Conditions Which Imply Metrizability in Some Generalized Metric Spaces*

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Abstract

In this paper we show that two important generalized metric properties are generalizations of first countability. We give some conditions on these generalized metric properties which imply metrizability. We prove that, a space X is metrizable if and only if X is a strongly–quasi–N–space, quasi–γ–space; a quasi–γ space is metrizable if and only if it is a pseudo wN–space or quasi–Nagata–space with quasi–G_δ^*–diagonal; a space X is a metrizable space if and only if X has a CWBC–map g satisfying the following conditions:

1. g is a pseudo–strongly–quasi–N–map;
2. for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.

1 Introduction

A **COC-map** (= countable open covering map) for a topological space X is a function from $\mathbb{N} \times X$ into the topology of X such that for every $x \in X$ and $n \in \mathbb{N}, x \in g(n,x)$ and $g(n+1,x) \subseteq g(n,x)$.

Consider the following conditions on g.

(A) If $x \in g(n,x_n)$ for every $n \in \mathbb{N}$, then x is a cluster point of the sequence $\langle x_n \rangle$.

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(B) If for each $n \in \mathbb{N}$, $x \in g(n, y_n)$ and $y_n \in g(n, x_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(C) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(D) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, x)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(E) If for each $n \in \mathbb{N}$, $x_n \in g(n, y_n)$ and the sequence $\langle y_n \rangle$ converges in X, then the sequence $\langle x_n \rangle$ has a cluster point.

(F) If for each $n \in \mathbb{N}$, $y_n \in g(n, x_n)$ and the sequence $\langle y_n \rangle$ converges to x in X, then x is a cluster point of the sequence $\langle x_n \rangle$.

Let (S) be any of the conditions (A), (B), (C), (D), (E), or (F), and (S^{-1}) be the statement obtained by formally interchanging all memberships (e.g., (C^{-1}) is the condition: If for each $n \in \mathbb{N}$, $y_n \in g(n, x) \cap g(n, x_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$). If the COC–map g satisfies condition (S)(resp. (S^{-1})) for $S = A$, B, C, D, E, or F, we say that g is an $\boldsymbol{S}$**–map** (resp. $\boldsymbol{S^{-1}}$**–map**). If there is an S–map (resp. S^{-1}–map) for X then we say that (X, τ) is an $\boldsymbol{S}$**–space** (resp. $\boldsymbol{S^{-1}}$**–space**). Corresponding to each of the conditions S above except (E) is the weaker condition, denoted wS, in which 'then x is a cluster point of the sequence $\langle x_n \rangle$' is replaced by 'then the sequence $\langle x_n \rangle$ has a cluster point'. If g satisfies wS, we say that g is an wS–map. If there is an wS–map for X then we say that (X, τ) is a wS–space. wS^{-1}–maps and wS^{-1}–spaces are defined analogously. The following are known, $A =$**semi–stratifiable space**, $B =\boldsymbol{\sigma}$**–space**, $C =$**developable space**, $D =\boldsymbol{\theta}$**–space**, $E =$**quasi–**$\boldsymbol{\gamma}$**–space**, $F =$**strongly–quasi Nagata space (= strongly–quasi–N space)**, $A^{-1} =$**first–countable space**, $B^{-1} =\boldsymbol{\gamma}$**–space**, $C^{-1} =$**Nagata space (= N–space)**, $E^{-1} =$**quasi–Nagata space (= quasi–N space)**, $wA =\boldsymbol{\beta}$**–space**, $wB =\boldsymbol{w\sigma}$**–space**, $wD =\boldsymbol{w\theta}$**–space**, $wA^{-1} =\boldsymbol{q}$**–space**, $wB^{-1} =\boldsymbol{w\gamma}$**–space**, $wC^{-1} =\boldsymbol{wN}$**–space**.

A **CWBC–map** (= countable weak base covering map) for a topological space X is a function from $\mathbb{N} \times X$ into $\mathcal{P}(X)$ such that for every $x \in X$ and $n \in \mathbb{N}$ we have $x \in g(n, x), g(n+1, x) \subseteq g(n, x)$ and a subset U of X is open if and only if for every $x \in U$there is an $n \in \mathbb{N}$ such that $g(n, x)$ is contained in U. A space with a CWBC-map is called **weakly first countable**.

