SCIENTIFIC LEGACY OF YA. L. GERONIMUS

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1. Introduction

Ya. L. Geronimus was born in Rostov on February 6, 1898 into a family of minor office workers. In 1913 he graduated from Rostov high school with honors. An excellent knowledge of mathematics, history and languages, both ancient (Latin and Greek) and modern, was noted in his certificate of good work and conduct. Among his works, we find papers in five languages (English, French, German, Russian and Ukrainian), all of them written on his own. He was well acquainted with Russian literature and frequently quoted by heart poems by Blok and Mayakovskii.

After graduating from Kharkov University in 1917, Ya. L. Geronimus worked for the Kharkov Institute of Technology, and then for the Kharkov Aviation Institute from the very moment of its inception in 1930 until his retirement in 1978, as the Head of the Theoretical Mechanics Department.

I was aware of three groups of people familiar with the name of Ya. L. Geronimus. The first and the largest of these consists of experts on theoretical and applied mechanics all over the former USSR. As a matter of fact, most of Ya. L.'s professional and public activities had nothing to do with orthogonal polynomials and mathematics at all. Only two out of his seven books and monographs are devoted to orthogonal polynomials. Among the others we find two monographs on the application of Chebyshev's methods to the problems of balancing and dynamic synthesis of mechanisms (1948 and 1958), a thorough historical treatise "Surveys on the Works of Leading Lights of Russian Mechanics" (1952), a textbook "Theoretical Mechanics" with a mathematically rigorous exposition of the subject (1973), and others. Geronimus was the leader of the Kharkov scientific school on the theory of mechanisms and machines, and an editor-in-chief of a scientific journal. In 1961, he published two papers on focusing fields, electromagnetic and mechanical, which were highly appreciated by experts.

The OP community constitutes the second group of people, wherein Ya. L. Geronimus was rightfully regarded as one of the top experts, and his classical monograph on orthogonal polynomials still remains quite up-to-date.

The third group, the smallest one which unfortunately decreases all too rapidly, comprises those who knew Ya. L. personally. Surprisingly enough, the intersection of all these groups is not empty and consists of a single individual – my father, B. L. Golinskii, whose assistance and encouragement in preparing this note can not be overestimated.

I was quite astonished to learn from Professor A. Zhedanov that there was yet another group of physicists who knew very well and appreciated the works of Ya. L. Geronimus. I was even more surprised to hear from Professor V. Spiridonov in the fall 1997 about the idea of commemorating the centenary of Ya. L. Geronimus. I immediately made up my mind to collaborate with them and to contribute to the progress of the meeting. I believe we were sufficiently successful.

Let me briefly explain Geronimus' scientific research in mathematics. I have selected four areas wherein his contribution is the most significant.

2. Extremal Problems for Polynomials and Analytic Functions in the Unit Disk

2.1. **Extremal L–Moment Problem and Duality Principle.** The first scientific interests of Ya. L. Geronimus were greatly influenced by the Kharkov mathematical school and notably by N. I. Akhiezer and M. G. Krein.

We begin with the definition of the L–moment problem for 2π–periodic functions. Let $B = B\{a_0, a_1, b_1, \dots, a_n, b_n\}$ be the class of piecewise continuous functions f on $[-\pi, \pi]$ with given first Fourier coefficients

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,\mathrm{d}t, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,\mathrm{d}t, \qquad k = 0, 1, \dots, n$$

($b_0 = 0$). The following problem is known as the "L– moment problem": given $L > 0$ does there exists a function $f \in B$ with $\|f\| = \max_{|t|\le\pi} |f(t)| \le L$?

Problem 1t (Extremal L–Moment Problem). Find the least value $L_{\min}$ for which the L–moment problem is solvable. Find the function $F \in B$ with $\|F\| = L_{\min}$.

The problem was solved by Akhiezer and Krein in the early 1930s by using ingenious methods from function theory. Ya. L. Geronimus suggested a totally different approach based on an idea called afterwards a "duality principle" (cf. [1, 2]).

