum a_n < \infty$. Their generating functions

$$\sum_{-\infty}^{\infty} a_n z^n$$

were determined by M. Aissen, Anne Whitney and myself in [**41**] and [**45***]. However, we left an important open problem which was elegantly solved by Albert Edrei in his paper [**Ed**]. Edrei had heard me state' the problem at the International Congress of 1950 at Harvard.

My family received a heavy blow when my beloved wife Dolli died of acute Leukemia on July 2, 1949. My older daughter Elizabeth left for College (Radcliffe) in the Fall of the same year. On December 2, 1950, I married Dolly van der Hoop, of Amsterdam, who was visiting Mark Dresden and his wife in Media PA.

Following a suggestion of Th. Motzkin, John Curtiss appointed me for 1951–1952 at the Institute of Numerical Analysis of the National Bureau of Standards at U.C.L.A., where my collaboration with Motzkin produced the papers [**44***], [**50***], [**153**]. The paper [**44***] anticipated the work of W. Dahmen and C. A. Micchelli [**DM**] on multi-dimensional splines. The paper [**50***] has connections with the notion of "Complexity" of Computer Science; cf. Goffin's commentary in these selecta. The paper [**153**] was only published as late as 1980.

Our son Michael Jan was born in Santa Monica CA on September 12, 1951.

Variation diminishing linear transformations also led to a study of curves convex in higher dimensions and to an extension of the classical isoperimetric inequality presented in [**51***].

At Penn I gave courses in Approximation Theory, Laplace Transforms, The Moment Problem, Functional Analysis, Divergent Series, Complex Analysis, Real Variables, and shared with Rademacher the weekly Problem Seminar. I also had a few Ph.D. students: Louis Brickman, Anne Whitney (Mrs. Calloway), S. J. Einhorn, Hyman Gabai, D. S. Greenstein, R. S. Johnson, T. P. G. Liverman, J. C. Mairhuber, R. E. Williamson,1 and Dorothy Wolfe.

In 1949 Hans Rademacher married my sister Irma who had divorced her first husband, the composer Stefan Wolpe.

During 1956–1957 I went on a sabbatical to Stanford to work with Pólya, Peter Scherk, of Toronto, replacing me at Penn. To introduce the main result of our collaboration, I first mention that I. P. Natanson in [**N**] uses the Bernstein polynomials to prove Weierstraß' approximation theorem in [0,1], and the de la Vallée Poussin means of the Fourier series to prove Weierstraß' theorem for periodic functions. In [**55***] Pólya and I showed that both these means are variation diminishing and cyclically variation diminishing, respectively.

Our main "result", however, was a conjecture: *If two power series* $\sum_0^\infty a_n z^n$ *and* $\sum_0^\infty b_n z^n$ *are univalent in* $|z| < 1$ *and map this unit circle onto convex domains, then also their Hadamard product*

$$\sum_0^\infty a_n b_n z^n$$

has the same property; see [**55***], also the largely expository paper [**57**].

A group of Analysts took a fancy to this "Pólya-Schoenberg conjecture". Only after 15 years of efforts our conjecture was finally established by S. Ruscheweyh and T. Sheil-Small in their paper [**RS**].

A fine addition to our department was A. S. Besicovitch who had retired from Cambridge University in the late fifties. He used to scare students at Oral Exams with his "Tell me please" in a heavy Russian accent. For a joint paper see [**66**]; see also [**70***] and [**71**] on the Kakeya problem.

During 1960–1963 I was chairman of the Mathematics Department. For 1961–1962 I organized a Symposium on Number Theory in honor of Hans Rademacher who was to retire in June 1962. We had several guests, among them Charles Pisot of Paris, with whom I wrote the papers [**74**] and [**81**].

Following the initial papers [**31***], [**32***], I had the spline functions for myself from 1946 to about 1960, when they were rediscovered by several people. For 1963–1964 I went on sabbatical to the Institute for Advanced Studies where I developed an intensive activity writing eight papers on several subjects such as spline functions and function theory; see [**75***], [**76***], [**77**], [**78**], [**80***], [**82**], [**83**], and [**85**]. At the end of that year I felt that I needed another year's leave from Penn and I therefore accepted for 1964–1965 an appointment to the Mathematics Research Center at the University of Wisconsin-Madison.

6.**Madison, Wisconsin 1965–1973.** In June 1965 I resigned from Penn and joined the Mathematics Department of the UW and the MRC. During those years the main subject of my work was on Cardinal Spline Interpolation adequately described in my eight papers [**98**], [**108**], [**109**], [**110***], [**111**], [**112**], [**113***], and [**134**], some written jointly with P. R. Lipow, A. Sharma and C. de Boor. The results were summarized in the monograph [**114**].

Starting with the 4th paper [**110***] the Eulerian character of the theory becomes apparent. As pointed out in [**98**] the theory was initiated by Yu. N. Subbotin [**Su**] for bounded data. My series of papers pretty much solved the one-dimensional case. Generalizations to higher dimensional interpolation problems, not obtained by tensor product constructions, were obtained by C. de Boor, K. Höllig and S. Riemenschneider. In their paper [**BHR**], they write: "In this paper, we carry Schoenberg's beautiful cardinal spline theory over to a two-dimensional context which is not just the tensor product of the univariate situation. We find that we must work harder, yet must be satisfied with less precise results."