H.W. Martin in [33] introduced weakly developable spaces. A space X is called a **weakly developable** space if there is a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of covers of X such that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for all n and $\{st(x, \mathcal{G}_n)\}_{n\in\mathbb{N}}$ is a local weak base at x for each $x \in X$; the sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is said to be a weak–development for the space X.

A space X has a **quasi–G_δ^*–diagonal** (resp. **quasi–S_2–diagonal**) (resp. **quasi–α_1–diagonal**) if there exists a countable family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets (resp. of collections of subsets and for each $x \in X, st(x,\mathcal{G}_n)$ is open for all $n \in \mathbb{N}$) (resp. of collections of subsets and for each $x \in X, x \in$ Int $\overline{st(x,\mathcal{G}_n)}$) such that for any distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in \overline{st(x,\mathcal{G}_n)} \subset X - \{y\}$.

A space X is called ***c*–semi–stratifiable** [34] if there a sequence $\langle g(n,x)\rangle$ of open neighborhoods of x such that for each compact set $K \subset X$, if $g(n,K) = \bigcup\{g(n,x) : x \in K\}$, then $\bigcap\{g(n,K) : n \geq 1\} = K$. The COC–map $g : \mathbb{N} \times X \to \tau$ is called a c–semi–stratification of X.

A space X which has a CWBC-map that satisfies condition (wC^{-1}) is called **pseudo–wN space**.

A space X which has a CWBC-map that satisfies condition (C^{-1}) is called **pseudo–N space**.

A space X which has a CWBC-map that satisfies condition (wB^{-1}) is called **pseudo–quasi–γ space**.

A space X which has a CWBC-map that satisfies condition (B^{-1}) is called **pseudo–γ space**.

From the papers [15], [17], [26] and [31], the relationship between the classes of spaces above can be summarized in the following diagram:

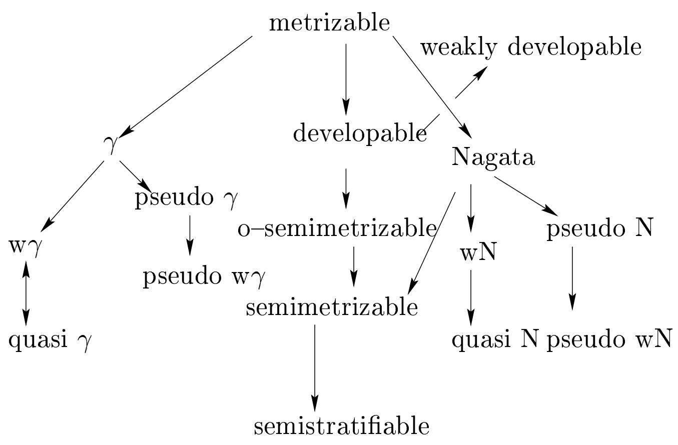

Figure 1: Relationships between some generalized metric spaces.

A space X is called an **$\aleph$–space** if it has a σ–locally finite K–network, where a collection $\mathcal{B}$ of subsets of X is a **K–network** if for any compact set C and open neighborhood U of C there is a finite subcollection $\mathcal{B}'$ of $\mathcal{B}$ such that $C \subset \mathcal{B}'^* \subset U$, where $\mathcal{B}'^* = \bigcup\{B : B \in \mathcal{B}'\}$. The following implications are well–known.

Frechet $\aleph \Rightarrow$ Lasnev $\Rightarrow$ stratifiable $\Rightarrow$ strongly–quasi-$N \Rightarrow \sigma \Rightarrow$ semi–stratifiable.

In this paper all spaces will be Hausdorff, unless we state otherwise.

2 Generalization of first countable spaces

A space X is **sequential** [7] if every sequentially open set is open, where a set U is said to be sequentially open if every sequence converging to a point in U is eventually in U. A space is **Frechet** [7] if every accumulation point of a set is the limit of a sequence in the set. X is called **strongly Frechet** if, whenever $\{F_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with a cluster point x, then there are $x_n \in F_n, n \in \mathbb{N}$ such that $\langle x_n \rangle$ converges to x.

Lemma 2.1 [40] *A space X is first countable if and only if X is Frechet and weakly first countable.*

Example 2.2 [40] *A Frechet space which is not weakly first countable and so not first countable.*

The space of rational numbers with the integers identified to a point and the quotient (or identification) topology. The one–point compactification of an uncountable discrete space. ■

Example 2.3 [40] *A q, and weakly first countable space which is not Frechet and so not first countable.*

Let X be obtained from $[0, \infty)$ by identifying $1/n$ and n for all $n \in \mathbb{N}$. We denote by x_n the point $\{1/n, n\}$ in the identification space X. All other points of X are singleton equivalence classes, i.e. real numbers.