Consider a linear functional $\omega(g)$ defined on the space of trigonometric polynomials of order n which is generated by the sequence $\{a_k, b_k\}$:

$$\omega(g) = \sum_{k=0}^{n}(a_k\alpha_k + b_k\beta_k), \qquad g(t) = \sum_{k=0}^{n}(\alpha_k\cos kt + \beta_k \sin kt).$$

Denote

$$\Lambda(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} |g(t)|\,\mathrm{d}t.$$

Problem 2t. Find a trigonometric polynomial T such that $\omega(T) = 1$ and

$$\Lambda(T) = \min\{\Lambda(g) \mid \omega(g) = 1\}.$$

Ya. L. Geronimus discovered an intimate relation between the two problems.

Theorem 2.1.

$$F(t) = L_{\min}\,\mathrm{signT(t)}, \qquad \mathrm{L_{\min}} = \Lambda^{-1}(\mathrm{T}).$$

Geronimus investigated uniqueness in Problem $2t$. It transpires that the solution is unique if and only if the number of sign changes is maximal (and equals $2n$), and there is an infinite family of solutions otherwise. It is worth noting that all the solutions change sign at the same points.

A similar duality principle holds for algebraic polynomials.

Problem 1a. Given real numbers $s_0, s_1, \dots, s_n$ find a piecewise continuous function G on the interval $[0, A]$ such that

$$\int_0^A G(x)x^k \, dx = s_k, \qquad k = 0, 1, \dots, n$$

which least deviates from zero. Find $\|G\|$.

Problem 2a. Find an algebraic polynomial $P(x) = \sum_{k=0}^n p_k x^k$ such that

$$\omega(P) = \sum_{k=0}^n p_k s_k = 1$$

and

$$\|P\|_1 = \min_{Q\in\Pi_n} \{\|Q\|_1 \mid \omega(Q) = 1\}, \qquad \|Q\|_1 = \int_0^A |Q(x)| \, dx,$$

where the minimum is taken over the set Π_n of algebraic polynomials of degree $\le n$

Theorem 2.2.

$$G(x) = \|G\| \, \mathrm{sign} P(x), \qquad \|G\| = \|P\|_1^{-1}.$$

Only two years later, M. G. Krein came up with the general duality principle in normed spaces which looks much the same as the particular one above (see [3]).

2.2. **Examples.** One of the main features of Ya. L. Geronimus as a mathematician was the constant and deep interplay between his research in pure mathematics and his activity in applied mechanics. Let me illustrate the power of the extremal L–moment problem in optimal control.

Example 2.3. *A body is moving along a straight line, starting from a point A at time $t = 0$. It reaches another point B which is h meters away from A at $t = T$ min. Denote by $s(t)$ the distance of the body from A at time t (the law of motion). We are looking for $s(t)$ such that*

$$s(0) = 0, \quad s(T) = h; \qquad \dot{s}(0) = \dot{s}(T) = 0$$

which has the minimal norm $\|\ddot{s}\|$ of acceleration of the moving body.

Put $w(t) = \ddot{s}(t)$. Integrating by parts gives

$$\int_0^T w(t) \, dt = 0, \qquad \int_0^T t w(t) \, dt = -h.$$

By Theorem 2.2, $w(t) = C \mathrm{sign} P(t)$, $p(t) = t - \alpha$ and $0 < \alpha < T$ (w must change sign inside $[0, T]$). Hence

$$\int_0^T w(t) \, dt = C(T - 2\alpha) = 0, \qquad \alpha = \frac{T}{2},$$
$$\int_0^T t w(t) \, dt = \frac{CT^2}{4} = -h, \qquad C = -\frac{4h}{T^2},$$

and, finally,

$$w(t) = \begin{cases} \frac{4h}{T^2}, & \text{for } 0 \le x < \frac{T}{2}; \\ -\frac{4h}{T^2}, & \text{for } \frac{T}{2} < x \le T; \end{cases}, \quad s(t) = \begin{cases} \frac{2h}{T^2} t^2, & \text{for } 0 \le x < \frac{T}{2}; \\ -\frac{2h}{T^2} t^2 + \frac{4h}{T} t - h, & \text{for } \frac{T}{2} < x \le T. \end{cases}$$

Example 2.4. (*Oscillation of a Moving Pendulum*).