For the solution of some extremum problems suggested by work of G. Glaeser and R. Louboutin see [**105***]. Long ago in [**21**], I made a peripheral contribution towards the problem of determining the so-called J. M. Whittaker constant of entire function theory. For a refinement of [**21**] see [**106***]. In 1966 I revived interest in Birkhoff's Interpolation Problem in [**89***]. Jointly with M. Marsden the variation diminishing spline approximation methods were developed in [**90***]. With S. D. Silliman I studied semi-cardinal interpolation and quadrature formulae in our paper [**117**].

In the sixties Spline Theory became one of the most active topics in Approximation Theory: already 1967 saw the appearance of the first book on the subject [**AN**] by J. H. Ahlberg, E. N. Nilson and J. L. Walsh, which was translated into Russian by Yu. N. Subbotin. This was followed by two Symposia on the subject at the MRC: 1) The Symposium of October 1968 whose Proceedings [**G69**] were edited by T. N. E. Greville, 2) The Symposium of May 1969 with its Proceedings [**S69**] edited by I. J. Schoenberg. Greville's article "Introduction to Spline Functions" in [**G69**] is still the best introduction to the subject. More recently we have the more specialized books [**Bo**] by C. de Boor and [**Sc**] by L. L. Schumaker.

My Ph.D. students in Madison were D. R. Ferguson, Alfred Cavaretta, F. R. Loscalzo, Martin Marsden, Franklin Richards, Sherwood Silliman, and I. Yuan Houng.

In 1966 I attended the International Congress in Moscow where the big experience was meeting M. G. Krein after many years of correspondence.

Much traveling connected with lectures was a result of my activities. Not all these trips need be mentioned, but here are a few: During 1967 I traveled for two months in Europe visiting Romania for the first time since 1930 (Bucharest, Cluj, Jassy). I was a guest of the Romanian Academy with a chauffeured car at my disposal. In 1969, from October to December, a visit to Haifa with lectures also in Jerusalem, Tel-Aviv, and the

Weizmann Institute in Rehovot. In 1971 a month's visit to the University of Dundee for a Symposium.

7. **After retiring in June 1973**, I remained at the MRC as a part-time Honorary Member and for this I warmly thank its Director, Professor John Nohel. In four consecutive years I spent one semester (or an entire year) at the following four institutions: 1) *The Weizmann Institute* in Rehovot, Israel, first semester, and [**KM**] is an outgrowth of that; 2) *The State University of California in San Diego*; 3) *The University of Pittsburgh*; 4) the entire year from June 1977 to June 1978 I was a Visiting Professor at the *United States Military Academy* at West Point NY. Out of a weekly seminar given at West Point grew my little book [**S83**] "Mathematical Time Exposures" already mentioned in Section 1.

Of the nearly fifty papers written since 1973 I will mention only a few. I begin with [**157***]. Everybody knows that Newton's general polynomial interpolation formula simplifies if the points of interpolation are in arithmetic progression. However, even Scottish mathematicians have forgotten that James Stirling found in 1730 that a similar simplification occurs if the points are in geometric progression. To remind them of it and the work of Schellbach and Runge, I wrote [**157***]. I base the discussion on the so-called q-divided differences; this led to my collaboration with D. J. Newman producing the joint paper [**126**].

The remaining papers that I wish to mention may be divided into two classes each of which deal with related problems. The first deals with

1) *The Landau-Kolmogorov inequality and related extremum problems*: They are described in the nine papers: [**100***] (jointly with A. Cavaretta), [**122**] (which received the Lester Ford Prize), [**124**], [**135**], [**136**], [**141***], [**149***], [**150***] and [**152**].

With the paper [**100***], which solved the Landau problem for the interval [0, oo), we had some excitement in 1970. We presented it to the International Conference on Constructive Function Theory in Varma (Bulgaria), May 19-25, 1970. As I did not go to Varma, the paper was read by Penkov. I received a postcard from Paul Turan in which he wrote the following: After the presentation of [**100***] by Penkov, S. B. Ste6kin announced that he had promised some years ago 4 bottles of cognac to the solver of the Landau problem for [0, oo). As I did not go to Russia after 1970, I did not receive any cognac.

2) *Extremum problems for Billiard Ball Motions in Cubes*: These problems can be traced to the paper [**KS**] by D. König and A. Szücs of 1913. Natural generalizations for higher dimensions are contained in the papers [**127**], [**131**], [**132**], [**146**], [**147**], [**148**] and [**164**]. Notice Coxeter's paper [**Co**]. Also notice that in [**148**] I state a conjecture in the Geometry of Numbers which I cannot establish.

During May 1977 I was in Italy (Rome, Florence, Pisa), spending three weeks at the Istituto "Mauro Picone" in Rome. There I wrote the paper [**143**] on Cardinal Spline Smoothing.

This is the end of my account. Let me thank most warmly my colleague and successor Carl de Boor for agreeing to be the Chief Editor of my Selected Papers in two volumes.

March 1986

I. J. Schoenberg

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