This example is also quasi–N space but neither wN nor strongly–quasi–N. ■

Note that every Nagata space is first countable; every γ space is first countable and every Frechet, pseudo wN–space is a wN–space.

The proof of the following theorem is straightforward:

Theorem 2.4 *(1) Every quasi–N–space is β.*
(2) Every quasi–γ space is q.

Theorem 2.5 *The following are equivalent for a first countable space X*

1. *X is a quasi–γ–space.*

2. *X is a pseudo γ–space.*

3. *X is a pseudo quasi–γ–space.*

Proof. It is clear that, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We prove that every Frechet, pseudo quasi–γ–space is a quasi–γ–space. Let $g : \mathbb{N} \times X \to \mathcal{P}(X)$ be a pseudo quasi–γ–map. We can use same proof as for Lemma 2.1 to prove that for each $x \in X, g(n,x)$ is a neighborhood of x for each $n \in \mathbb{N}$. Thus x is in the interior of $g(n,x)$. Now, put $h(n,x) = Int\ g(n,x)$ for each $n \in \mathbb{N}$ and $x \in X$, then $h : \mathbb{N} \times X \to \tau$ satisfies the quasi–γ–condition. ∎

Y. Inui and Y. Kotake [22] proved the following result:

Theorem 2.6 *The following are equivalent for a first countable space X*

1. *X is a wN–space.*

2. *X is a quasi N–space.*

3. *X is a pseudo N–space.*

4. *X is a pseudo wN–space.*

Lemma 2.7 *A q space with quasi–S_2 is first countable.*

Proof. Let f be a q–map and $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ a quasi–S_2–sequence of X. Define g by

$$g(n,x) = \begin{cases} st(x, \mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^* . \end{cases}$$

For each $x \in X$ and $n \in \mathbb{N}$, let $h(n,x) = f(n,x) \cap g(n,x)$. Then h is a first countable map. Let $x_n \in h(n,x)$. Then $\langle x_n \rangle$ has a cluster point, say y (because g is q–map). For all $n \in \mathbb{N}, y$ is a cluster point of $\{x_m : m \geq n\}_{n \in \mathbb{N}}$, so $y \in \overline{h(n,x)}$ as $x_m \in h(n,x)$ for all m. Thus $y \in \bigcap_{n \in \mathbb{N}} \overline{h(n,x)} \subset \bigcap_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_n)} = \{x\}$, so $y = x$ and x is a cluster point of $\langle x_n \rangle$. ∎

Theorem 2.8 *(Lutzer [29]) Let X be a regular q space. If every point in X is a G_δ then X is first countable.*

Corollary 2.9 *A regular q space with quasi–α_1–diagonal is first countable.*

3 Stability of Strongly–quasi–N Spaces

Theorem 3.1 *Every subspace of a strongly–quasi–N–space is a strongly–quasi–N–space.*

Proof. Let g be a COC–map on X satisfying the condition for a strongly–quasi–N–space. Let Y be a subspace. Then the restriction h of g on $\mathbb{N} \times Y$, $h(n,x) = g(n.x) \cap Y$ is a COC–map. ■

Theorem 3.2 *Every countable product of strongly–quasi–N–spaces is a strongly–quasi–N–space.*

Proof. For each i, let X_i be a strongly–quasi–N space with a COC–map g_i satisfying the strongly–quasi–N condition. Let $X = \prod X_i$ be the product space, and let $\pi_i : X \to X_i$ be the projection. For each i, n and $x \in X$, let $h_i(n,x) = g_i(n, \pi_i(x))$ if $i \leq j$, and X_i if $i > j$. Now let $g(n,x) = \prod_{i=1}^{\infty} h_i(n,x)$ for each $(n,x) \in \mathbb{N} \times X$. That is, $g(n,x) = g_1(n,x_1) \times g_2(n,x_2) \times g_3(n,x_3) \times ...g_n(n,x_n) \times \prod_{j>n} X_j$ for each $n \in \mathbb{N}$, where $x = (x_1, x_2, x_3, ..)$.

Clearly each $g(n,x)$ is open, $x \in g(n,x)$ and $g(n+1,x) \subset g(n,x)$ for each $(n,x) \in \mathbb{N} \times X$.