The trolley of a crane is moving along a horizontal line with a mass suspended from it at length l. Denote by $s(t)$ the law of motion and by $\phi(t)$ the angle that pendulum makes with the vertical direction at time t. Then, if $\phi(t)$ is small, the oscillation is governed by the second order differential equation

$$\ddot{\phi}(t) + \frac{G}{l}\,\phi(t) = \frac{\ddot{s}(t)}{l}\,.$$

The problem is to find $s(t)$ with given initial and final data

$$s(0) = 0, \quad s(T); \qquad \dot{s}(0), \quad \dot{s}(T); \qquad \ddot{s}(0) = \ddot{s}(T) = 0$$

so as to minimize $\|\dddot{\phi}\|$ (the so-called angle jerk) for the solution $\phi(t)$ with zero initial and final data.

The point is that the first five moments of $\dddot{\phi}$ can be easily computed and thereby the problem is reduced to an extremal L–moment problem.

2.3. **Analytic Functions in the Unit Disk.** Ya. L. Geronimus turned to extremal problems of function theory on and off during his life. Here is a typical example of such a problem.

Let A be the class of analytic functions f in the open unit disk $\mathbb{D}$, that are continuous in the closed disk $\bar{\mathbb{D}}$, with norm $\|f\|_A = \max_{\bar{\mathbb{D}}} |f(z)|$. Given n points $\alpha_1, \alpha_2, \dots, \alpha_n$ in $\mathbb{D}$ and n complex numbers $c_1, c_2, \dots, c_n$, define the linear functional

$$\Omega(f) = \sum_{k=1}^{n} c_k f(\alpha_k).$$

Problem 3. Find the norm of Ω in the Banach space A.

Ya. L. Geronimus solved this problem (and several similar ones) (cf. [4]) by giving an explicit expression for the extremal function f_{ex}, such that

$$|\Omega(f_{\text{ex}})| = \|\Omega\| \, \|f_{\text{ex}}\|_A\,.$$

3. Orthogonal Polynomials

Ya. L. Geronimus was by far one of the outstanding experts on orthogonal polynomials and their applications. His monograph [9] and appendix to the Szegő fundamental treatise still remain an invaluable sources of information on the subject.

3.1. **General Orthogonal Polynomials.** Historically the first area of Geronimus' interest in orthogonal polynomials concerned *orthogonality with respect to a given sequence* (cf. [5, 6]).

Let $\{c_n\}_0^\infty$ be a sequence of complex numbers with

$$\Delta_n = \det \|c_{i+j}\|_0^{n-1} \neq 0, \qquad \Delta_0 = 1.$$

It gives rise to a linear functional on the set of algebraic polynomials

$$\sigma\{T\} = \sum_{j=0}^{m} t_j c_j, \qquad T(z) = \sum_{j=0}^{m} t_j z^j.$$

The sequence of polynomials $P_0(z) = 1$,

$$P_n(z) = \frac{1}{\Delta_n} \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-1} \\ 1 & z & \dots & z^n \end{vmatrix} = z^n + \dots, \qquad n = 1, 2, \dots,$$

is easily seen to be orthogonal with respect to $\{c_k\}$, that is,

$$\sigma\{P_n P_m\} = \frac{\Delta_{n+1}}{\Delta_n}\, \delta_{n,m}.$$

The sequence P_n is known to satisfy the three-term recurrence relation

$$P_n(z) = (z - \alpha_n)P_{n-1}(z) - \lambda_n P_{n-2}(z), \quad n = 1, 2, \dots, \qquad P_{-1} = 0,\ P_0 = 1.$$

The second linearly independent solution $\{Q_n\}_0^\infty$ of this relation with initial conditions $Q_{-1} = 1,\ Q_0 = 0$ is called the *second kind polynomials*. Among the most significant achievements in this area I would like to mention the following three problems which were completely examined by Ya. L. Geronimus.

Problem 4. Is it true that $\{Q_n\}$ forms a sequence of orthogonal polynomials (OPS)? If so, find the corresponding sequence $\{c'_n\}$.

Problem 5. Find conditions on the OPS $\{P_n\}$ and a sequence $\{A_i^{(n)}\}_{i=0}^s$ such that

$$\tilde{P}_n(z) = \sum_{i=0}^{s} A_i^{(n)} P_{n-i}(z)$$

is also an OPS.

Problem 6. Find all OPS's $\{P_n\}$ such that the derivatives $\{\frac{1}{n}P'_n\}$ are also OPS, possibly with respect to another sequence $\{c'_n\}$.