To verify g is a strongly–quasi–N–map for X, let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two sequences in $X = \prod X_i$ such that $y_n \in g(n, x_n)$ and the sequence $\langle y_n \rangle$ converges to x in X, we only need to prove that x is a cluster point of the sequence $\langle x_n \rangle$. Put, $x_n = (x_n)_i$, $y_n = (y_n)_i$ and $x = (x)_i$. For each fixed $i \in \mathbb{N}$, we have $(y_n)_i \in g(n, (x_n)_i)$ when $n \geq i$ and and the sequence $\langle (y_n)_i \rangle$ converges to x_i in X_i. Since each X_i is a strongly–quasi–N space, x_i is a cluster point of $\langle (x_n)_i \rangle$. Thus, x is a cluster point of $\langle (x_n) \rangle$ in X. Hence, $X = \prod X_i$ is a strongly–quasi–N–space. ■

Theorem 3.3 *Closed images of regular strongly–quasi–N–spaces are strongly–quasi–N–spaces.*

Proof. Let $f : X \to Y$ be closed surjective map such that X is a strongly–quasi–N–space. We want to show that Y is also a strongly–quasi–N–space. Since X is a strongly–quasi–N–space, there is a COC–map g satisfying the strongly–quasi–N–condition. In other words, if $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $\langle x_n \rangle$ converges to x, then $\langle y_n \rangle$ converges to x. Define $h(n,y) = Y - f(X - (\bigcup\{g(n,x) : x \in f^{-1}(y)\}))$. It is clear that h is a COC–map. Let $x_n{}' \in h(n, y_n{}')$. Suppose $\langle x_n{}' \rangle$ converges to x'. We want to prove that $\langle y_n{}' \rangle$ converges to x'.

Let $x_n \in f^{-1}(x_n{}')$ for each $n \in \mathbb{N}$, so every subsequence of $\langle x_n \rangle$ has at least a cluster point in $f^{-1}(x')$ since f is closed. Note that, since X is a strongly–quasi–N–space, any singleton set is a G_δ–set, in other words, $\{x\}$ is a G_δ–set for each $x \in X$ and since f is closed, $\{x'\}$ is a G_δ–set for every $x' \in Y$. Now every subsequence of $\langle x_n \rangle$ has at least one cluster point because f is closed. So, $x \in f^{-1}(x')$ is a cluster point of $\langle x_n \rangle$. Note that, $\{x\} = \bigcap_{n=1}^{\infty} G_n$, where G_n is a closed neighborhood of x (X is regular).

Choose $x_{n_m} \in \{x_n\} \cap G_m$ (because x is a cluster point of $\langle x_n \rangle$ and G_n is a neighborhood of x), where we may assume $n_1 < n_2 < ...$, then x is a unique cluster point of $\langle x_{n_m} \rangle$, (if $z \neq x$, then there is a G_{m_0} such that $z \notin G_{m_0}$, hence z is not cluster point of $\langle x_{n_m} \rangle$). But $\langle x_{n_m} \rangle$ has a cluster point, therefore, y is a unique cluster point of $\langle x_{n_m} \rangle$. Since every subsequence of $\langle x_{n_m} \rangle$ has cluster point, we have that the sequence $\langle x_{n_m} \rangle$ converges to x for every $m \in \mathbb{N}$.

Now, we have $x_{n_m} \in g(n_m, y_{n_m}) \subset g(n, y_{n_m})$ and $\langle x_{n_m} \rangle$ converges to x. Since X is a strongly–quasi–N–space, $\langle y_{n_m} \rangle$ converges to x. Since f is closed, $\langle {y_{n_m}}' \rangle$ converges to x' and hence x' is a cluster point of $\langle {y_n}' \rangle$. This completes the proof that Y is a strongly–quasi–N–space. ∎

Theorem 3.4 *Every strongly–quasi–N–space is σ–space.*

Proof. Let g be a COC–map on X satisfying the condition for a strongly–quasi–N–space. Let $x \in g(n, y_n)$ and $y_n \in g(n, x_n)$, for each $n \in \mathbb{N}$. Then $\langle y_n \rangle$ converges to x and since g is strongly–quasi–N–map, x is a cluster point of the sequence $\langle x_n \rangle$. ∎

Example 3.5 *The converse of Theorem 3.4 is not true. There is a σ–space (and so semi–stratifiable) which is not a strongly–quasi–N–space.*

Proof. Let X (Heath space [16]) be the upper half plane including the real axis $\mathbb{R}$. Let each point of $X - \mathbb{R}$ be open and take as a neighborhood basis of points $x \in \mathbb{R}$ a V–vertex at x, sides of slopes $= 1$ and height $1/n$, which a V–vertex at x is the set $W = \{(\xi, \eta) : \eta = |\xi - x| \text{ and } \eta < \frac{1}{n}\}$.

We define a COC–map by:

$$h(n, x) = \begin{cases} \{x\} & \text{if } x \in X - \mathbb{R}. \\ \text{the } V\text{–vertex at } x \text{ of height } 1/n & \text{if } x \in \mathbb{R}\ . \end{cases}$$

Clearly h is a COC–map and satisfies the condition for a σ–space. Thus X is a σ–space. It is known that X is a Moore space [16], and hence first countable. If X is a strongly–quasi–N–space, it would be stratifiable by 4.1 and hence it would be paracompact. However X is not even normal: consider the two closed sets consisting of the rationals and irrationals in $\mathbb{R}$ respectively. ∎

In [12], Z. Gao proved the following result:

Theorem 3.6 *Every regular k–semi–stratifiable space is a strongly–quasi–N–space.*

Example 3.7 *There is a strongly–quasi–N–space which is not an N–space (it is not even stratifiable).*

Proof. In [38], O'Meara constructs an example of a non–normal (and hence not stratifiable) $\aleph$–space which is completely regular, and by Lemma 2.4 [29], any $\aleph$–space is k–semi–stratifiable and hence a strongly–quasi–N–space. ■

4 Metrizability Results

Theorem 4.1 *A space X is N if and only if it is a first countable strongly–quasi–N–space.*

Proof. It is well–known that every Nagata–space is a paracompact first countable space. Now, let f and g be, respectively, a first countable–map and a strongly–quasi–N–map on X. Let $h(n,x) = f(n,x) \cap g(n,x)$. It is easy to see that h is a first countable and strongly–quasi–N–map. To prove h is a Nagata–map, suppose that $h(n,x_n) \cap h(n,x) \neq \emptyset$. Then there is a sequence $\langle y_n \rangle$ such that $y_n \in h(n,x_n) \cap h(n,x)$. Since h is a first countable–map, $\langle y_n \rangle$ converges to x and h is strongly–quasi–N–map, $\langle x_n \rangle$ converges to x. Hence X is a Nagata space. ■

Corollary 4.2 *A space X is N (and stratifiable) if and only if it is a q strongly–quasi–N–space with quasi–G^*_δ–diagonal.*

Proof. The 'if' part is obvious. The 'only if' follows from Theorem 4.1 and Lemma 2.7 . ■

From Theorem 2.8, Theorem 3.6 and Theorem 4.1 we get the following result:

Corollary 4.3 *A space X is N (and stratifiable) if and only if it is a regular q k–semi–stratifiable space.*

Theorem 4.4 *A space X is metrizable if and only if X is a strongly–quasi–N–space, quasi–γ–space.*

Proof. We shall show that the space X is developable. This will complete the proof since developable spaces are first countable and first countable strongly–quasi–Nagata spaces are Nagata, hence paracompact, and paracompact developable spaces are metrizable [3]. Let $f : \mathbb{N} \times X \to \tau$ and g :

$\mathbb{N} \times X \to \tau$ be, respectively, quasi–γ and strongly–quasi–N maps for X. Let $h(n,x) = f(n,x) \cap g(n,x)$ for each $(n,x) \in \mathbb{N} \times X$. Then $h : \mathbb{N} \times X \to \tau$ is both a quasi–γ and a strongly–quasi–N map for X. Suppose $\{p, x_n\} \in h(n, y_n)$ for each $n \in \mathbb{N}$. Since h is a semistratifiable map, y_n converges to p. Also x_n converges to q (because h is quasi–γ) and since h is a strongly–quasi–N map y_n converges to q, so $p = q$. Hence x_n converges to p. ■

From [17, Corollary 4.6 (a space X is a Moore space if and only if it is a regular semi–stratifiable $w\theta$–space)], Corollary 4.3 and Nagata's famous double sequence theorem (every Nagata developable space is metrizable), we have the following:

Corollary 4.5 *A space is metrizable if and only if it is a regular k–semi–stratifiable $w\theta$–space.*

Martin proved the following result:

Theorem 4.6 *[31] Every γ quasi–Nagata–space is metrizable.*

He asked in [32, Question 1]: Is every quasi–N, quasi–γ space with a G_δ^*–diagonal metrizable?