In the latter problem (aka Hahn's problem), $\{c_n\}$ satisfies the difference equation

$$[a(n+3)+b]c_{n+2} + [c(n+2)+d]c_{n+1} + e(n+1)c_n = 0, \quad n = 0, 1, \dots, \qquad c_{-1} = 0$$

with arbitrary real parameters $a,\ b,\ c,\ d,\ e$ such that $a(n+3)+b \neq 0$. For some particular choice of parameters we come to the well known solutions of Hahn's problem (Jacobi, Laguerre, Hermite polynomials).

3.2. **Orthogonal Polynomials on the Unit Circle (Asymptotic Properties and Reflection Coefficients).** The first paper by Ya. L. Geronimus [7] wherein orthogonal polynomials on the unit circle emerged came out in the same 1940. I want to mention also his thorough survey [8] and the fundamental monograph [9].

Let μ be a probability measure on the unit circle $\mathbb{T} = \{|\zeta| = 1\}$ such that its support $\operatorname{supp}(\mu)$ is an infinite set. The polynomials $\phi_n(z) = \phi_n(z, \mu) = \kappa_n(\mu)z^n + \dots$, orthonormal on the unit circle with respect to μ, are uniquely determined by the requirement that $\kappa_n = \kappa_n(\mu) > 0$ and

$$\int_{\mathbb{T}} \phi_n(\zeta)\overline{\phi_m(\zeta)}\, \mathrm{d}\mu = \delta_{n,m}, \qquad n, m = 0, 1, \dots, \qquad \zeta \in \mathbb{T}. \tag{3.1}$$

The monic orthogonal polynomials Φ_n are defined by

$$\Phi_n(z) = \Phi_n(z, \mu) = \kappa_n^{-1}\phi_n(z) = z^n + \cdots, \qquad n = 0, 1, \dots.$$

The numbers $a_n = \Phi_n(0, \mu),\quad n = 1, 2, \dots$, which play a key role throughout the whole theory, are known as the *reflection coefficients*. Since all zeros of Φ_n are inside the unit circle (cf. [9, Section 8]), we have $|\Phi_n(0,\mu)| < 1, \quad n = 1, 2, \ldots$. What is more to the point, given an arbitrary sequence of complex numbers $\{a_n\}$ with the only restriction $|a_n| < 1$, there is a unique probability measure μ with infinite support such that $a_n = \Phi_n(0,\mu)$ for $n = 1, 2, \ldots$. This result is usually referred to as Favard's Theorem for the unit circle.

Ya. L. Geronimus put forward the problem of studying the properties of measures and orthogonal polynomials depending upon the behavior of their reflection coefficients.

Theorem 3.1. *The following statements are equivalent.*

(1) $\sum |a_n|^2 < \infty$;
(2) $\sum |\phi_n(z)|^2 < \infty$ *at least for one point* $z \in \mathbb{D}$;
(3) *there is a subsequence* n_k *such that* $\phi^*_{n_k}(z)$ *converges at least for one point* $z \in \mathbb{D}$;
(4) $\lim_{n\to\infty} \phi^*_n(z) = S(z)$ *uniformly inside* $\mathbb{D}$.

Theorem 3.2. *If* $\sum |a_n| < \infty$, *then* μ *is absolutely continuous and* $\mu' > 0$ *and continuous. The relation*

$$\max_{|z|\le 1} |\phi^*_k(z) - S(z)| \le C \sum_{n=k}^{\infty} |a_n|$$

holds.

Theorem 3.3. *Let* E *be the closed support of* μ *and denote by* $d(E)$ *the transfinite diameter of* E. *Then*

$$d(E) \ge \limsup_{n\to\infty} \prod_{k=1}^{n} \left(1 - |a_n|^2\right)^{\frac{1}{2n}} .$$

The bound is sharp: for $a_n = \rho, \ 0 < \rho < 1$, *we have*

$$\text{supp}\mu = \{e^{it} \mid \alpha \le t \le 2\pi - \alpha\}, \qquad \alpha = 2 \arcsin \rho$$

and $d(E) = \cos\frac{\alpha}{2} = \sqrt{1-\rho^2}$.