Noting that, every space with a G_δ^*–diagonal has a quasi–S_2–diagonal, we answer this question in the affirmative by the following:

Theorem 4.7 *A quasi–γ space is metrizable if and only if it is a pseudo wN–space or quasi–Nagata–space with quasi–S_2–diagonal.*

Proof. Let X be a quasi-γ, pseudo wN-space or quasi–Nagata–space with quasi–S_2. Since every quasi-γ space is a q-space [22] then by lemma 2.7, X is first countable. From Theorem 2.6 and [17, Proposition 3.2], X is countably paracompact, so by [2], X is regular. Since every wN–space is β and every β space with quasi–S_2–diagonal is a semistratifiable space [36], X is a Nagata-space which is therefore a strongly–quasi–Nagata space. Applying Theorem 4.4 completes the proof. ■

The following is a well–known characterization of γ–spaces:

Proposition 4.8 *A space X is γ if and only if X has a COC–map g such that if $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences in X such that $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $\langle y_n \rangle$ converges to x in X, then x is a cluster point of the sequence $\langle x_n \rangle$.*

Definition 4.9 *A space X has a* **quasi–$G_\delta^*(2)$–diagonal** *if there exists a sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of open families of X such that for distinct points x, y there exists some $\mathcal{G}_m$ such that $y \notin \overline{st^k(x, \mathcal{G}_m)}(y \notin st^k(x, \mathcal{G}_m))$.*

Theorem 4.10 *A space X with a quasi–$G_\delta^*(2)$–diagonal is Nagata if and only if it is a q, quasi–N–space.*

Proof. Suppose that X is a q quasi–N–space with a quasi–$G_\delta^*(2)$ sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$. Since the space X is a q and has a quasi–G_δ^*–diagonal, by Lemma 2.7, X is a first countable space. From Theorem 4.1, we need only to prove that X is a strongly–quasi–N space. Let f be a quasi–N–map. Define $g : \mathbb{N} \times X \to \tau$ as follows:

$$g(n, x) = \begin{cases} st(x, \mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^* \ . \end{cases}$$

Let $h(n, x) = \bigcap_{i=1}^n g(i, x)$. Set $k(n, x) = f(n, x) \cap h(n, x)$. We show that k is a strongly–quasi–N–map for X. Let $y_n \in k(n, x_n)$ and suppose $\langle y_n\rangle$ converges to p. Since f is a quasi–N–map, $\langle x_n\rangle$ has a cluster point, say q. The proof ends if $p = q$. Suppose $p \neq q$. Fix $n \in c_{\mathcal{G}}(x)$. Then there are infinitely many integers $m \geq n$ such that $x_m \in k(n, q)$. Let $m \geq n$ with $x_m \in k(n, q)$. Then $x_m \in g(n, q) = st(q, \mathcal{G}_n)$. Thus $\{y_m : m \geq n\} \subseteq st^2(q, \mathcal{G}_n)$ for all $n \in c_{\mathcal{G}}(x)$. So, $p \in \overline{\{y_m : m \geq n\}} \subseteq \overline{st^2(q, \mathcal{G}_n)}$ for all $n \in c_{\mathcal{G}}(x)$. It follows that $p \in \bigcap_{n\in c_{\mathcal{G}}(x)} \overline{st^2(q, \mathcal{G}_n)} = \{q\}$. Thus $p = q$, as required. ■

From Theorem 4.2 and Theorem 4.10 we get the following result:

Corollary 4.11 *Let X be a q–space with quasi–G_δ^*–diagonal, then the following are equivalent:*

1. *X is a quasi–N space.*
2. *X is a strongly–quasi–N.*
3. *X is a wN space.*
4. *X is a N space.*

Theorem 4.12 *A space X with a quasi–G_δ^*–diagonal is γ if and only if it is a quasi–γ–space.*

Proof. Suppose that X is a quasi–γ–space with a quasi–G_δ^*–diagonal. From Lemma 2.7, X is a first countable space and by Theorem 2.6 [36], X is c–semistratifiable space. Applying Proposition 4.8, we need only to prove that

X has a COC–map g such that if $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences in X such that $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $\langle y_n \rangle$ converges to x in X, then x is a cluster point of the sequence $\langle x_n \rangle$. Let g be a c–semistratifiable–map that satisfies the condition of quasi–γ–space. Suppose that for each $n \in \mathbb{N}$, $x_n \in g(n, y_n)$ and q is a cluster point of the sequence $\langle y_n \rangle$. Since X is first countable, there is a convergent subsequence $\langle y_{j_n} \rangle$ of $\langle y_n \rangle$ such that for each $n \in \mathbb{N}, y_{j_n} \in g(n, q)$. Then $x_{j_n} \in g(j_n, y_{j_n}) \subset g(n, y_{j_n})$. Since g is a quasi–γ–map, the sequence $\langle x_n \rangle$ has a cluster point, say p. If $p = q$, then the proof is completed. Suppose that $p \neq q$. Then there is a $k \in \mathbb{N}$ such that if $n \geq k$ then $y_{j_n} \neq p$. Let $K = q \cup \{y_{j_n}\}_{n \geq k}$. Then $p \in \bigcap_{n=1}^{\infty} g(n, K) = K$, a contradiction. ■

The relationships between the classes of spaces considered in this section can be summarized in the following diagram:

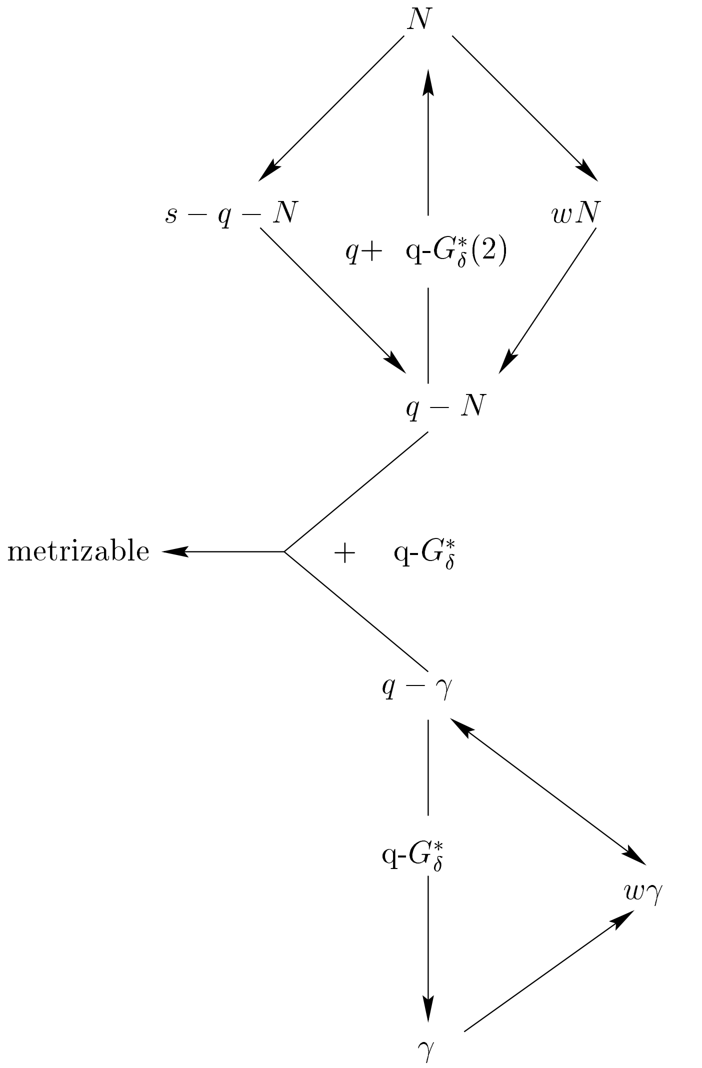

Figure 2: Relationships between generalizations of N and γ spaces.

5 Difference Between Metrizability and Strongly–quasi–N and γ Spaces

In this section we discuss and answer the question: What is the difference (in terms of g–maps) between metrizable spaces and various generalized metric spaces, like strongly–quasi–N and γ spaces. First we start with the following result which gives the difference between Lasnev (= the closed continuous image of a metric space) and strongly–quasi–N–spaces.

The proofs of the following theorems can be found in [12] and [37].