In [10], Ya. L. Geronimus gave a complete treatment of the measures with periodic and asymptotically periodic reflection coefficients. He showed that in the periodic case (i.e., when $a_{j+N} = a_j, \ j \ge s$), the support of the orthogonality measure consists of a finite number of circular arcs and a finite number of mass points off these arcs. He obtained an explicit expression for the density function μ' and the mass points.

3.3. **Orthogonal Polynomials on the Unit Circle and Bounded Analytic Functions in the Unit Disk.**

Consider the class of analytic functions f in $\mathbb{D}$ with $|f(z)| < 1, \ z \in \mathbb{D}$. Such functions are known as the *Schur functions*. Each Schur function f generates a sequence $\{f_n\}$ of Schur functions by means of the following Schur algorithm

$$f_{n+1}(z) = \frac{f_n(z) - \gamma_n}{z(1 - \bar{\gamma}_n f_n(z))}, \quad n = 0, 1, \ldots, \qquad \gamma_n = f_n(0), \quad f_0 = f.$$

This sequence is actually an infinite one if and only if f is not a finite Blaschke product. The complex numbers $\gamma_n, \ |\gamma_n| < 1$, are called the *Schur parameters* corresponding to the Schur function f. They arise in various problems of complex analysis and applications.

According to the fundamental Schur's Theorem, there is a one-to-one correspondence between all Schur functions (that are not finite Blaschke products) and all sequences of complex numbers γ_n with $|\gamma_n| < 1$. We thereby come to a quite natural parametrization of the class of Schur functions.

It was Ya. L. Geronimus who first discovered an intimate relation between Schur functions and orthogonal polynomials on the unit circle (cf. [11]).

Given a Schur function f, put

$$F(z) = \frac{1 + zf(z)}{1 - zf(z)}$$

so that

$$F(0) = 1, \qquad \Re F(z) = \frac{1 - |zf(z)|^2}{|1 - zf(z)|^2} > 0.$$

By the Riesz–Herglotz Theorem, F admits the representation

$$F(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, \mathrm{d}\mu(\zeta),$$

where μ is a probability measure on $\mathbb{T}$ with infinite support.

Next, let Φ_n be monic orthogonal polynomials with respect to μ with reflection coefficients $a_n = \Phi_n(0)$.

Theorem 3.4. *Let γ_n be Schur parameters of the original Schur function f. Then*

$$\gamma_n = -\bar{a}_{n+1}, \qquad n = 0, 1, \dots.$$

This result enabled Ya. L. to invoke the whole machinery of orthogonal polynomials on the unit circle for studying the Schur functions in terms of their Schur parameters (see [12]).

4. Embedding of Classes of Functions

There is another area of research wherein the contribution of Ya. L. Geronimus is indisputable.[1] The subject is now well-elaborated upon thanks to P. L. Ul'yanov and his school.

The following problem is under consideration: given $f \in L^p[0, 2\pi]$, what additional conditions on f imply $f \in L^q[0, 2\pi]$ with $q > p$? In particular, when is $f \in L^p[0, 2\pi]$ equivalent to some continuous function f_0?

The first result in this direction was obtained by G. Hardy and J. Littlewood [13].

Theorem 4.1. *Let $f \in L^p[0, 2\pi]$, $p > 1$ and $\omega_p(x, f) = O(x^\alpha)$ with $\alpha > 1/p$. Then $f \sim f_0 \in C[0, 2\pi]$ and $\omega(x, f_0) = O\left(x^{\alpha - 1/p}\right)$, where $\omega_p(x, g)$ $(\omega(x, g))$ is the modulus of continuity in L^p (C), respectively.*

In 1958 Ya. L. Geronimus extended this result in the following way (see [14]).

Theorem 4.2. *Let $f \in L^p$, $p > 1$ and*

$$\int_0^1 \frac{\omega_p(x, f)}{x^{1+1/p}}\, \mathrm{d}x < \infty.$$

[1] To the best of my knowledge, the name "embedding of classes of functions" itself belongs to Ya. L.

Then $f \sim f_0 \in C$. *Furthermore, if in addition*

$$\int_0^1 \frac{\omega_p(x,f)}{x^{1+1/p}} \log\frac{1}{x}\,\mathrm{d}x < \infty$$

then

$$\omega(x,f_0) \le C \int_0^x \frac{\mathrm{d}y}{y} \int_0^y \frac{\omega_p(t,f)}{t^{1+1/p}}\,\mathrm{d}x.$$

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