Theorem 5.1 *A space X is Lasnev (metrizable) if and only if X is Frechet (strongly Frechet), strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that if the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfy either:*

1. *$x_i \in g(n, y_i)$ for all $i \in \mathbb{N}$, and $x_j \in X - g(n, y_i)$ for all $j > i$ or*

2. *$x_i \in X - g(n, y_i)$ for all $i \in \mathbb{N}$, and $x_j \in g(n, y_i)$ for all $j > i$,*

then $\{x_i : i \in \mathbb{N}\}$ is discrete in X.

Theorem 5.2 *A space X is metrizable if and only if X is strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n, x) : x \in A\}$.*

Theorem 5.3 *(Nagata) A space X is metrizable if and only if X is strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g^2(n, x) : x \in A\}$, where $g^2(n, x) = \bigcup\{g(n, y) : y \in g(n, x)\}$.*

Theorem 5.4 *A space X is an $\aleph$–space if and only if it is strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$ and for each $x \in X, n \in \mathbb{N}$ $|\{g(n, y) : y \in g(n, x), x \notin g(n, y)\}| < \aleph_0$.*

The following theorem is due (independently) to Hung [20] and Hodel [19].

Theorem 5.5 *A space X is metrizable if and only if X has a COC–map g satisfying the following conditions:*

1. *g is a γ–map;*

2. *for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n, x) : x \in A\}$.*

The proof of our next results relies on a metrisation theorem of H. Martin [33].

Theorem 5.6 *(Martin) A necessary and sufficient condition that a topological space X be metrizable is that X has a weak development $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ such that $\{st^2(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of X.*

Definition 5.7 *A space X is called a* **pseudo–strongly–quasi–N–space** *if there is a CWBC–map $g : \mathbb{N} \times X \to \mathcal{P}(X)$ such that if for each $n \in \mathbb{N}, y_n \in g(n, x_n)$ and the sequence $\langle y_n \rangle$ converges to p in X, then p is a cluster point of the sequence $\langle x_n \rangle$. The CWBC–map g is called a pseudo–strongly–quasi–map.*

Theorem 5.8 *A space X is metrizable if and only if X has a $CWBC$–map g satisfying the following conditions:*

1. *g is a pseudo–strongly–quasi–N–map;*
2. *for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.*

Proof. The only if part is obvious. We now prove the if part. Assume that X has a a $CWBC$–map g satisfying the conditions (1) and (2). Let $h(n,x) = \{y \in X : x \in g(n,y)\}$ and $k(n,x) = g(n,x) \cap h(n,x)$ for each $(n,x) \in \mathbb{N} \times X$. Let $\mathcal{G}_n = \{k(n,x) : (n,x) \in \mathbb{N} \times X\}$. Then $st(x, \mathcal{G}_n) = \bigcup\{k(n,y) : x \in k(n,x)\}$ and $st^2(x, \mathcal{G}_n) = \bigcup\{k(n,y) : k(n,y) \cap st(x, \mathcal{G}_n) \neq \emptyset, (n,x) \in \mathbb{N} \times X\}$.

By condition (2), $h(n,x)$ is a neighborhood (not necessarily open) of x and so is $k(n,x)$. Therefore, in virtue of the Martin metrization theorem 5.6, we only need prove that $\{st^2(x, \mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of X. If $\{st^2(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is not a local weak base for some $x \in X$, then there exists an open neighbourhood U of x such that $st^2(x, \mathcal{G}_n) - U \neq \emptyset$ for each $n \in \mathbb{N}$. Take $y_n \in st^2(x, \mathcal{G}_n) - U, n \in \mathbb{N}$. That means we can find $z_n, w_n \in X$ such that $y_n \in k(n, z_n), k(n, z_n) \cap k(n, w_n) \neq \emptyset, x \in k(n, w_n)$. Take $v_n \in k(n, z_n) \cap k(n, w_n)$. By $x \in k(n, w_n) \subseteq g(n, w_n)$ and condition (1), we conclude that $\langle w_n \rangle$ converges to x, and by $v_n \in k(n, w_n) \subseteq h(n, w_n)$ and the definition of h, we get $w_n \in g(n, v_n)$. Using condition (1) again, we have that $\langle v_n \rangle$ converges to x. Similarly, from $v_n \in k(n, z_n) \subseteq g(n, z_n)$, we have that $\langle z_n \rangle$ converges to x, and by $y_n \in k(n, z_n) \subseteq h(n, z_n)$, we get that $\langle y_n \rangle$ converges to x. But $y_n \notin U$ for each $n \in \mathbb{N}$, which is a contradiction. ■

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